Vector Calculus with Vector Algebra

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Vector Calculus
with Vector Algebra

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Preface

Vector Calculus with Vector Algebra is intended primarily for a one-year sequence in linear algebra and multivariable calculus, to follow a sequence of calculus of one variable. Since a linear-algebra approach to vector calculus necessarily involves a sophistication that strongly challenges most college sophomores, I have included many aids for the reader, the most basic of which is the "dual track" organization of the book.

In each chapter, the discussions of theory are augmented by problem sets correlated with the mathematical structure. These sections in many ways duplicate the information presented in the main body of the chapter. However, they also provide an independent development of the major ideas with emphasis on self-study and techniques of manipulation. The problem sets can be used to obtain a prior working knowledge of the symbols and formulas which accompany the development of the theory; by giving meaning to abstract formulations, they also provide a natural motivation for the student.

In addition to the problem sets at the end of each chapter, there are completion-type questions at the end of each section which underscore important concepts of newly learned theory; exercises that promote a deeper understanding of the material; and proof exercises which may be considered optional should they require more time than is available in the course.

The first eight chapters are devoted to a development of vector algebra that blends linear algebra and vector concepts from geometry and physics. Because of its role as a background for the unfolding of the vector calculus, more emphasis is placed on functions and vector sets—such as lines, planes, and parallelograms—than is usually found in linear-algebra texts. In Chapter VII, the absolute determinant is introduced for its later use as a valuable link between linear algebra and the integral on curves and surfaces. A special technique is given in Chapter VIII for testing the definite property of symmetric matrices; this becomes useful in the study of maxima and minima.

The vector calculus builds upon the vector derivative; as a natural extension of the ordinary derivative, the vector derivative yields helpful geometric interpretations for real functions of two variables, and the basic theory of the differential and the affine approximation becomes a corollary of its development. Through application of the magnification theorem, introduced in the vector-algebra section, the integral on curves, regions, surfaces, and solids is seen as a special case of the single integral. The final chapter includes the basic study of line and surface integrals and a clarification of the often troublesome relationship between the symbolism used for vector analysis and that used for differential forms.

Many of the results in the text have proofs which, although general in idea, are made for specific cases. This makes the notation simpler and the proof easier to read. Some of the more difficult proofs are in an appendix.

> In deciding how to use the text for a given class, first consideration should probably be given to the problem sets. Each set is assigned to a related reading section, as shown in the outlines at the beginning of each chapter. However, because of some duplication of definitions and formulas, each set can be worked independently of its section. One might, of course, use these problems in the conventional manner, as assignments at the end of each section. An alternative method, which I have used many times with satisfaction, separates the treatment of problem sets in this way: devote one class per week to a question-and-answer session on the problems, followed by a short quiz to test students on routine problem techniques, rather than on depth of understanding or imaginative talent. The remaining classes of the week, then, are concerned with the mathematical structure and exercise sets, which demand deeper understanding. It is usually best to pace the problem set slightly ahead of its related material when using this dual-track approach.

Asterisks are used in the table of contents to indicate the level and importance of various sections. A single asterisk indicates a level of difficulty higher than the average among the sections required for proper understanding of later discussions; an instructor might give these sections light treatment with emphasis on definitions and results. Sections marked by a double asterisk are optional in that they are not needed for later sections of the text; they might, however, have relevance to other course work.

I would like to thank all of those who have contributed to the development of this book. Many helpful ideas have come from my colleagues, particularly Herman Meyer, who class tested preliminary versions. George Pedrick, California State College, Hayward, besides uncovering numerous flaws during his review of the manuscript, made excellent suggestions on the motivational material in the text. To the staff of Wadsworth Publishing Company, I am grateful for valuable assistance and their concern that this book be directed primarily to the interests of the student. A most significant contribution has come from classroom students who have freely questioned and criticized the preliminary manuscripts. Finally, I am grateful to my wife Julia for typing the manuscript.

Paul McDougle

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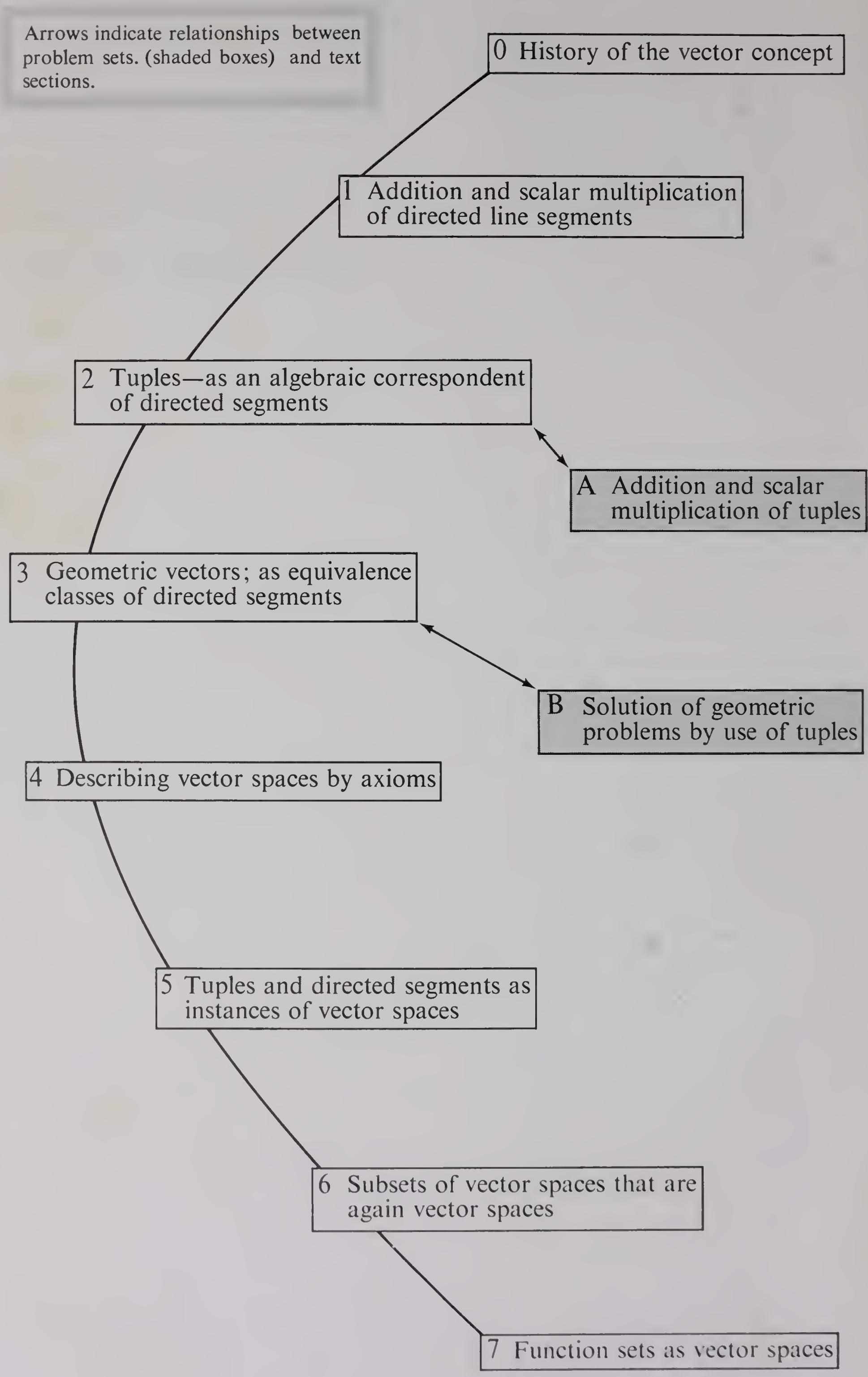
Vector Calculus
with Vector Algebra


Arrows indicate relationships between problem sets. (shaded boxes) and text sections.
0 History of the vector concept
1 Addition and scalar multiplication of directed line segments
2 Tuples—as an algebraic correspondent of directed segments
A Addition and scalar multiplication of tuples
3 Geometric vectors; as equivalence classes of directed segments
B Solution of geometric problems by use of tuples
4 Describing vector spaces by axioms
5 Tuples and directed segments as instances of vector spaces
6 Subsets of vector spaces that are again vector spaces
7 Function sets as vector spaces

I

Vector Spaces

No one individual can be named as the inventor of vector algebra. The *parallelogram law*, which states a fundamental property of geometric vectors, was recognized by the ancient Greeks in their study of velocity problems. The velocity of an object is specified by its rate or speed and its direction of movement; it is described geometrically by a directed line segment. According to this law, the combined effect, called the *resultant*, of two velocities in different directions may be determined as the diagonal of a parallelogram whose sides denote the given velocities. An example will illustrate the principle.

> **Example 1** A boat which moves at a rate of 10 miles per hour in still water is to cross a stream from A to B along a straight-line path (see Figure 1.1). The stream current is 3 miles per hour. The direction of the boat's course must be chosen so that the parallelogram having sides 3 and 10 in the respective directions of the current and course will have a diagonal in the direction from A to B. The resulting speed of the boat is the length of the diagonal.

Much later, in the seventeenth century, other physical concepts evolved that could be represented as vectors. Isaac Newton of England (1642–1727)

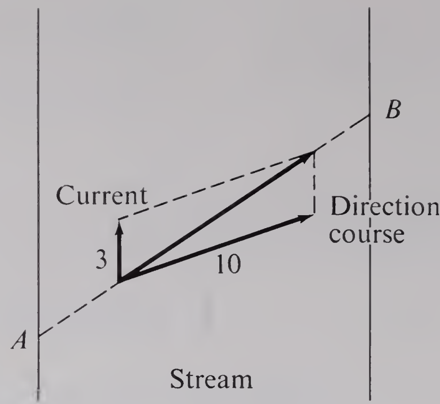


Figure 1.1

gave significance to the concept of force as a quantity possessing both magnitude and direction. He formally asserted that two forces give rise to a resultant (equivalent) force which is determined by the parallelogram law. He also said that a force can be split into component forces along any two nonparallel lines so that the resultant of the two components is the given force. This splitting is described geometrically by representing the force as a directed line segment, and then projecting this segment onto the two lines.

Example 2 A load is pulled up an inclined plane by two forces A, B acting in different directions (see Figure 1.2). The resultant is represented by the diagonal of a parallelogram with sides A, B. The forces A, B may each be split into a component along the inclined

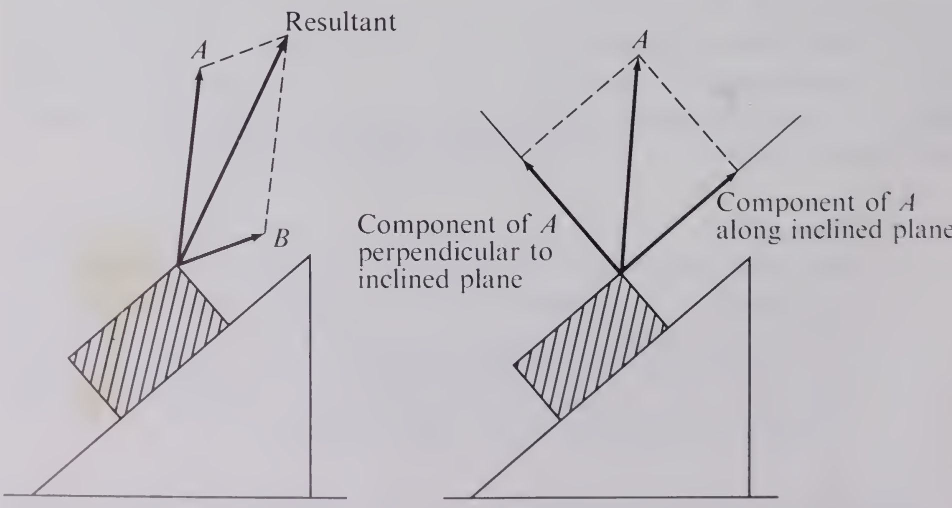


Figure 1.2

plane and a component perpendicular to the inclined plane. From physics we know that the work done by each of the forces, A, B, is the product of the distance the load moves and the component of the force along the inclined plane.

Early in the nineteenth century, the next important phase in the development of vector algebra arose in connection with the study of complex numbers. The complex number $a + b\sqrt{-1}$ may be geometrically represented by the point (a, b) in the Cartesian plane. In order to get a geometric study of the product operation on complex numbers, however, mathematicians often found it more useful to represent $a + b\sqrt{-1}$ by the directed segment from the origin to the point (a, b). Many mathematicians sought a geometric system with properties similar to those of the complex numbers.

The beginning of vectors as a mathematical system in the 1830's is credited to W. R. Hamilton of Ireland (1805–1865) and H. G. Grassmann of Germany (1809–1877), who worked independently. Hamilton, a physicist and mathematician, sought a 3-dimension analogue of complex numbers. His efforts resulted in a system whose elements he called *quaternions*. A quaternion is algebraically 4-dimensional, but it can be split into a real-number part, which Hamilton termed *scalar*, and a 3-dimensional part, which Hamilton called *vector*. The word scalar, which has persisted to the present day, arose from Hamilton's visualization of the real numbers as a time scale. The term vector, which is based on the Latin word *vehere* meaning "to carry," had previously been used in the expression radius vector.

Grassmann was a philospher and mathematician. Although his development of vectors is generally considered mathematically superior to Hamilton's, it was so couched in abstraction and philosophical discussion that it was not understood. Grassmann's work contains a large portion of modern vector analysis and was far ahead of his time.

Questions

1. The ancient Greeks obtained the resultant of two velocities using the ________ law.
2. Physical quantities which could be described by vectors were conceived in the seventeenth century by ________.
3. Vector operations were employed in the geometric study of ________ numbers.
4. Hamilton's study of space analysis led to a system whose elements he named ________.
5. The most comprehensive development of vectors in the nineteenth century was made by ________.

Exercises

For each of the following problems, sketch a suitable parallelogram and determine the answer by trigonometry.

1. An airplane steers a course due east at 500 mph and encounters a wind blowing 25 mph in the direction 60° north of east. Find the resultant speed and direction.
2. A ship which moves at a rate of 20 mph in still water must travel in a direction due north. If the ocean current is 4 mph in an easterly direction, in what direction should the ship steer?
3. A load is pulled up a plane inclined at 30° to the horizontal by forces of 20 and 30 lbs applied, respectively, at angles of 30° and 45° with respect to the plane.
   (a) Find the resultant of the two forces.
   (b) Find for each of the forces the components parallel to and perpendicular to the plane.

1. Directed Line Segments

In this section we shall study addition and multiplication operations on the set of directed line segments in a plane and in space. Definitions of these operations are motivated by force and velocity examples. For instance, a force may be geometrically represented by a directed line segment whose direction and length, respectively, correspond to the direction and magnitude of the force. It will then follow that the combined effect of two forces is described by addition of their directed segments. A change in the magnitude of a force will then be described by the multiplication of the directed segment by a real number. First to be considered are segments in the Euclidean plane, which is the plane studied in high school geometry. A fixed point O is selected to be the *initial point* of all directed segments. This restriction of a common initial point is imposed to simplify the analysis. If P is any other point in the plane, then the symbol $\overline{OP}$ will denote the line segment directed from O to P. The point P is called the *terminal point* of the segment. *Addition* is defined according to the parallelogram law:

(a) $\overline{OP} + \overline{OQ} = \overline{OR}$, *where* $\overline{OR}$ *is the diagonal of the parallogram with sides* $\overline{OP}$ *and* $\overline{OQ}$.

This definition is incomplete, for if $\overline{OP}$ and $\overline{OQ}$ are *collinear*, meaning they lie on the same line, then there is no parallelogram. For the collinear case we have the following definition, which is consistent with our motivating examples of velocity and force:

(b) *If $\overline{OP}$ and $\overline{OQ}$ have the same direction, then $\overline{OP}+\overline{OQ}$ also has that same direction and has length equal to the sum of the lengths of $\overline{OP}$ and $\overline{OQ}$.*

(c) *If $\overline{OP}$, $\overline{OQ}$ have opposite directions, then $\overline{OP}+\overline{OQ}$ has the same direction as the longer of them and has length equal to the difference of their lengths.*

We have not yet included the case in which $\overline{OP}$, $\overline{OQ}$ have the same length and opposite directions. Two forces of equal magnitude acting in opposite directions will produce a null resultant, which cannot be described by a directed segment. However in mathematics it is desirable to have operations completely defined—such operations are called *closed*—whenever this can be done without introducing inconsistencies. We therefore introduce the *zero segment* $\overline{OO}$, which is geometrically described by the single point O. The distinction between O and $\overline{OO}$ is subtle, but worth noting. The definition of addition can now be made complete:

(d) *If $\overline{OP}$ and $\overline{OQ}$ have equal length and opposite directions, then $\overline{OP}+\overline{OQ}=\overline{OO}$.*

(e) $\overline{OP}+\overline{OO}=\overline{OO}+\overline{OP}=\overline{OP}$ *for each segment* $\overline{OP}$.

Various cases of the addition operation are shown in Figure 1.3.

Multiplication of a directed segment by a real number, called *scalar multiplication,* is defined as follows (see Figure 1.4).

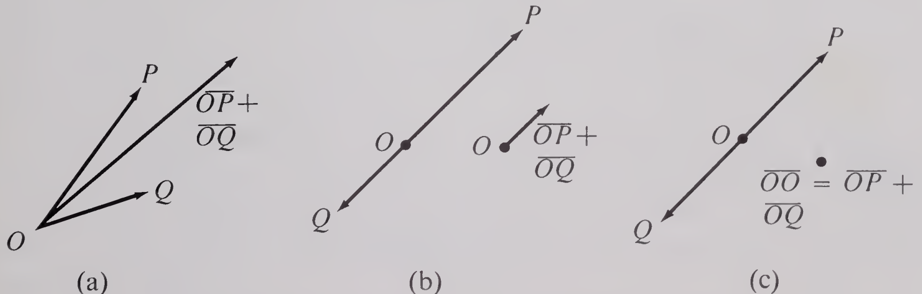


Figure 1.3

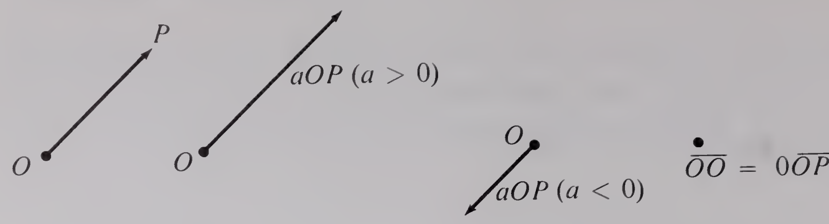


Figure 1.4

(a) *If $a = 0$ or $P = O$, then $a\overline{OP} = \overline{OO}$.*
(b) *If $a > 0$ and $P \neq O$, then $a\overline{OP}$ has the same direction as $\overline{OP}$ and length a times the length of $\overline{OP}$.*
(c) *if $a < 0$ and $P \neq O$, then $a\overline{OP}$ has the opposite direction of $\overline{OP}$ and length $(-a)$ times the length of $\overline{OP}$.*

From these definitions it is possible to prove various properties by geometric methods. The proofs of some, such as $\overline{OP} + \overline{OQ} = \overline{OQ} + \overline{OP}$, are trivial. Other properties can be proved only by considering various cases and using cumbersome techniques from high school geometry. An illustration of an associative and a distributive property is shown in Figure 1.5. As we shall see later in this chapter, most fundamental properties are more easily proved through algebraic methods made possible by the introduction of a Cartesian coordinate system.

Any two noncollinear directed segments induce a coordinate system for the plane. For instance, let $\overline{OP}$ and $\overline{OQ}$ be noncollinear, as shown in Figure 1.6, and let R be an arbitrary point in the plane. A line drawn through R and parallel to $\overline{OQ}$ intersects the line containing $\overline{OP}$ in a point P'. Similarly a line drawn through

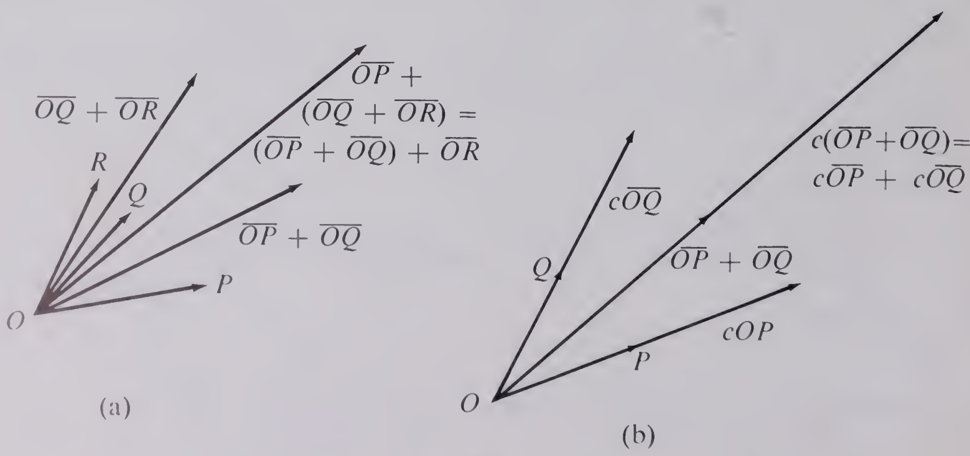


Figure 1.5

R and parallel to $\overline{OP}$ intersects the line containing $\overline{OQ}$ in a point Q'. Since $\overline{OP'}$ is collinear with $\overline{OP}$, there exists a number a such that $\overline{OP'} = a\overline{OP}$. Also there exists a number b such that $\overline{OQ'} = b\overline{OQ}$. The numbers a, b are called the *coordinates* of R relative to $\overline{OP}$, $\overline{OQ}$. It is evident from Figure 1.6 that

$$\overline{OR} = \overline{OP'} + \overline{OQ'} = a\overline{OP} + b\overline{OQ}.$$

We have shown the coordinate axis property for a plane.

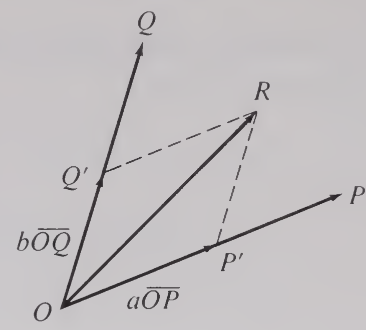


Figure 1.6

Coordinate Axis Property (Plane)

If $\overline{OP}$ and $\overline{OQ}$ are not collinear, then for each point R there exist unique numbers a, b such that

$$\overline{OR} = a\overline{OP} + b\overline{OQ}.$$

We now seek to describe the property of collinearity in terms of our two operations. From the definition of scalar multiplication it appears that $\overline{OQ}$ is collinear with $\overline{OP}$ if and only if $\overline{OQ} = c\overline{OP}$ for some number c. This conclusion is not valid, however, if $\overline{OP} = \overline{OO}$ and $\overline{OQ} \neq \overline{OO}$. All cases are included when collinearity is defined according to the following criterion.

Criterion for Collinear Segments

$\overline{OP}$ and $\overline{OQ}$ are collinear if and only if there exist real numbers a, b not both 0 *such that*

$$a\overline{OP} + b\overline{OQ} = \overline{OO}.$$

It should be noted that the numbers a, b are not unique. If $2\overline{OP} + 3\overline{OQ} = \overline{OO}$, then also $4\overline{OP} + 6\overline{OQ} = \overline{OO}$.

We shall next look at directed line segments in space. A large portion of the study in the plane may be carried over with little or no change. A fixed point O is again chosen, and operations of addition and multiplication are defined as before. Three segments are *coplanar* if they lie in the same plane. The defined operations imply

(a) $\overline{OP} + \overline{OQ}$ is coplanar with $\overline{OP}$ and $\overline{OQ}$,
(*b*) $a\overline{OP}$ is collinear with $\overline{OP}$.

Analogous to the coordinate axis property for the plane is the coordinate axis property for space.

Coordinate Axis Property (Space)

If $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ are not coplanar, then for each point S in space there exist unique numbers a, b, c such that

$$\overline{OS} = a\overline{OP} + b\overline{OQ} + c\overline{OR}.$$

The criterion for collinear segments remains the same as for directed segments in the plane. We shall now derive a similar description for the coplanar property of three segments in space, $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$. Consider first the case in which $\overline{OP}$, $\overline{OQ}$ are not collinear and $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ are coplanar. Application of the coordinate axis property to the plane containing $\overline{OP}$, $\overline{OQ}$ shows that $\overline{OR} = a\overline{OP} + b\overline{OQ}$ for suitable numbers a, b. This may be written in the form

$$a\overline{OP} + b\overline{OQ} + (-1)\overline{OR} = \overline{OO}.$$

If $\overline{OP}$, $\overline{OQ}$ are collinear, then there exist a, b such that

$$a\overline{OP} + b\overline{OQ} = \overline{OO}$$

by the criterion for collinear segments. From $0\overline{OR} = \overline{OO}$, we obtain the equation

$$a\overline{OP} + b\overline{OQ} + 0\overline{OR} = \overline{OO}.$$

We have proved the necessity portion of the following criterion (for the sufficiency see Proofs, exercise 3).

Criterion for Coplanar Segments

$\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ are coplanar if and only if there exist numbers a, b, c not all 0 such that

$$a\overline{OP} + b\overline{OQ} + c\overline{OR} = \overline{OO}.$$

Questions

1. The parallelogram law is used in defining the ________ operation.
2. The parallelogram law does not apply to ________ segments.

3. If $\overline{OP}$, $\overline{OQ}$ have opposite directions and equal length, then $\overline{OP} + \overline{OQ} =$ ________.

4. $\overline{OP}$ and $a\overline{OP}$ have the same direction provided a is ________.

5. A coordinate system for the plane may be obtained from any two ________directed segments.

6. A coordinate system for space may be obtained from any three ________ directed segments.

7. $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ are coplanar provided there exist numbers a, b, c not all 0 such that ________.

8. $a\overline{OP} = \overline{OO}$ provided either ________ or ________.

Exercises

1. In Figure 1.7, sketch
   (a) $3\overline{OP}$, (b) $\overline{OP} + \overline{OQ}$,
   (c) $\overline{OP} + (-2)\overline{OR}$, (d) $3\overline{OP} + 5\overline{OQ}$,
   (e) $\overline{OP} + (\overline{OQ} + \overline{OR})$.

2. In Figure 1.7, find (approximately) a, b such that
   (a) $\overline{OR} = a\overline{OP} + b\overline{OQ}$,
   (b) $\overline{OP} = a\overline{OQ} + b\overline{OR}$,
   (c) $\overline{OQ} = a\overline{OP} + b\overline{OR}$.

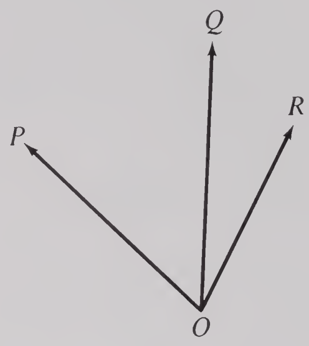


Figure 1.7

Proof and Illustrations

1. Construct a diagram to illustrate

$$2(\overline{OP} + \overline{OQ}) = 2\overline{OP} + 2\overline{OQ}.$$

2. Construct a diagram to illustrate

$$(\overline{OP} + \overline{OQ}) + (-1)\overline{OR} = \overline{OP} + (\overline{OQ} + (-1)\overline{OR}).$$

3. Prove that if $a\overline{OP} + b\overline{OQ} + c\overline{OR} = \overline{OO}$ in space, where a, b, c are not all 0, then $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ are coplanar.

2. *Tuples*

A unification of geometry and algebra is achieved through the introduction of coordinate axis systems. Rene Descartes of France (1596–1650)

is generally credited with the basic idea of the coordinate system. The Euclidean plane with the standard right-angle axis system is called the *Cartesian plane*; and space with an axis system of three perpendicular lines is called *Cartesian space.* We get the desired relation between algebra and geometry by placing the origin of the Cartesian axes at the fixed point O. For any point P in the plane, there is a directed segment $\overline{OP}$ and a number pair $\langle a, b\rangle$, where a, b are, respectively, the abscissa and the ordinate of P. The point P is, as usual, denoted by (a, b) (see Figure 1.8). Thus the symbol $\langle a, b\rangle$ denotes a number pair, whereas (a, b) denotes a point. The distinction is delicate, but important, and it will be maintained throughout the text. The number pair $\langle a, b\rangle$ is called a *2-tuple*.

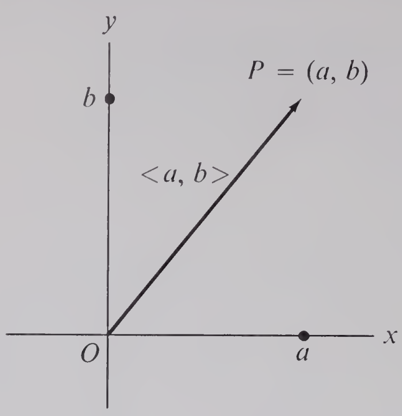


Figure 1.8

The correspondence between $\overline{OP}$ and $\langle a, b\rangle$ is *one-to-one*, meaning that to each directed segment $\overline{OP}$ there is associated exactly one 2-tuple $\langle a, b\rangle$ and to each 2-tuple there is associated exactly one segment. This correspondence enables us to study directed segments via associated 2-tuples, and thus to use algebraic methods. We need to define operations on 2-tuples which correspond to the two operations on directed segments. The boldface symbol **OP** will denote the 2-tuple corresponding to $\overline{OP}$. Therefore:

$$\mathbf{OP} = \langle a, b\rangle \quad \text{if and only if} \quad P = (a, b).$$

It is desired to find an additive operation on 2-tuples so that the sum of 2-tuples is associated with the sum of their corresponding segments. From Figure 1.9 we see this is accomplished by the definition

$$\langle a, b\rangle + \langle a', b'\rangle = \langle a + a', b + b'\rangle.$$

Next we wish to define scalar multiplication so that the product of a number and a 2-tuple is associated with the product of the number and the segment corresponding to the 2-tuple. From Figure 1.10 this definition is accomplished by the following equation:

$$c\langle a, b\rangle = \langle ca, cb\rangle.$$

A similar analysis applies to space. There is a one-to-one correspondence between directed segments and *3-tuples* $\langle a, b, c\rangle$. We set $\mathbf{OP} = \langle a, b, c\rangle$ whenever $P = (a, b, c)$ (see Figure 1.11). The operations on *3-tuples* are an obvious extension.

$$\langle a, b, c\rangle + \langle a', b', c'\rangle = \langle a + a', b + b', c + c'\rangle,$$
$$d\langle a, b, c\rangle = \langle da, db, dc\rangle.$$

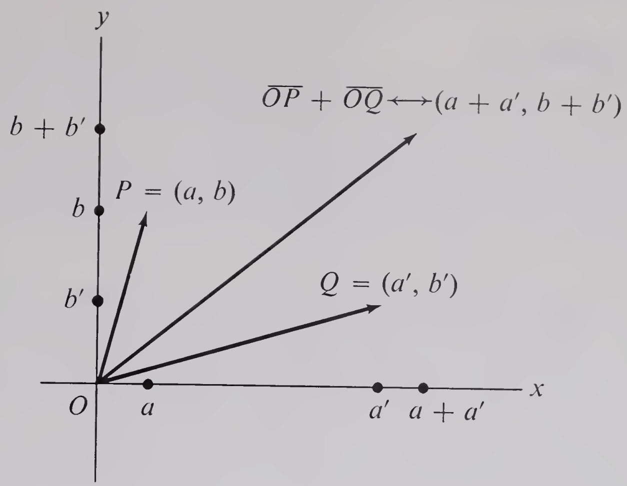


Figure 1.9

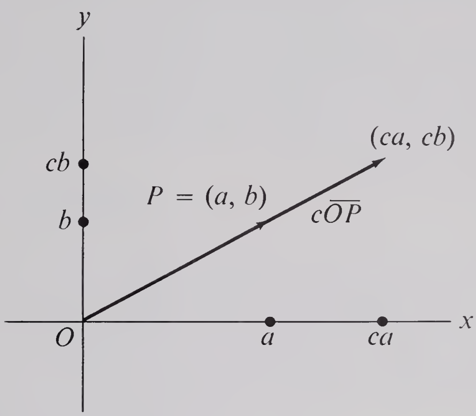


Figure 1.10

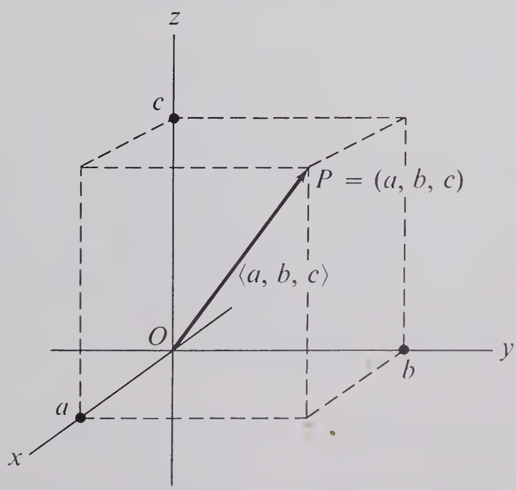


Figure 1.11

Example 2.1

(a) $\langle 1, 3\rangle + \langle 2, -6\rangle = \langle 1 + 2, 3 - 6\rangle = \langle 3, -3\rangle$,
(b) $4\langle 0, 3\rangle = \langle 4(0), 4(3)\rangle = \langle 0, 12\rangle$,
(c) $\langle 1, 0, 3\rangle + \langle 2, -9, 6\rangle = \langle 1 + 2, 0 - 9, 3 + 6\rangle = \langle 3, -9, 9\rangle$,
(d) $7\langle 2, -1, 3\rangle = \langle 7(2), 7(-1), 7(3)\rangle = \langle 14, -7, 21\rangle$.

There is an advantage to the development of tuples, rather than directed segments, in addition to that afforded by the use of algebraic techniques for computation. Although geometric studies of directed segments cannot proceed beyond dimension three, there is no such limitation to the concept of tuples. All objections that large size tuples have no relevance to a 3-dimensional world have been demolished by developments in physical science in the past one hundred years. The best known example, though only one of many, occurs in Einstein's theory of relativity, which merges time and space into a 4-dimensional mathematical system.

We now come to the concept of an *n-tuple* as an ordered set of n real numbers, where n is a positive integer. Subscript notation is necessary for adequate symbolization of an n-tuple. The following definitions are obvious extensions of the cases $n = 2$ and $n = 3$.

Addition of n-Tuples (add tup)

$$\langle a_1, a_2, \ldots, a_n\rangle + \langle b_1, b_2, \ldots, b_n\rangle = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\rangle.$$

Scalar Multiplication of n-Tuples (sc mp tup)

$$c\langle a_1, a_2, \ldots, a_n\rangle = \langle ca_1, ca_2, \ldots, ca_n\rangle.$$

The number a_i is called the ith *coordinate*, or *component*, of $\langle a_1, a_2, \ldots, a_n\rangle$. The tuples $\langle a_1, a_2, \ldots, a_n\rangle$ and $\langle b_1, b_2, \ldots, b_n\rangle$ are *equal* provided $a_i = b_i$ for every i, $i = 1, 2, \ldots, n$. The case $n = 1$ merits special attention. There is a trivial correspondence $\langle a\rangle \leftrightarrow a$ between 1-tuples and real numbers. Consequently 1-tuples are generally regarded as real numbers, although there is an essential difference which will at times need recognition.

Questions

1. The number of 2-tuples associated with each directed segment in the plane is ________.

2. If $P = (2, 3)$, then $\mathbf{OP} =$ ______.
3. There are no corresponding directed segments for n-tuples if n is greater than ______.
4. The third coordinate of $\langle 3, 4, 5, 6\rangle$ is ______.
5. It is conventional to replace n-tuples by real numbers when $n =$ ______.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. Using the commutative property for real numbers, prove the commutative property of addition for 2-tuples.

3. Geometric Vectors

Thus far, the study of directed segments has been restricted to the case in which all segments have the same initial point O. In most physical situations the vector quantities do not all have a single point of action. At times it is convenient to have the terminal point of one segment be the initial point of another. An example will illustrate.

Example 3.1 An object moves along straight-line paths from O to S, as shown in Figure 1.12. We wish to find the position of S relative to O. Corresponding to each of the segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ are segments originating at O and having the same length and direction. Adding the 2-tuples associated with each of these four segments gives

$$\begin{aligned}&\langle 10\cos 45°, 10\sin 45°\rangle + \langle 15\cos 330°, 15\sin 330°\rangle\\ &\qquad + \langle 20\cos 60°, 20\sin 60°\rangle + \langle 10\cos 180°, 10\sin 180°\rangle\\ &\qquad = \langle 20.1, 16.9\rangle.\end{aligned}$$

Therefore S is 20.1 units east and 16.9 units north of O.

In spite of a need for adding directed segments not having the same initial point, there is no plausible definition for the addition of *any* two directed segments in the plane. For instance, there is no natural choice for the initial point of the sum of segments $\overline{PQ}$ and $\overline{RS}$ if $P \neq R$. The dilemma is resolved by

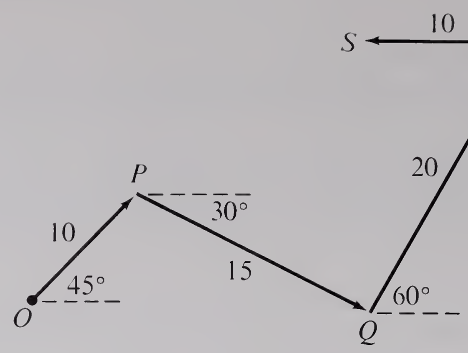


Figure 1.12

introducing the concept of *equivalence*. Two directed line segments $\overline{PQ}$ and $\overline{RS}$ in the plane or in space are said to be equivalent, written $\overline{PQ} \sim \overline{RS}$, provided they have the same length and direction. Corresponding to each segment $\overline{PQ}$ is the *geometric vector* $[\overline{PQ}]$ consisting of all directed segments equivalent to $\overline{PQ}$. Thus $[\overline{PQ}] = [\overline{RS}]$ if and only if $\overline{PQ} \sim \overline{RS}$. The notion of the geometric vector leads to natural definitions of addition and scalar multiplication which apply to any directed segments. By geometric considerations, for each P and Q there is a unique point Q' such that $\overline{OQ'} \sim \overline{PQ}$. A definition for addition and scalar multiplication of geometric vectors is therefore given by the following:

(a) $[\overline{OP}] + [\overline{OQ}] = [\overline{OP} + \overline{OQ}]$,
(b) $a[\overline{OP}] = [a\overline{OP}]$.

Example 3.2 Referring to Figure 1.13, we have $\overline{OQ'} \sim \overline{PQ}$, $\overline{OS'} \sim \overline{RS}$ and hence

$$[\overline{RS}] + [\overline{PQ}] = [\overline{OS'}] + [\overline{OQ'}] = [\overline{OT}],$$

where $\overline{OT}$ is the diagonal of a parallelogram with sides $\overline{OQ'}$, $\overline{OS'}$.

An important property of the addition operation is

$$[\overline{PQ}] + [\overline{QR}] = [\overline{PR}].$$

This can be easily illustrated by a sketch.

Algebraic techniques may also be used in the study of geometric vectors. We have already seen a correspondence between tuples and directed segments originating at O and a correspondence between segments originating at O and geometric vectors. Combining these gives a one-to-one correspondence between

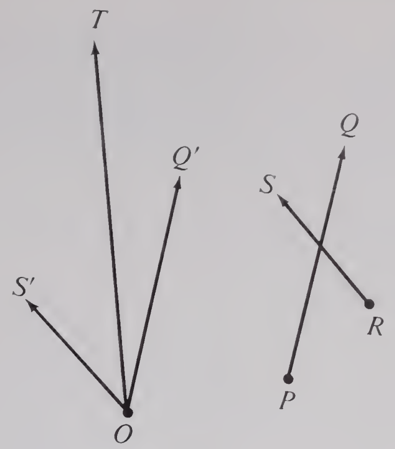


Figure 1.13

tuples and geometric vectors. We let $[\mathbf{PQ}]$ denote the tuple associated with the geometric vector $[\overline{PQ}]$. It may be verified that if $P=(a, b)$ and $Q=(a', b')$ then

$$[\mathbf{PQ}] = \langle a' - a, b' - b \rangle.$$

In space, if $P=(a, b, c)$ and $Q=(a', b', c')$, then

$$[\mathbf{PQ}] = \langle a' - a, b' - b, c' - c \rangle.$$

Example 3.3 If $P=(1, 2, 6)$ and $Q=(3, 2, -5)$, then

$$[\mathbf{PQ}] = \langle 3-1, 2-2, -5-6 \rangle = \langle 2, 0, -11 \rangle.$$

The geometric vector $[\overline{PQ}]$ is usually written $\overline{PQ}$ (without brackets). Its corresponding tuple is then written $\mathbf{PQ}$. We shall use this simpler notation. The ambiguous use of $\overline{PQ}$ as both a directed segment and a geometric vector will lead to no confusion.

These concepts will now be applied to an abstract physical problem. Let a point particle move in space from P to Q along a straight line and at a constant rate r, such that the change in position requires c units of time. We assume a coordinate system with origin O. By definition the *original position vector* is $\mathbf{OP}$, the *final position vector* is $\mathbf{OQ}$, and the *final displacement vector* is $\mathbf{PQ}$. From the equation

$$\text{rate} = \frac{\text{distance}}{\text{time}},$$

it follows that the *velocity vector* is $\mathbf{v} = \frac{1}{c}\mathbf{PQ}$ and the *displacement vector* after time t is $t\mathbf{v} = \frac{t}{c}\mathbf{PQ}$. The *position vector* after time t is $\mathbf{OP} + \frac{t}{c}\mathbf{PQ}$.

Example 3.4 A point particle moves from $P = (2, 1, 0)$ to $Q = (10, -3, 4)$ along a straight-line path and at a constant rate during a time interval of 4 units. The original position vector is $\langle 2, 1, 0 \rangle$, and the final position vector is $\langle 10, -3, 4 \rangle$. The final displacement vector is

$$\mathbf{PQ} = \langle 10 - 2, -3 - 1, 4 - 0 \rangle = \langle 8, -4, 4 \rangle.$$

The velocity vector is

$$\mathbf{v} = \frac{1}{4}\mathbf{PQ} = \langle 2, -1, 1 \rangle.$$

After t units of time the displacement vector is

$$t\mathbf{v} = t\langle 2, -1, 1 \rangle = \langle 2t, -t, t \rangle,$$

and the position vector is

$$\langle 2, 1, 0 \rangle + \langle 2t, -t, t \rangle = \langle 2 + 2t, 1 - t, t \rangle.$$

Questions

1. Equivalent directed segments have the same ______ and ______.
2. A total set of equivalent directed segments is called a ______.
3. The physical vector representing a change in position is called a ______ vector.
4. If $P = (1, 3)$ and $Q = (2, 5)$, then $\overline{PQ} \sim \overline{OR}$ where $R =$ ______.
5. The sum of the original position vector and final displacement vector is the ______ vector.

Problems

1. Do Problem set B at the end of the chapter.

Exercises

1. A ship moves 50 miles in a direction 30° south of east, then 100 miles in a northeast direction, and then 30 miles in a direction 60° west of north. The final position is how many miles north and east of the original position?
2. A point particle moves from $(3, 2, 5)$ to $(11, -6, 9)$ along a straight line and at a constant rate during a time interval of 2 units. Find

(a) the final displacement vector,
(b) the velocity vector,
(c) the displacement vector after time t,
(d) the position vector after time t.

Proofs

1. Prove $\overline{PQ} + \overline{QR} = \overline{PR}$
   (a) by geometric methods,
   (b) by first showing $\mathbf{PQ} + \mathbf{QR} = \mathbf{PR}$.
2. Using the property $\overline{PQ} + \overline{QR} = \overline{PR}$, prove
   (a) $(\overline{PQ} + \overline{QR}) + \overline{RS} = \overline{PS}$,
   (b) $\overline{PQ} - \overline{PR} = \overline{RQ}$ (where $\overline{PQ} - \overline{PR} = \overline{PQ} + (-1)\overline{PR}$).
3. A relation $\sim$ is an *equivalence* relation if it satisfies the following three properties: (1) *reflexive*: $a \sim a$, (2) *symmetric*: $a \sim b$ implies $b \sim a$, and (3) *transitive*: $a \sim b$ and $b \sim c$ implies $a \sim c$. Prove the relation $\overline{PQ} \sim \overline{RS}$ is an equivalence relation.

4. *Abstract Vector Spaces*

Thus far we have studied the primitive vector systems, which are of a physical and geometrical nature, and the tuples, which are algebraic. The word vector may be correctly applied to a directed segment, a geometric vector, or a tuple. Furthermore, other mathematical objects such as functions and matrices may also be rightfully called vectors. What makes an object a vector, in the modern sense, is not the specific properties of the object, but rather the role of the object within a mathematical system.

A principal requirement of a modern mathematical study is a *form* or *structure* capable of rigorous development. The classical model is Euclidean geometry, which was written by Euclid in Egypt 300 BC. Since 1900 much attention has been given to axiomatic characterizations of mathematical systems. The procedure is to define an abstract system in terms of a set, operations on the elements of that set, and imposed properties of those operations. The specific nature of the elements in the set is immaterial in the definition.

We are interested in the abstract system called a (*real*) *vector space*. The set, whose elements are called *vectors*, will usually be denoted by $\mathbf{V}$. Elements of $\mathbf{V}$ will be symbolized by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and so forth. There are two basic operations, called *addition* and *scalar multiplication*. Real numbers are called *scalars* when related to vector spaces.

There are nine properties which will be used to characterize a vector space.

(1) *Closure* Closure of addition means that if **u**, **v** are vectors in **V** then also $\mathbf{u} + \mathbf{v}$ is in **V**. Closure of scalar multiplication means that if **u** is in **V** and c is a real number then also $c\mathbf{u}$ is in **V**.

In general an operation on a system is closed if it always produces elements in that system. The set of all polynomials of degree 3 is not closed under addition, as is shown by the equation

$$x^3 + (-1)x^3 = 0.$$

(2) *Commutativity of Addition* If **u**, **v** are vectors in **V**, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Until about 1830, when Hamilton developed the system of quaternions, it was thought by most mathematicians that commutativity was a necessary property of all mathematical operations. In modern mathematics noncommutative operations are commonplace. Two examples of noncommutative operations, the matrix product and cross product of 3-tuples, will be studied later in this text.

(3) *Associativity of Addition* If **u**, **v**, **w** are in **V**, then

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

The lack of associativity is more rare than that of commutativity. Nevertheless, there is one example found in this text, the cross product of 3-tuples.

(4) *Existence of (Additive) Identity* There exists a vector, **0**, such that

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$$

for every **u** in **V**. The vector **0** is called the *zero vector*, and is unique.

(5) *Existence of (Additive) Inverse* For each **u** in **V** there exists a vector, $(-\mathbf{u})$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. The vector $(-\mathbf{u})$ is called the *inverse* of **u**, and is unique.

Properties (1)–(5) are often called the *additive group* properties of **V**. The remaining properties involve scalar multiplication.

(6) *Vector Distributivity* If **u**, **v** are in **V**, then

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

for each number a.

(7) *Scalar Distributivity* If **u** is in **V**, then

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$$

for all numbers a, b.

(8) *Multiplicative Associativity* If **u** is in **V**, then

$$(ab)\mathbf{u} = a(b\mathbf{u})$$

for all numbers a, b.

(9) *Identity of 1* If $\mathbf{u}$ is in $\mathbf{V}$, then $1\mathbf{u} = \mathbf{u}$

Subtraction is defined as usual by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$. Division of a vector by a scalar is not defined, although it will sometimes be convenient to write $\mathbf{u}/a$ in place of $(1/a)\mathbf{u}$. For ease of reference, the definition of vector space will now be repeated unencumbered by explanations. Convenient abbreviations are included.

Definition of Vector Space

A vector space is a set $\mathbf{V}$ *together with operations of addition and scalar multiplication which satisfy*

(1)	*Closure*	*cl*
(2)	*Commutativity of Addition*	*comm add*
(3)	*Associativity of Addition*	*as add*
(4)	*Existence of Identity*	*ex id*
(5)	*Existence of Inverse*	*ex inv*
(6)	*Vector Distributivity*	*vec dist*
(7)	*Scalar Distributivity*	*sc dist*
(8)	*Multiplicative Associativity*	*mp as*
(9)	*Identity of 1*	*id 1*

Other elementary properties of abstract spaces can be proved from this definition. A few are given by the following result.

Proposition 4.1 If $\mathbf{u}$ is in $\mathbf{V}$ and a is a number, then
(a) $0\mathbf{u} = \mathbf{0}$,
(b) $a\mathbf{0} = \mathbf{0}$,
(c) $(-1)\mathbf{u} = -\mathbf{u}$,
(d) $a\mathbf{u} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{u} = \mathbf{0}$.

A proof will be given for (a). Each step will be justified by a property of real numbers or a defining property of a vector space.

$$
\begin{aligned}
0\mathbf{u} &= 0\mathbf{u} + \mathbf{0} && \text{ex id}\\
&= 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u})) && \text{ex inv}\\
&= (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) && \text{as add}\\
&= (0 + 0)\mathbf{u} + (-0\mathbf{u}) && \text{sc dist}\\
&= 0\mathbf{u} + (-0\mathbf{u}) && 0 + 0 = 0\\
&= \mathbf{0} && \text{ex inv}
\end{aligned}
$$

Because of the defining properties, vectors may be manipulated in much the same way as real numbers. For example, the equation $3\mathbf{u} + \mathbf{v} = \mathbf{0}$ may be solved for $\mathbf{u}$ as follows:

$$
\begin{array}{rll}
(3\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{0} + (-\mathbf{v}) & & \text{ex inv} \\
3\mathbf{u} + (\mathbf{v} + (-\mathbf{v})) = \mathbf{0} + (-\mathbf{v}) & & \text{as add} \\
3\mathbf{u} + \mathbf{0} = \mathbf{0} + (-\mathbf{v}) & & \text{ex inv} \\
3\mathbf{u} = -\mathbf{v} & & \text{ex id} \\
\frac{1}{3}(3\mathbf{u}) = \frac{1}{3}[(-1)\mathbf{v}] & & \text{Prop. 4.1(c)} \\
[\frac{1}{3}(3)]\mathbf{u} = [\frac{1}{3}(-1)]\mathbf{v} & & \text{mp as} \\
\mathbf{u} = -\frac{1}{3}\mathbf{v} & & \text{id 1}
\end{array}
$$

Properties of real numbers and equality relationships are also used in this solution for $\mathbf{u}$. It is evident that step-by-step manipulations and justifications are generally impractical since so many details are necessary. Therefore we shall make future manipulations with vectors in the same loose style as is done with numbers in elementary algebra. Such vector computation must be done with the understanding that each step can be justified by the vector space properties.

Questions

1. A vector is necessarily ______.
   (a) a physical concept, (b) a directed segment,
   (c) a tuple, (d) an element of a vector space.

2. The fundamental operations on a vector space are ______ and ______.

3. An operation which does not give elements outside the system is called ______.

4. The equality $5\langle 1, 4\rangle = 3\langle 1, 4\rangle + 2\langle 1, 4\rangle$ illustrates the ______ property.

5. The equality $24\overline{OP} = 6(4\overline{OP})$ illustrates the ______ property.

6. The equality $2\langle 1, 3, 0, 2\rangle = 2\langle 1, 3, 0, 0\rangle + 2\langle 0, 0, 0, 2\rangle$ illustrates the ______ property.

Proofs

1. Justify each step of the following proof of Proposition 4.1(b): $a\mathbf{0} = a(0\mathbf{0}) = 0\mathbf{0} = \mathbf{0}$.

2. Justify each step of the following proof of Proposition 4.1(c): $(-1)\mathbf{u} + \mathbf{u} = \mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathbf{u} = \mathbf{0}$, $(-1)\mathbf{u} = -\mathbf{u}$.

5. Tuples and Directed Line Segments as Vector Spaces

Any set **V** with operations of addition and multiplication by real numbers is entitled to be called a vector space if it satisfies the nine properties listed in the previous section. Some candidates are the sets of directed segments and the sets of tuples. The set of 2-tuples will be given first attention.

The closure property for 2-tuples follows immediately from the definitions of addition and scalar multiplication. In order to verify the commutative property of addition, we must show that for two arbitrary 2-tuples, say $\langle a, b\rangle$ and $\langle a', b'\rangle$, the equality

$$\langle a, b\rangle + \langle a', b'\rangle = \langle a', b'\rangle + \langle a, b\rangle$$

holds. This is proved by the chain of equalities

$$\begin{aligned} \langle a, b\rangle + \langle a', b'\rangle &= \langle a + a', b + b'\rangle && \text{add tup} \\ &= \langle a' + a, b' + b\rangle && a + b = b + a \\ &= \langle a', b'\rangle + \langle a, b\rangle && \text{add tup} \end{aligned}$$

Verification of some other properties follows the same pattern. In each case the defining properties of 2-tuple addition and scalar multiplication are used in conjunction with a property of real numbers which corresponds to the vector-space property. It is easily verified that $\langle 0, 0\rangle$ is the identity, and $\langle -a, -b\rangle$ is the inverse of $\langle a, b\rangle$.

For each positive integer n, the symbol $\mathbf{R}^n$ will denote the system which consists of the set of all n-tuples and the operations of addition and scalar multiplication on this set. By the methods of the previous paragraph it can be shown that each $\mathbf{R}^n$ is a vector space.

The previously defined operations of addition and scalar multiplication on the set of all directed line segments in the Euclidean plane, originating at a fixed point O, give a vector space. A direct proof of this using geometric methods would be painfully tedious. We can simplify matters by introducing a Cartesian coordinate system and using the correspondence between $\overline{OP}$ and **OP**. This correspondence is called an *isomorphism*, since it is one-to-one and "preserves" the addition and scalar multiplication operations. Literally "isomorphism" means "same shape." Since an isomorphism preserves the basic operations, it preserves all the properties described in terms of these operations.

The $\mathbf{R}^n$ spaces will be the primary vector spaces studied in this text. Their algebraic structure makes them more accessible to analysis than spaces of directed segments. Corresponding directed segments will be used throughout, however, to provide geometric significance and to aid the reader's intuition. Additional geometry is provided by the *graphs* of sets of 2-tuples or 3-tuples. The

graph of a subset of $\mathbf{R}^2$ is defined to consist of all points (a, b) such that $\langle a, b\rangle$ is in the subset. Graphs in $\mathbf{R}^3$ are defined similarly. The manner in which tuples, directed segments, and graphs are used will be illustrated by an example.

Example 5.1 (a) Let $\mathbf{L} = \{\langle x, 2x\rangle : x$ is a real number$\}$. The graph of $\mathbf{L}$ is the line $y = 2x$ (see Figure 1.14). This line may also be described as the line containing the segment $\overline{OP}$, where P is any point other than O on $\mathbf{L}$. The graph is an aggregate of points, whereas $\overline{OP}$ is a single entity.
(b) If $\mathbf{L} = \{\langle x, 2x, -3x\rangle : x$ is a real number$\}$, then the graph of $\mathbf{L}$ is a line in space. It contains $\overline{OP}$ where $P = (1, 2, -3)$. Other choices for P are $(-1, -2, 3)$, $(2, 4, -6)$, and so forth.

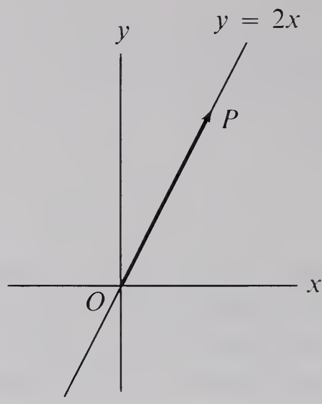


Figure 1.14

Questions

1. The identity of $\mathbf{R}^3$ is ______.
2. The identity of the set of directed line segments in the plane is ______.
3. The inverse of $\langle 1, -3, 2, 4\rangle$ is ______.
4. The points in the Cartesian plane corresponding to a set of 2-tuples is called the ______ of the set.
5. A one-to-one correspondence which preserves operations is called an ______.

Exercises

1. Let $\mathbf{L}$ denote the set $\{\langle x, -x\rangle : x$ is a real number$\}$.
   (a) Sketch the graph of $\mathbf{L}$.
   (b) Give a point P such that the line in (a) contains $\overline{OP}$.

2. Given $P = (1, 5, 2)$, give a set $\mathbf{L}$ in $\mathbf{R}^3$ whose graph is a line which contains $\overline{OP}$.

Proofs

1. Justify each step:

$$\begin{aligned}(a+b)\langle c, d\rangle &= \langle (a+b)c, (a+b)d\rangle \\ &= \langle ac+bc, ad+bd\rangle \\ &= \langle ac, ad\rangle + \langle bc, bd\rangle \\ &= a\langle c, d\rangle + b\langle c, d\rangle.\end{aligned}$$

2. Justify each step:

$$\begin{aligned}(ab)\langle c, d, e\rangle &= \langle (ab)c, (ab)d, (ab)e\rangle \\ &= \langle a(bc), a(bd), a(be)\rangle \\ &= a\langle bc, bd, be\rangle \\ &= a(b\langle c, d, e\rangle).\end{aligned}$$

3. Prove the "as add" property for $\mathbf{R}^2$.
4. Prove the "sc dist" property for directed segments in the plane using the "sc dist" property for 2-tuples and the isomorphic correspondence. (*Hint:* Let $\mathbf{OP} = \langle a, b\rangle$. Show that $(c+d)\overline{OP}$ and $c\overline{OP} + d\overline{OP}$ correspond to the same 2-tuple.)

6. Subspaces

There are many instances of vector spaces besides those of tuples and directed segments. In this section we shall see that every vector space, with trivial exceptions, contains within itself infinitely many vector spaces, called *subspaces*. Even the most superficial study of vector spaces requires an investigation of these subspaces.

Let $\mathbf{S}$ denote a nonempty subset of $\mathbf{V}$. Since vectors in $\mathbf{S}$ are also in $\mathbf{V}$, they can be added and multiplied by numbers using the operations of $\mathbf{V}$. The operations on $\mathbf{V}$ are said to *induce* operations on $\mathbf{S}$. We wish to investigate the possibility that $\mathbf{S}$, with the induced operations, is a vector space. This means that the nine defining properties are satisfied by $\mathbf{S}$. Many of these properties are obviously true for $\mathbf{S}$, since they are true for $\mathbf{V}$. For example if $\mathbf{u}$, $\mathbf{v}$ are in $\mathbf{S}$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds by the commutative property for $\mathbf{V}$. Proceeding through the other properties, we see that $\mathbf{S}$ must receive all but three properties from $\mathbf{V}$. The three not necessarily inherited by $\mathbf{S}$ are

(1) closure,
(2) existence of identity,
(3) existence of inverse.

Therefore we have reduced to three the number of properties to be checked in testing whether S is a vector space. This list can be further reduced to just one, closure. To see this, let it be assumed that the induced operations on S are closed. Choosing $\mathbf{u}$ arbitrary in S (nonempty), we see that $0\mathbf{u} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$. This shows the zero vector of $\mathbf{V}$ is in S and the inverse (in $\mathbf{V}$) of $\mathbf{u}$ is also in S. It is obvious that the zero vector of $\mathbf{V}$ is an additive identity for S, and hence S is a vector space. This suggests the next definition.

Definition of Subspace

A nonempty subset S *of* $\mathbf{V}$ *is a* subspace *if and only if the induced operations on* S *are closed.*

By the definition of subspace and closure, to prove a set S in $\mathbf{V}$ is a subspace we must show:

(1) if $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S,
(2) if $\mathbf{u}$ is in S then $a\mathbf{u}$ is in S for every number a.

Example 6.1 (a) We wish to show that

$$S = \{\langle x, y\rangle : 2x + 3y = 0\}$$

is a subspace. It must be verified that if $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle a', b'\rangle$ are in S, then

$$\mathbf{u} + \mathbf{v} = \langle a + a', b + b'\rangle \quad \text{and} \quad c\mathbf{u} = \langle ca, cb\rangle$$

are in S for each number c. It is known from $\mathbf{u}$, $\mathbf{v}$ in S that $2a + 3b = 0$ and $2a' + 3b' = 0$. Using elementary algebra it can be verified that

$$2(a + a') + 3(b + b') = 0 \quad \text{and} \quad 2(ca) + 3(cb) = 0,$$

from which the desired conclusion follows.

(b) The set $S = \{\langle x, y\rangle : x^2 - y^2 = 0\}$ is not a subspace. It is easily verified that $\langle 1, 1\rangle$, $\langle -1, 1\rangle$ are in S and $\langle 1, 1\rangle + \langle -1, 1\rangle = \langle 0, 2\rangle$ is not in S. Therefore, S is not closed with respect to addition. It might be noted that S is closed with respect to scalar multiplication. If $\langle a, b\rangle$ is in S, then $a^2 - b^2 = 0$, from which we see that $(ca)^2 - (cb)^2 = 0$; hence $\langle ca, cb\rangle$ is in S for each number c.

The definition of subspace implies that every vector space is a subspace of itself. Also the set $\{\mathbf{0}\}$ consisting only of the zero vector is a subspace, since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for each number a. The two subspaces $\mathbf{V}$ and $\{\mathbf{0}\}$ of $\mathbf{V}$ carry little interest in their role as subspaces and are called *improper*. All other subspaces are called *proper*. We shall next investigate the geometric characteristics of proper subspaces of $\mathbf{R}^2$.

Suppose $\mathbf{S}$ is a proper subspace of $\mathbf{R}^2$ containing $\mathbf{u} = \langle a, b \rangle \neq \mathbf{0}$. Since $\mathbf{S}$ is closed under scalar multiplication, the set $\mathbf{L} = \{c\mathbf{u}: c \text{ is a real number}\}$ must be included in $\mathbf{S}$. The equations

$$c\mathbf{u} + d\mathbf{u} = (c + d)\mathbf{u} \quad \text{and} \quad c(d\mathbf{u}) = (cd)\mathbf{u}$$

show that $\mathbf{L}$ is a subspace of $\mathbf{R}^2$. We shall now see that the proper subspace $\mathbf{S}$ cannot have any vectors not in $\mathbf{L}$. If $\mathbf{v}$ were in $\mathbf{S}$ and not in $\mathbf{L}$, then $\mathbf{S}$ would have to include all vectors of the form $c\mathbf{u} + d\mathbf{v}$ by the closure property. By the coordinate-axis property for the plane, the graph of all such vectors is the entire Cartesian plane. Since the graph of $\mathbf{L}$ is a line through the origin, we have shown the following proposition.

Proposition 6.1 A set $\mathbf{S}$ in $\mathbf{R}^2$ is a proper subspace if and only if its graph is a line through the origin.

The analysis of proper subspaces of $\mathbf{R}^3$ is similar. If $\mathbf{S}$ is a subspace of $\mathbf{R}^3$ containing $\mathbf{u} = \langle a, b, c \rangle$, then $\mathbf{L} = \{c\mathbf{u}: c \text{ is a real number}\}$ is a subspace contained in $\mathbf{S}$, and its graph is a line through the origin. The inclusion in $\mathbf{S}$ of a vector $\mathbf{v} = \langle a', b', c' \rangle$ not in $\mathbf{L}$, however, does not imply that $\mathbf{S}$ is all of $\mathbf{R}^3$. Instead, $\mathbf{S}$ may have for its graph the plane $\mathbf{P}$ containing the origin, (a, b, c), and (a', b', c'). The addition to $\mathbf{S}$ of a vector not in $\mathbf{P}$ would force $\mathbf{S}$ to be all of $\mathbf{R}^3$ by the closure condition and coordinate axis property of space. We may conclude:

Proposition 6.2 A set $\mathbf{S}$ in $\mathbf{R}^3$ is a proper subspace if and only if its graph is a line through the origin or a plane through the origin.

Questions

1. $\mathbf{S}$ is a subspace if the induced operations on $\mathbf{S}$ are ______.
2. $\mathbf{S}$ is a subspace of $\mathbf{V}$ if for each $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{S}$ and every real number a we have ______ and ______.
3. Every vector space $\mathbf{V}$ has as improper subspaces ______ and ______.
4. The graph of a proper subspace of $\mathbf{R}^2$ is a ______.
5. The graph of a proper subspace of $\mathbf{R}^3$ is either a ______ or ______.

Exercises

1. Show $\mathbf{S} = \{\langle x, y \rangle: x - 3y = 0\}$ is a subspace by demonstrating that it is closed under
   (a) addition, (b) scalar multiplication.

2. Show that $S = \{\langle x, y\rangle : x > 0\}$ is not a subspace. Verify that it is closed under addition but is not closed under scalar multiplication.
3. Determine if each of the following sets is closed under addition and scalar multiplication, and state whether or not it is a subspace.
   (a) $\{\langle x, y\rangle : x = 3\}$, (b) $\{\langle x, y\rangle : x - 3y = 1\}$,
   (c) $\{\langle x, y, z\rangle : z = 0\}$, (d) $\{\langle x, y, z\rangle : x^2 = 3y^2\}$.
4. Determine whether or not each of the sets in Exercises 3(a), (b), (c), (d) is a subspace by considering its graph.

Proofs

1. Prove from the definition of a subspace that $\{\langle x, y\rangle : ax + by + c = 0\}$ is a subspace of $\mathbf{R}^2$ if and only if $c = 0$.

7. *Function Spaces*

In the last section we studied vector spaces included in a given vector space **V**. This section will be concerned with the construction of new vector spaces from **V**. The elements of the new spaces will be functions.

The first mathematical use of the word "function" is credited to the philosopher-mathematician G. Leibniz of Germany (1646–1716). In the eighteenth century a function meant a formula or rule. This concept, however, was inadequate for many applications to mathematics and physics. Today functions are usually described as either correspondences or relations. The definition of a function as a relation consisting of a set of ordered pairs is used in the more abstract developments in mathematics. For our purposes it is more convenient to use the correspondence description, which provides us with a working concept rather than a definition.

A *function* f involves a *domain* set X, a *range* set Y, and a *rule* which assigns to each element of X exactly one element of Y. The element in Y assigned by f

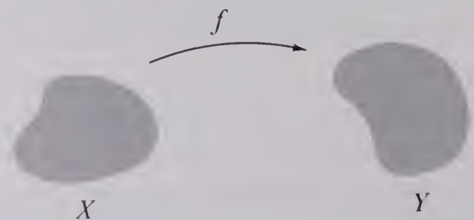


Figure 1.15

to an x in X is denoted $f(x)$ and called the *f–image* of x. When X and Y are sets of numbers (not generally the case in this text), a function from X to Y may be pictured in the usual way by a graph in the Cartesian plane. The point (x, y) is on the graph provided $f(x) = y$. More generally, a function f is pictured by an abstract diagram as shown in Figure 1.15. The value of such a picture will become apparent later.

We shall be interested in the case in which Y is a vector space $Y = \mathbf{V}$. It is then possible to define addition and scalar multiplication operations on functions from X to $\mathbf{V}$.

Definition of Function Operations

$$(f+g)(x) = f(x) + g(x) \qquad \textit{add fcn}$$
$$(cf)(x) = c(f(x)) \qquad \textit{sc mp fcn}$$

The operations on the right side of the defining equalities are in $\mathbf{V}$; the operations on the left side are the operations, on functions, that are being defined.

Two functions are *equal* if they have the same domain, range, and rule. Functions are added, or multiplied by numbers, only when they have the same domain and same range; the result has again the same domain and range, with rule given by the above definitions.

The symbol $\mathbf{F}[X, \mathbf{V}]$ will denote the set of all functions from X to $\mathbf{V}$. This set, together with its defined operations, forms a vector space. We show this by verifying the nine properties of a vector space. Consider, for example, the vector distributive property $c(f+g) = cf + cg$. A proof must show $c(f+g)$ and $cf + cg$ assign the same vector to an arbitrary x in X. We have

$$\begin{aligned}
(c(f+g))(x) &= c((f+g)(x)) && \text{sc mp fcn}\\
&= c(f(x) + g(x)) && \text{add fcn}\\
&= c(f(x)) + c(g(x)) && \text{vec dist}\\
&= (cf)(x) + (cg)(x) && \text{sc mp fcn}\\
&= (cf + cg)(x) && \text{add fcn}
\end{aligned}$$

Again, this shows how a property is proved using a corresponding property of another system. It is easily verified that the zero element of $\mathbf{F}[X, \mathbf{V}]$ assigns the zero vector in $\mathbf{V}$ to every x in X.

A *real function* is a function whose range is the set of real numbers. It has already been observed that the real numbers correspond to $\mathbf{R}^1$, the set of 1-tuples, and may, therefore, be regarded as a vector space. We shall now show that $\mathbf{R}^2$, and indeed any $\mathbf{R}^n$, can be conceived as a vector space of real functions. If $X = \{1, 2\}$ and $\mathbf{V} = \mathbf{Re}$, the set of real numbers, then a function f from X to $\mathbf{V}$ is described by the rule $f(1) = a, f(2) = b$ where a, b, are real numbers. We can

denote f by the symbol $\langle a, b\rangle$, and one can verify that when this symbol is used, addition and scalar multiplication of functions are exactly the same as they are for 2-tuples. We therefore have a new proof that $\mathbf{R}^2$ is a vector space.

As a second illustration let $X = [0, 1]$, the closed unit interval of real numbers, and $\mathbf{V} = \mathbf{Re}$. Certain elements of $\mathbf{F}[X, \mathbf{V}] = \mathbf{F}[[0, 1], \mathbf{Re}]$ may then be graphed in the usual manner. Particular subsets of this function space are of special interest. For instance, let $\mathbf{S}$ be the set of all continuous functions on $[0, 1]$. Then, $\mathbf{S}$ is a subspace, and hence a vector space, since it is closed under the two operations. In this case closure means:

(1) *if f and g are continuous on* $[0, 1]$, *then $f + g$ is continuous on* $[0, 1]$;
(2) *if f is continuous on* $[0, 1]$ *then cf is continuous on* $[0, 1]$.

These properties are normally proved in the first-year calculus of one variable. It can also be shown from results of first-year calculus that the following classes of functions on $[0, 1]$ form vector spaces:

(a) differentiable functions,
(b) polynomial functions,
(c) polynomial functions of degree $\leq n$ (n fixed).

Questions

1. A function involves a ______ set, a ______ set, and a ______.
2. The element which a function associates with x is called the ______ of x.
3. Addition and scalar multiplication operations can be defined for functions provided the ______ set is a vector space.
4. The statement that the differentiable functions on $[0, 1]$ are closed under addition is ______.
5. The statement that the polynomial functions on $[0, 1]$ are closed under scalar multiplication is ______.

Exercises

1. For each of the following subsets of the space of real functions on $[0, 1]$, state the meaning of closure under addition and scalar multiplication and determine whether or not the set is a subspace.
   (a) polynomials of degree 5,
   (b) constant functions,
   (c) $\{f: f(1) = 0\}$,
   (d) $\{f: f(0) = 1\}$,

(e) functions with a straight-line graph,

(f) $\left\{f: \dfrac{d^2f}{dx^2} + 2\dfrac{df}{dx} + f = 0\right\}$.

Proofs

1. Justify each step:

$$((a+b)f)(x) = (a+b)(f(x)) = a(f(x)) + b(f(x)) = (af)(x) + (bf)(x) = (af+bf)(x).$$

2. Justify each step:

$$((ab)f)(x) = (ab)f(x) = a(b(f(x))) = a((bf)(x)) = (a(bf))(x).$$

3. Prove the associative property of addition for $\mathbf{F}[X, \mathbf{V}]$.

Problems

A. *Addition and Scalar Multiplication of Vectors*

The *sum* of 2-tuples and 3-tuples is given respectively by

A.1 (a) $\langle a, b\rangle + \langle a', b'\rangle = \langle a+a', b+b'\rangle$,
(b) $\langle a, b, c\rangle + \langle a', b', c'\rangle = \langle a+a', b+b', c+c'\rangle$.

The *scalar product* of 2-tuples and 3-tuples is given by

A.2 (a) $c\langle a, b\rangle = \langle ca, cb\rangle$,
(b) $d\langle a, b, c\rangle = \langle da, db, dc\rangle$.

1. Evaluate
(a) $\langle 1, 2\rangle + \langle 3, -4\rangle$, (b) $2\langle 1, 3\rangle$, (c) $\langle 1, 0, 3\rangle + \langle -2, 1, 4\rangle$,
(d) $3\langle 1, 1, 0\rangle - 2\langle 3, 0, 1\rangle$.

An alternative symbolism for 2-tuples and 3-tuples is commonly used. Symbols **i**, **j**, **k** are defined by

A.3 (a) $\mathbf{i} = \langle 1, 0\rangle$, $\mathbf{j} = \langle 0, 1\rangle$ for 2-tuples,
(b) $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$ for 3-tuples.

The dual usage of **i** and **j** in A.3 will be unambiguous if it is understood by context which size tuples are being considered. The following equations are easily verified:

A.4 (a) $\langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$,
(b) $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

From A.1(a), A.2(a), and A.4(a) we see that

A.5 (a) $(a\mathbf{i} + b\mathbf{j}) + (a'\mathbf{i} + b'\mathbf{j}) = (a + a')\mathbf{i} + (b + b')\mathbf{j}$,
(b) $c(a\mathbf{i} + b\mathbf{j}) = (ca)\mathbf{i} + (cb)\mathbf{j}$.

Similar equations are valid for 3-tuples. Thus **i**, **j**, **k** may be manipulated like real numbers.

2. Evaluate:
(a) $(\mathbf{i} + \mathbf{j}) + (3\mathbf{i} - \mathbf{j})$, (b) $4(2\mathbf{i} + 3\mathbf{j})$,
(c) $(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, (d) $4(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$,
(e) $2(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k})$.

Addition and scalar multiplication of tuples of larger size follow the same pattern.

3. Evaluate:
(a) $\langle 4, 1, 3, 2\rangle + \langle 2, -1, 5, 6\rangle$,
(b) $3\langle 2, 1, 0, 7\rangle$,
(c) $\langle 2, 1, 0, 5, 3\rangle + \langle 1, 0, 6, 5, 4\rangle$,
(d) $3\langle 1, 0, 2, 1, 4\rangle + 2\langle 1, -1, 2, 1, 6\rangle$.

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4. Evaluate:
(a) $\langle 1, 6\rangle + 3\langle 7, -4\rangle$, (b) $\langle 4, 0, 7\rangle - 2\langle 6, 1, 3\rangle$,
(c) $3(4\mathbf{i} + \mathbf{j} - \mathbf{k}) + 2(\mathbf{i} + \mathbf{j})$, (d) $5\mathbf{i} + 3(\mathbf{j} - \mathbf{k}) + 2(7\mathbf{i} + \mathbf{j})$,
(e) $6\langle 1, 0, 3\rangle + \langle 0, 0, 1\rangle - \langle 2, 0, 5\rangle$,
(f) $4(\mathbf{i} - \mathbf{k}) + 3\mathbf{j} - 2(\mathbf{j} - 4\mathbf{k})$, (g) $\langle 1, 0, 6, 5\rangle - \langle 1, 1, 7, 3\rangle$,
(h) $4\langle 1, 1, 1, 0, 2\rangle + 3\langle 1, 0, 5, 6, -3\rangle$.

B. *Geometric Vectors*

If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points in the Cartesian plane, then there is associated with the directed line segment $\overline{PQ}$ from P to Q a 2-tuple $\mathbf{PQ}$, given by

B.1 $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$.

Similarly, for points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in space there is associated with the directed segment $\overline{PQ}$ from P to Q a 3-tuple **PQ** given by

B.2 $\quad \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$

The associated tuples may also be written in the **i**, **j** and **i**, **j**, **k** form.

1. (a) Find **PQ** if $P = (1, 1)$, $Q = (2, 3)$.
   (b) Find Q if $P = (1, 2)$ and $\mathbf{PQ} = 3\mathbf{i} - \mathbf{j}$.
   (c) Find S if $P = (1, 2)$, $Q = (-1, 3)$, $R = (2, 1)$, and $\mathbf{PQ} = \mathbf{RS}$.
2. (a) Find **PQ** if $P = (2, 0, 1)$, $Q = (3, -2, 6)$.
   (b) Find P if $Q = (1, 0, 3)$ and $\mathbf{PQ} = \langle 4, 2, -6 \rangle$.
   (c) Find R if $P = (1, 3, 2)$, $Q = (-1, 6, 5)$, $S = (3, 0, 7)$, and $\mathbf{PQ} = \mathbf{RS}$.

A fundamental property is

B.3 $\quad \mathbf{PQ} + \mathbf{QR} = \mathbf{PR}.$

This has, in the plane and in space, a geometrical interpretation that $\overline{PR}$ is the diagonal of the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$ (see Figure 1.16).

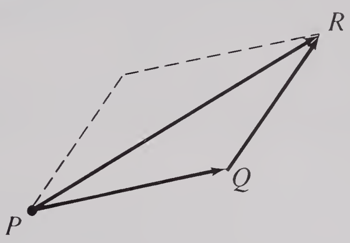

Figure 1.16

3. Find the fourth vertex S of the parallelogram with sides $\overline{PQ}$, $\overline{PR}$ where $P = (1, 0)$, $Q = (2, 3)$, $R = (6, 5)$. (*Hint:* Use $\mathbf{PQ} = \mathbf{RS}$ or $\mathbf{PR} = \mathbf{QS}$.)
4. Find the fourth vertex S of the parallelogram with sides $\overline{PQ}$, $\overline{PR}$ where $P = (2, 1, 6)$, $Q = (-3, 0, 7)$, and $R = (2, 1, 5)$.
5. Find the remaining four vertices of the parallelepiped with sides $\overline{PQ}$, $\overline{PR}$, $\overline{PS}$ where $P = (1, 0, 2)$, $Q = (0, 1, 3)$, $R = (2, 0, 5)$, $S = (2, 3, 1)$ (see Figure 1.17). (*Hint:* One of the answers is the remaining vertex of the parallelogram with sides $\overline{PQ}$, $\overline{PR}$.)

Two segments $\overline{PQ}$ and $\overline{RS}$ have the same direction if $\mathbf{PQ} = c\mathbf{RS}$ for some number $c > 0$. This is true provided the respective coordinates of **PQ** and **RS** have the same positive ratio. Thus if $\mathbf{PQ} = \langle a, b \rangle$ and $\mathbf{RS} = \langle a', b' \rangle$, then

B.4 $\quad \overline{PQ}$ and $\overline{RS}$ have the same direction if $a'/a = b'/b > 0$ (if $a = 0$ then $a' = 0$; if $b = 0$ then $b' = 0$).

6. Given $P = (1, 6)$, $Q = (2, 0)$, $R = (7, 5)$, find S on the x axis so that $\overline{PQ}$ and $\overline{RS}$ have the same direction.

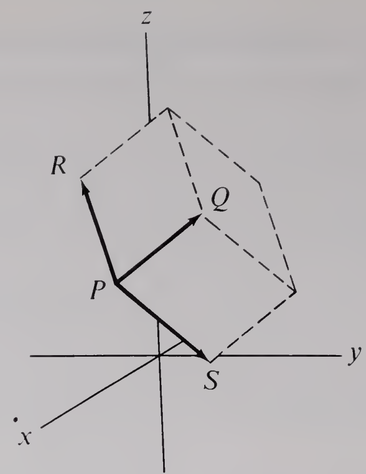

Figure 1.17

7. Given $P = (3, 0)$, $Q = (2, 5)$, and $R = (6, 4)$, find S on the y axis so that $\overline{PQ}$ and $\overline{RS}$ have the same direction.

If $\mathbf{PQ} = \langle a, b, c\rangle$ and $\mathbf{RS} = \langle a', b', c'\rangle$, then

B.5 $\overline{PQ}$ and $\overline{RS}$ have the same direction if $a'/a = b'/b = c'/c > 0$ (zero coordinates correspond as in B.4).

8. Given $P = (2, 0, 1)$, $Q = (6, 3, 5)$, and $R = (-1, 7, -2)$, find S in the xy plane so that $\overline{PQ}$ and $\overline{RS}$ have the same direction. (*Hint:* Let $S = (x, y, 0)$ and solve for x, y.)
9. Given $P = (3, 2, -5)$, $Q = (2, 1, 7)$, and $R = (4, 1, 3)$, find S in the yz plane so that $\overline{PQ}$ and $\overline{RS}$ have the same direction.

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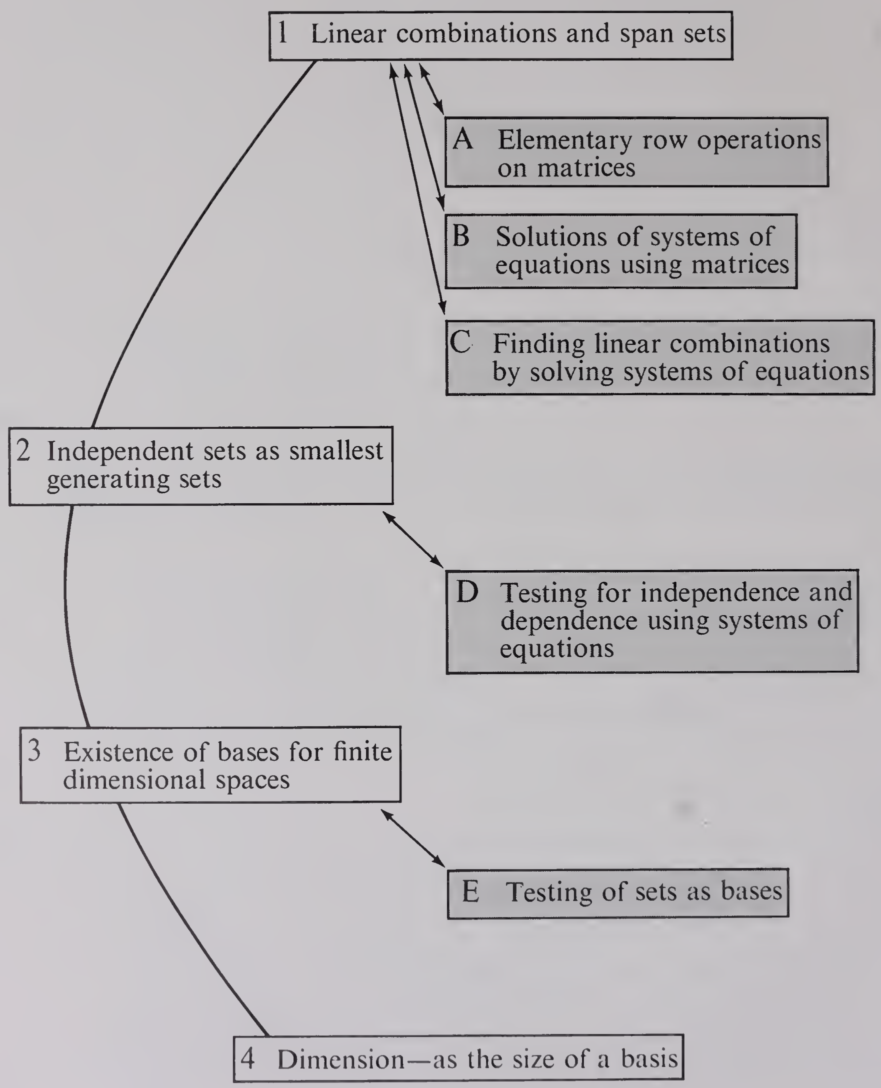
10. Given $P = (2, 0)$, $Q = (4, 3)$, and $R = (6, -1)$, find:
    (a) $\mathbf{PQ}$, (b) S if $\mathbf{PQ} = \mathbf{RS}$,
    (c) the fourth vertex of the parallelogram with sides $\overline{PQ}$ and $\overline{PR}$,
    (d) S on the x axis so that $\overline{PQ}$ and $\overline{RS}$ have the same direction.
11. Repeat Problem 10, given $P = (6, 3)$, $Q = (2, -1)$, and $R = (7, 2)$.
12. Given $P = (3, 1, 6)$, $Q = (2, -1, 5)$, $R = (2, 1, -3)$, and $S = (4, 0, 2)$, find:
    (a) $\mathbf{PQ}$, (b) T if $\mathbf{PQ} = \mathbf{RT}$,
    (c) the remaining vertices of the parallelepiped with sides $\overline{PQ}$, $\overline{PR}$, $\overline{PS}$,
    (d) T in the xz plane so that $\overline{PQ}$ and $\overline{RT}$ have the same direction.
13. Repeat Problem 12, given $P = (2, 1, 7)$, $Q = (3, 6, 0)$, $R = (2, -7, 5)$, and $S = (6, 2, 5)$.


1 Linear combinations and span sets
A Elementary row operations on matrices
B Solutions of systems of equations using matrices
C Finding linear combinations by solving systems of equations
2 Independent sets as smallest generating sets
D Testing for independence and dependence using systems of equations
3 Existence of bases for finite dimensional spaces
E Testing of sets as bases
4 Dimension—as the size of a basis

II

Dimension of a Vector Space

Dimension is usually regarded as a non-negative integer associated with certain physical or geometrical objects. Some examples are shown below.

ink dot	0	point	0
thin wire	1	line	1
thin sheet of paper	2	plane	2
block of wood	3	space	3

Although dimension does not exceed 3 in our geometric experience, there is no such limitation in mathematics or mathematical physics, where any non-negative integer dimension can be made meaningful.

In this chapter we seek to obtain for certain vector spaces a characterizing dimension number which is compatible with the ordinary geometry of the plane and space. The relative simplicity of vector-space structure makes possible a much easier solution than is possible with nonlinear mathematical spaces. We shall be able to assign to a large class of vector spaces, called *finite dimensional*, a dimension number which completely determines the structure of the vector space. Some preliminary terminology and conclusions must precede our desired goal.

1. Linear Combinations, Span Set

By a *linear combination* of a nonempty set $\mathbf{T}$ in a vector space $\mathbf{V}$ we mean any vector $\mathbf{v}$ in $\mathbf{V}$ which can be written in the form $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$, where n is a positive integer, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are vectors in $\mathbf{T}$, and $c_1, c_2, \ldots, c_n$ are real numbers. We will be most interested in the case where $\mathbf{T}$ is *finite*, meaning that $\mathbf{T}$ has an integer number of elements. Some examples will illustrate.

Example 1.1 (a) If $\mathbf{T} = \{\mathbf{0}\}$ consists only of the zero vector, then $c\mathbf{0} = \mathbf{0}$ for all c implies that only $\mathbf{0}$ is a linear combination of $\mathbf{T}$.

(b) If $\mathbf{T} = \{\langle 2, 1\rangle\}$, then any vector of the form $c\langle 2, 1\rangle$ is a linear combination of $\mathbf{T}$. Instances are

$$\langle 4, 2\rangle = 2\langle 2, 1\rangle,\ \langle 0, 0\rangle = 0\langle 2, 1\rangle,\ \langle -2, -1\rangle = (-1)\langle 2, 1\rangle.$$

(c) If $\mathbf{T} = \{\langle 1, 3, 0\rangle, \langle 2, 1, 5\rangle\}$, then an arbitrary linear combination of $\mathbf{T}$ has the form $a\langle 1, 3, 0\rangle + b\langle 2, 1, 5\rangle$. Thus, for instance,

$$1\langle 1, 3, 0\rangle + (-4)\langle 2, 1, 5\rangle = \langle -7, -1, -20\rangle$$

is a linear combination of $\mathbf{T}$.

The set of all linear combinations of $\mathbf{T}$ is called the *span set* of $\mathbf{T}$ and denoted Sp $\mathbf{T}$. This set is also called the space *generated* by $\mathbf{T}$; it will be seen that Sp $\mathbf{T}$ is a subspace of $\mathbf{V}$. The span set of $\mathbf{T}$ consists of all vectors which can be constructed by using only vectors in $\mathbf{T}$ and the addition and scalar multiplication operations.

Example 1.2 (a) If $\mathbf{T} = \{\langle 2, 1\rangle\}$, then Sp $\mathbf{T}$ consists of all vectors of the form $c\langle 2, 1\rangle$ for c a real number. The graph of $\mathbf{T}$ is the line $x = 2y$ in the Cartesian plane.

(b) If $\mathbf{T} = \{\overline{OP}, \overline{OQ}\}$, where P, Q are points in space not collinear with O, then

$$\text{Sp } \mathbf{T} = \{a\overline{OP} + b\overline{OQ}\colon a \text{ and } b \text{ are real numbers}\}.$$

Thus Sp $\mathbf{T}$ consists of all $\overline{OR}$ where R is in the plane containing $\overline{OP}$ and $\overline{OQ}$.

(c) Let $\mathbf{T} = \{1, x, x^2\}$ be a subset of $\mathbf{F}[[0, 1]\colon \mathbf{Re}]$. An arbitrary linear combination of $\mathbf{T}$ has the form $a + bx + cx^2$. Thus Sp $\mathbf{T}$ consists of all polynomials of degree 2 or less on $[0, 1]$.

It may be observed that in each of the preceding examples Sp **T** is a subspace. We now see this is always true.

Proposition 1.1 Sp **T** is a subspace of **V**.

We must show that addition and scalar multiplication are closed on Sp **T**. If **u**, **v** are in Sp **T**, then adding a linear combination expression for **v** to one for **u** gives a linear combination expression for $\mathbf{u} + \mathbf{v}$. For example if

$$\mathbf{u} = 3\mathbf{u}_1 + 2\mathbf{u}_2 \quad \text{and} \quad \mathbf{v} = 4\mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3$$

are linear combination expressions of **T**, then so also is

$$\mathbf{u} + \mathbf{v} = 3\mathbf{u}_1 + 2\mathbf{u}_2 + 4\mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3 .$$

The closure property for scalar multiplication is shown similarly (see Proofs, exercise 1).

Questions

1. The subspace generated by **T** is called the _______ of **T**.
2. Sp **T** consists of all _______ of **T**.
3. An arbitrary linear combination of $\{\mathbf{u}, \mathbf{v}\}$ has the form _______, where a, b are real numbers.

Problems

1. Do Problem Sets A, B, and C at the end of the chapter.

Exercises

1. Describe the graphs of

   (a) Sp $\{\langle 2, 1\rangle\}$, (b) Sp $\{\langle 1, 3\rangle, \langle -1, 2\rangle\}$, (c) Sp $\{\langle 1, 3\rangle, \langle 2, 6\rangle\}$.
2. From $2\mathbf{u} + 3\mathbf{v} - 7\mathbf{w} = \mathbf{0}$ write **w** as a linear combination of $\{\mathbf{u}, \mathbf{v}\}$.
3. From $\mathbf{v} - \mathbf{u} = 4\mathbf{v} + 5\mathbf{w}$ write **0** as a linear combination of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ with at at least one nonzero coefficient.
4. (a) Write $3\mathbf{u} + 4\mathbf{v}$ as a linear combination of $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$.
   (b) Write $2\mathbf{u} + \mathbf{v} - \mathbf{w}$ as a linear combination of $\{\mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{w}, \mathbf{v} - \mathbf{w}\}$.
5. (a) Write $3x + 5$ as a linear combination of $\{x - 2, 2x + 1\}$.
   (b) Show that every polynomial of degree ≤ 1 is a linear combination of $\{x - 2, 2x + 1\}$.

6. Show that $\mathbf{T} = \{1, x, x^2 - 2, x^3\}$ generates the polynomials of degree ≤ 3 on [0, 1].
7. Find three vectors in the set

$$\mathrm{S} = \{\langle x_1, x_2, x_3, x_4\rangle : x_1 - 2x_2 + x_3 + 4x_4 = 0\}$$

which generate S, and verify your answer. (*Hint:* Assign values to x_2, x_3, x_4 and determine x_1.)

Proofs

1. Show that if $\mathbf{u}$ is a linear combination of $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then so is $c\mathbf{u}$, where c is an arbitrary real number.

2. *Dependence, Independence*

The previous section was primarily concerned with subspaces which can be generated by finite sets. In this section we seek to determine the character of the smallest sets that generate a given vector space. The size of such sets will be an indicator of the dimension of the generated space.

The coordinate-axis property for the Euclidean plane asserts that the vector space of all directed segments with initial point 0 is the span set of any two noncollinear directed segments. It is geometrically evident that no single directed segment and no two collinear segments can generate all the directed segments in a plane. Therefore, the smallest sets which generate the entire vector space in this case are the sets of two noncollinear segments. A characterization of these is given by the criterion for collinear segments, which says that $\overline{OP}$ and $\overline{OQ}$ are collinear if and only if there exist numbers a and b not both 0 such that $a\overline{OP} + b\overline{OQ} = \overline{OO}$. Similarly for directed segments in space, the smallest generating sets consist of any three noncoplanar segments; also the set $\{\overline{OP}, \overline{OQ}, \overline{OR}\}$ is noncoplanar if and only if there do not exist numbers a, b, c not all 0 such that

$$a\overline{OP} + b\overline{OQ} + c\overline{OR} = \overline{OO}.$$

These conclusions motivate the study of our next concepts. In the following definitions it will be assumed that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are distinct vectors in $\mathbf{V}$.

Definition of (Linear) Dependence

The set $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is* dependent *if and only if there exist numbers* $c_1, c_2, \ldots, c_n$ *not all* 0 *such that* $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$.

Definition of (Linear) Independence

The set $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is* independent *if and only if it is not dependent.*

Thus a finite set is dependent or independent in accordance with whether or not $\mathbf{0}$ can be written in a nontrivial way as a linear combination of the set. It follows from preceding discussions that in the case of directed segments the independence property holds for the smallest sets which generate the entire vector space. The next proposition gives a useful alternate description of dependence.

Proposition 2.1 $\mathbf{T}$ is dependent if and only if some vector in $\mathbf{T}$ is a linear combination of the other vectors in $\mathbf{T}$.

We shall make a proof for the case $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. If $\mathbf{T}$ is dependent, then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0},$$

where some c_i, say c_1, may be taken not equal to 0. Then the equation may be solved for $\mathbf{u}_1$ to give

$$\mathbf{u}_1 = -\frac{c_2}{c_1}\mathbf{u}_2 - \frac{c_3}{c_1}\mathbf{u}_3$$

as a linear combination of $\{\mathbf{u}_2, \mathbf{u}_3\}$. Conversely if $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathbf{u}_1$ is a linear combination of $\{\mathbf{u}_2, \mathbf{u}_3\}$, then $\mathbf{u}_1 = c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for some numbers c_2 and c_3. Therefore

$$(-1)\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

and $\mathbf{T}$ is dependent.

Every set which contains the zero vector is dependent. This is shown by selecting the linear combination expression having 1 as coefficient of $\mathbf{0}$ and 0 as the coefficient of the other vectors in the set. On the other hand, if $\mathbf{T}$ consists of a single nonzero vector $\mathbf{u}$, then $\mathbf{T}$ is independent. This follows from proposition 4.1(d) in Chapter 1, which says that $a\mathbf{u} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{u} = \mathbf{0}$. A set of two vectors is dependent if and only if one is a scalar multiple of the other.

The set $\{\overline{OP}, \overline{OQ}\}$ is dependent provided $\overline{OP}$ and $\overline{OQ}$ are collinear. The set $\{\overline{OP}, \overline{OQ}, \overline{OR}\}$ is dependent provided $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ are coplanar. The corresponding result for $\mathbf{R}^2$ and $\mathbf{R}^3$ is that any two 2-tuples or any 3-tuples form a dependent set if their graph points are coplanar with the origin. It follows from proposition 2.1 and the coordinate-axis property that any three directed

segments in the plane and any four directed segments in space form a dependent set. Thus any set of three 2-tuples is dependent, as is any set of four 3-tuples.

Example 2.1 (*a*) In the space $\mathbf{V} = \mathbf{F}[[0, 2]: \mathbf{Re}]$ the set $\{1, x, x^2\}$ is independent. This is a consequence of the fundamental theorem of algebra, which says that the equation $ax^2 + bx + c = 0$ has at most two roots and, therefore, cannot hold for all x in $[0, 1]$.

(b) In the space of (a), the set

$$\{1 + x + 2x^2, 2x + 3x^2, 2 + x^2\}$$

is dependent since

$$2(1 + x + 2x^2) - (2x + 3x^2) - (2 + x^2) = 0 \text{ for all } x.$$

An equivalent formulation of the definition of independence is generally used for proving that a set is independent. The procedure is to assume $\mathbf{0}$ is a linear combination of the set and then show all coefficients of the linear combination must be zero. We state it now for reference purposes.

Procedure for Proving Independence of a Set

To prove $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ *is independent, assume*

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$$

and deduce

$$c_1 = c_2 = \cdots = c_n = 0.$$

Questions

1. A(n) ________ set is a smallest generating set of the space it spans.
2. $\{\mathbf{u}\}$ is dependent provided ________.
3. A set of two directed segments is dependent if the segments are ________.
4. A set of three directed segments is independent if the segments are ________.
5. A set of four 3-tuples is necessarily ________.
6. If $\{\mathbf{u}, \mathbf{v}\}$ is independent and $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then ________.

Problems

1. Do Problem Set D at the end of the chapter.

Exercises

1. Determine whether or not the following subsets of $\mathbf{F}[[0, 1], \mathbf{Re}]$ are dependent or independent.
   (a) $\{1, x, x^2 + 1\}$, (b) $\{2, x, 3x + 6\}$,
   (c) $\{0, x, x^2\}$, (d) $\{e^x, e^{2x}, e^{3x}\}$,
   (e) $\{e^x, xe^x, (x - 3)e^x\}$.

Proofs

1. Prove $\{\mathbf{u}, \mathbf{v}\}$ is dependent if and only if one of the two vectors $\mathbf{u}$ and $\mathbf{v}$ is a scalar multiple of the other.
2. Prove $\{\langle a, b\rangle, \langle c, d\rangle\}$ is dependent if and only if $ad - bc = 0$.

3. *Bases of V*

Our search for smallest generating sets of vector spaces has thus far led to the property of independence, which is an abstraction of the noncollinear and noncoplanar properties. However, we have not yet established the extent to which there exist independent sets which generate an abstract vector space. This will now be studied.

If $\mathbf{T} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbf{V}$, then an arbitrary vector $\mathbf{u}$ in $\mathbf{V}$ may be written

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$$

for suitable numbers a_1, a_2, a_3. If $\mathbf{T}$ is dependent, then by proposition 2.1 some vector in $\mathbf{T}$, say $\mathbf{u}_3$, is a linear combination of the rest of $\mathbf{T}$, and we may write $\mathbf{u}_3 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$. Substitution gives

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = (a_1 + a_3c_1)\mathbf{u}_1 + (a_2 + a_3c_2)\mathbf{u}_2,$$

and hence $\mathbf{u}$ is in the span set of $\{\mathbf{u}_1, \mathbf{u}_2\}$. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is independent, then we have shown that an independent subset of $\mathbf{T}$ generates $\mathbf{V}$. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is dependent, then the process may be repeated to write $\mathbf{u}$ as a linear combination of a single vector in $\mathbf{T}$, which constitutes an independent subset of $\mathbf{T}$. The argument may be generalized to prove the next proposition.

Proposition 3.1 If $\mathbf{T}$ is a finite set containing a nonzero vector, then there exists an independent subset $\mathbf{B}$ of $\mathbf{T}$ which spans the same vector space as $\mathbf{T}$.

We have made a giant step, though not the final one, toward our search for a dimension number for certain vector spaces. A vector space $\mathbf{V}$ is called *finite dimensional* if it is generated by some finite set. Proposition 3.1 says that every finite dimensional space, except the trivial space $\{\mathbf{0}\}$, is spanned by an independent subset. It is now convenient to introduce a new term:

Definition of Basis

A subset $\mathbf{B}$ *of a finite dimensional space* $\mathbf{V}$ *is a* basis *if and only if*
(a) $\operatorname{Sp} \mathbf{B} = \mathbf{V}$,
(b) $\mathbf{B}$ *is independent.*

Proposition 3.1 may now be restated.

Existence of Basis Theorem

Every finite dimensional vector space except $\{\mathbf{0}\}$ *contains a basis.*

It has already been observed that in the case of geometric spaces the dimension of the space equals the number of vectors in an independent set which generates the space. Thus for these spaces the dimension is the number of vectors in a basis. It is a natural generalization to consider the dimension of any finite dimensional space ($\neq \{\mathbf{0}\}$) as the number of vectors in a basis. This definition of dimension is premature, however, because we do not yet know that two different bases of a finite dimensional vector space $\mathbf{V}$ necessarily have the same number of elements. This has been shown, of course, for our geometric vector spaces. We shall postpone the study of this problem until the next section.

Example 3.1 Any two noncollinear segments form a basis of the directed segments in the Euclidean plane which originate at a fixed point 0. Any three noncoplanar directed segments form a basis of the segments in space. Any two 2-tuples form a basis of $\mathbf{R}^2$ provided their graph points are noncollinear with the origin. Any three 3-tuples form a basis of $\mathbf{R}^3$ if their graph points are noncoplanar with the origin.

Example 3.2 An arbitrary element of the vector space of polynomials of degree less than or equal to 2 on $[0, 1]$ has the form $a + bx + cx^2$. It is evident that $\mathbf{B} = \{1, x, x^2\}$ spans this space.

Since **B** is also independent, by Example 2.1(a), it follows that **B** is a basis of the given space.

If $\mathbf{B} = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$ then the equation

$$\langle a, b\rangle = a\langle 1, 0\rangle + b\langle 0, 1\rangle$$

shows that $\mathrm{Sp}\,\mathbf{B} = \mathbf{R}^2$. Also, **B** is independent, since $a\langle 1, 0\rangle + b\langle 0, 1\rangle = \langle 0, 0\rangle$ implies that $\langle a, b\rangle = \langle 0, 0\rangle$ and hence $a = b = 0$. Therefore, **B** is a basis of $\mathbf{R}^2$, called the *standard basis*. The standard basis of $\mathbf{R}^3$ is

$$\{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\}.$$

More generally, the standard basis of $\mathbf{R}^n$ is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where

$$\mathbf{e}_i = \langle 0, 0, \ldots, 0, 1, 0, \ldots, 0\rangle$$

has 1 in the ith position and 0 elsewhere. Conventional symbols for the standard basis vectors of $\mathbf{R}^2$ are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$; for $\mathbf{R}^3$ we write

$$\mathbf{i} = \langle 1, 0, 0\rangle, \mathbf{j} = \langle 0, 1, 0\rangle, \mathbf{k} = \langle 0, 0, 1\rangle.$$

Thus

$$\langle a, b\rangle = a\mathbf{i} + b\mathbf{j} \quad \text{and} \quad \langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The particular use of **i**, **j** as a 2-tuple or 3-tuple is usually understood by its context.

As a final result of this section, it will be shown that every vector can be written in only one way as a linear combination of a basis. Suppose, for instance, that $\{\mathbf{u}, \mathbf{v}\}$ is a basis and $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = c\mathbf{u} + d\mathbf{v}$. Then

$$(a - c)\mathbf{u} + (b - d)\,\mathbf{v} = \mathbf{0},$$

and therefore $a - c = 0$ and $b - d = 0$, by the independence of $\{\mathbf{u}, \mathbf{v}\}$; hence $a = c$ and $b = d$. A generalization of our argument leads to the proposition below.

Proposition 3.2 If **B** is a basis of **V**, then each vector in **V** can be uniquely written as a linear combination of **B**.

Questions

1. **V** is finite dimensional if it (a) is finite, (b) is generated by a finite set, (c) contains a finite independent set ________.
2. **B** is a basis of **V** provided ________ and ________.
3. The standard basis of $\mathbf{R}^4$ is ________.
4. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis and $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a'\mathbf{u} + b'\mathbf{v} + c'\mathbf{w}$, then ________.

Problems

1. Do Problem Set E at the end of the chapter.

Proofs

1. Prove that $\{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\}$ is a basis of $\mathbf{R}^3$.

4. Dimension

In this section we arrive at our goal of assigning a dimension number to finite dimensional spaces. The necessary preliminary result, that any two bases of a vector space have the same number of elements, is a consequence of the following theorem, which also has other important implications.

Exchange Theorem

Let **B** *be a basis of a finite dimensional space* **V** *and* **T** *an independent set in* **V**. *Then* **T** *has no more elements than* **B** *and may be substituted into* **B** *for some equal size set to obtain a new basis of* **V**.

Thus, if **B** has n elements and **T** has m elements, then $m \leq n$. Furthermore, **T** can replace some m-element subset of **B** to give another basis of **V**.

Example 4.1 The exchange theorem states that **T** can be substituted for some, not any, subset of **B** having equal size. Let $\mathbf{B} = \{\overline{OP}, \overline{OQ}\}$ be a basis of the directed segments in the Euclidean plane and let $\mathbf{T} = \{\overline{OR}\}$, where $R \neq O$ and $\overline{OR}$ is collinear with $\overline{OP}$. Then **T** may be substituted for $\{\overline{OP}\}$ to get a new basis; however, substitution of **T** for $\{\overline{OQ}\}$ will not give a basis.

A proof of the exchange theorem is found in the Appendix. We shall now see how it applies to our problem. Assume **V** has two bases, $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively, containing n_1, n_2 elements. Since a basis is an independent set, we may set $\mathbf{B} = \mathbf{B}_1$, $\mathbf{T} = \mathbf{B}_2$ in the exchange theorem. From the conclusion of the exchange theorem, $\mathbf{B}_2$ has no more elements than $\mathbf{B}_1$ and hence $n_2 \leq n_1$. Similarly, set-

ting $\mathbf{B} = \mathbf{B}_2$, $\mathbf{T} = \mathbf{B}_1$ shows $n_1 \leq n_2$, and therefore $n_1 = n_2$, which leads to the following proposition.

Proposition 4.1 Any two bases of a finite dimensional space have the same number of elements.

Another consequence of the exchange theorem will now be observed. Let $\mathbf{V}$ denote a finite dimensional space, and $\mathbf{T}$ denote an independent set in $\mathbf{V}$. Then $\mathbf{V}$ has a basis $\mathbf{B}$ and, by the exchange theorem, $\mathbf{T}$ may be substituted into $\mathbf{B}$ to obtain a new basis, $\mathbf{B}_1$, of $\mathbf{V}$. Since $\mathbf{T}$ is a subset of $\mathbf{B}_1$, we may make the following conclusion.

Proposition 4.2 Every independent subset of a finite dimensional vector space is contained in a basis.

The uniqueness of basis size from Proposition 4.1 justifies our next definition.

Definition of Dimension

If $\mathbf{V}$ *is a finite dimensional vector space containing a nonzero vector, then the* dimension *of* $\mathbf{V}$ *is the number of elements in each basis of* $\mathbf{V}$.

The trivial space $\{\mathbf{0}\}$ does not have a basis and it is assigned dimension zero. The symbol dim $\mathbf{V}$ will represent the dimension of $\mathbf{V}$.

Suppose dim $\mathbf{V} = n$, and $\mathbf{T}$ is an independent set in $\mathbf{V}$ having n elements. By Proposition 4.2. $\mathbf{T}$ is contained in a basis $\mathbf{B}$ of $\mathbf{V}$. Since $\mathbf{B}$ must have n elements, by the uniqueness of basis size, we conclude $\mathbf{T} = \mathbf{B}$ and therefore $\mathbf{T}$ is also a basis.

Proposition 4.3 If dim $\mathbf{V} = n$, then $\mathbf{B}$ is a basis of $\mathbf{V}$ if and only if $\mathbf{B}$ is an independent set of n vectors.

Not all vector spaces are finite dimensional. Consider $\mathbf{V} = \mathbf{F}[[0, 1], \mathbf{Re}]$, the space of real functions on $[0, 1]$. Every finite subset of $\{1, x, x^2, \ldots, x^n, \ldots\}$ is independent by the fundamental theorem of algebra, which says an equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

holds for at most n values of x. Suppose that $\mathbf{V}$ is finite dimensional and has a basis consisting of n vectors. Then $\mathbf{T} = \{1, x, x^2, \ldots, x^n\}$ is an independent set

of $(n+1)$ vectors and hence has more elements than **B**, contradicting the exchange theorem. Therefore **V** is not finite dimensional.

We are now ready to show that the mathematical structure of a finite dimensional vector space is completely determined by a single number, the dimension of the space. Let **V**, **V**$'$ each have dimension three, with respective bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$. To an arbitrary vector $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ in **V** we correspond the vector $c_1\mathbf{u}_1' + c_2\mathbf{u}_2' + c_3\mathbf{u}_3'$ in **V**$'$. This correspondence is one-to-one by the property that each vector is a unique linear combination of a basis. Furthermore, this correspondence preserves the addition and scalar multiplication operations, and is, therefore, an isomorphism (see Proofs, exercise 1). This implies that **V**, **V**$'$ have the same properties, and thus an identical structure. The argument generalizes to arbitrary dimension.

It has already been observed that $\mathbf{R}^n$ has a basis, the standard basis, consisting of n vectors; hence $\dim \mathbf{R}^n = n$. Therefore, by the preceding paragraph, every n dimensional vector space has the same structure as $\mathbf{R}^n$. This does not mean, however, that we should ignore other n-dimensional spaces. Specific attributes of the individual elements of vector spaces often have particular interest. Thus, although $\mathbf{R}^3$ and the set of polynomials of degree less than or equal to 2 on the interval [0, 1] have the same structure, by the following example, their vector members assume a very different form.

Example 4.2 Let **V** be the set of all polynomials of degree less than or equal to 2 on [0, 1]. A basis of **V**, by Example 3.2, is $\{1, x, x^2\}$. Therefore, $\dim \mathbf{V} = 3$ and **V** is isomorphic to $\mathbf{R}^3$.

Questions

1. The exchange theorem says that an independent set can have no more elements than a ______.
2. An example of a vector space which is not finite dimensional is ______. (a) $\mathbf{R}^n$ for n sufficiently large, (b) $\{\mathbf{0}\}$, (c) the continuous functions on [0, 1].
3. If $\dim \mathbf{V} = n$, then any set of n vectors is a basis provided it ______. (a) is dependent, (b) is independent, (c) contains $\mathbf{0}$, (d) does not contain $\mathbf{0}$.
4. The ______ number of **V** determines the structure of **V**.

Exercises

1. Show that $\{1, x-2, x^2+x\}$ is a basis of the set of polynomials of degree less than or equal to 2. (*Hint:* It suffices to show that the set is independent. Why?)

2. Find for the polynomials of degree less than or equal to 2 a basis which contains $\{x - 1, 2x + 1\}$ and justify your answer.
3. Show that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, then
   (a) $\{\mathbf{u}, 2\mathbf{u} - \mathbf{w}, \mathbf{v} + \mathbf{w}\}$ is a basis of Sp $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$,
   (b) $\{\mathbf{u} - \mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{v}\}$ is not a basis of Sp $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.
4. Let $\{\overline{OP}, \overline{OQ}, \overline{OR}\}$ be a basis of directed segments in space and let $\mathbf{T} = \{\overline{OS}, \overline{OT}\}$ be an independent set. Given that $\overline{OS}$ is collinear with $\overline{OP}$ while $\overline{OT}$ is coplanar with $\overline{OQ}$ and $\overline{OR}$ but collinear with neither $\overline{OQ}$ nor $\overline{OR}$, determine for which of the following sets $\mathbf{T}$ may be substituted to give a new basis:
   (a) $\{\overline{OP}, \overline{OQ}\}$, (b) $\{\overline{OP}, \overline{OR}\}$, (c) $\{\overline{OQ}, \overline{OR}\}$.

Proofs

1. Prove that if $\{\mathbf{u}, \mathbf{v}\}$ is a basis of $\mathbf{V}$ and $\{\mathbf{u}', \mathbf{v}'\}$ is a basis of $\mathbf{V}'$, then the correspondence $a\mathbf{u} + b\mathbf{v} \leftrightarrow a\mathbf{u}' + b\mathbf{v}'$ preserves the addition and scalar multiplication operations. (*Hint:* For the addition part it must be shown that the sum of the associated vectors of $a_1\mathbf{u} + b_1\mathbf{v}$ and $a_2\mathbf{u} + b_2\mathbf{v}$ is the associated vector of

$$(a_1\mathbf{u} + b_1\mathbf{v}) + (a_2\mathbf{u} + b_2\mathbf{v}).)$$

Problems

A. *Echelon Form of Matrices*

The study of certain concepts in this chapter is facilitated by a systematic approach to the solution of systems of linear equations. For this reason it is convenient to introduce

certain matrix techniques at this time. A more careful approach to matrices is delayed until a later chapter. For our present purposes, a *matrix* is simply a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 1 & -1 & 5 \\ 4 & 2 & -1 \end{bmatrix}.$$

A horizontal line of numbers in a matrix is called a *row*; a vertical line of numbers is called a *column*. The rows and columns may be regarded as tuples. Thus the *row vectors* of the given matrix are $\langle 1, -1, 5\rangle$ and $\langle 4, 2, -1\rangle$, and the *column vectors* are $\langle 1, 4\rangle$, $\langle -1, 2\rangle$, and $\langle 5, -1\rangle$. Row vectors are numbered from top to bottom and column vectors from left to right.

Application of matrices requires that certain manipulations be made to convert the matrices to specified forms. The row vector $\langle 0, 0, 0, 3, 0\rangle$ is said to have three zeros *initially*, meaning that the first three entries reading from left to right are each zero. The last zero does not count, since it is preceded by a nonzero entry. Similarly $\langle 0, 0, 3, 0\rangle$ has two zeros initially, and $\langle 1, 0, 0, 2\rangle$ has no zeros initially. A matrix is said to be in *echelon form* if each row either has at least one more zero initially than the row above it or has all entries zero. The following matrices have echelon form:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

whereas these matrices do not have echelon form:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

1. For each of the following matrices, give the number of zeros initially in each row and determine whether or not the matrix is in echelon form.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The *three elementary matrix row operations* are:

A.1 (a) interchange of two rows,
(b) addition of a scalar multiple of one row to another,
(c) multiplication of a row by a number not zero.

2. Given the matrix which has column vectors $\langle 0, 1, -1\rangle$ and $\langle 12, 2, 4\rangle$, do the following operations, using each time the matrix obtained in the previous step.
   (a) Interchange the first and second rows.
   (b) Add the first row to the third row.

(c) Multiply the second row by $-1/2$.
(d) Add the second row to the third row.

3. Given the matrix with column vectors $\langle 3, 1\rangle$, $\langle 1, 4\rangle$, and $\langle 2, 0\rangle$, do the following operations using the matrix obtained in the previous step.
(a) Interchange the two rows.
(b) Add (-3) times the first row to the second row.

Every matrix can be converted to echelon form by consecutive applications of the elementary row operations. The most important of these is A.1(b), which could be used exclusively except for certain possible occurrences of 0 entries in the process. The other operations are sometimes used to avoid fractions.

4. Convert to echelon form the matrix with column vectors $\langle 1, 2, 5\rangle$ and $\langle 4, 3, 0\rangle$ by applying the following steps consecutively.
(a) Add(-2) times the first row to the second row.
(b) Add (-5) times the first row to the third row.
(c) Add (-4) times the second row to the third row.
5. Convert to echelon form the matrix with column vectors $\langle 1, 0, 5\rangle$, $\langle 3, 1, 2\rangle$, $\langle 2, 1, 2\rangle$ with the following steps.
(a) Add (-5) times the first row to the third row.
(b) Add (13) times the second row to the third row.
6. Convert to echelon form the matrix with column vectors $\langle 0, 2, 3\rangle$ and $\langle 1, 3, 4\rangle$ as follows:
(a) Interchange the first and second rows.
(b) Multiply the first row by 3 and the third row by (-2) (to avoid fractions).
(c) Add the first row to the third row.
(d) Add (-1) times the second row to the third row.

In general, if the upper left entry of a matrix is not 0, then by adding suitable multiples of the first row to the other rows, we can make the first column 0 in each position except the top. This is the first stage of the conversion of a matrix to echelon form. If the upper left entry is 0, then an interchange of rows will produce a nonzero number, unless the first column is the zero vector, in which case the process starts at the second stage. The second stage proceeds as the first, with the first column and row being ignored. The process is continued until echelon form is reached.

The echelon form of a matrix is not unique. There is, however, one characteristic which is the same for the various echelon forms of a matrix. The *rank* $r(A)$ of a matrix A is defined to be the number of rows in an echelon form of A which are not the zero vector. It can be shown that the rank value is the same for every echelon form.

7. Convert the following matrices to echelon form and find their rank.

(a) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 0 \\ 3 & 5 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 4 \\ 5 & 6 & 7 & 4 \\ 0 & 4 & 8 & -4 \end{bmatrix}$.

A matrix is *square* if it has the same number of rows and columns. The *diagonal entries* of

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

are a_{11}, a_{22}; the diagonal entries of

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

are a_{11}, a_{22}, a_{33}. A *diagonal matrix* has 0 for all entries that are not diagonal. A square matrix can be converted to diagonal form by the elementary row operations, provided its rank equals the number of rows and columns.

8. Convert $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ to diagonal form by adding (-2) times the second row to the first row.

9. Convert $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ to diagonal form by the following procedure.

   (a) Add (-2) times the third row to the second.
   (b) Add (-3) times the third row to the first.
   (c) Add $(-1/2)$ times the second row to the first.

In general, the first step in converting a square matrix to diagonal form is the conversion to echelon form. Then, adding suitable multiples of the bottom right entry to the other rows converts all entries of the last column to 0, except, of course, the bottom entry. The operation is repeated with the last column and last row being ignored. The process continues until diagonal form is reached.

10. Convert these matrices to diagonal form.

    (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 5 & 2 \\ 1 & 7 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 0 \end{bmatrix}$,

    (d) $\begin{bmatrix} 6 & 1 & 2 \\ 3 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$.

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11. Convert to echelon form and find the rank of these matrices.

    (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 1 & 5 & 6 & 8 \end{bmatrix}$,

(d) $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 2 & -1 & -1 & 4 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$,

(f) $\begin{bmatrix} 0 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

12. Convert each matrix below to diagonal form.

(a) $\begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 8 & 5 & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} 2 & -1 & 6 \\ 3 & 4 & 7 \\ 5 & 0 & 2 \end{bmatrix}$.

B. *Systems of Linear Equations*

A system of two linear equations in two unknowns has the form

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 . \end{aligned}$$

We associate with this system a *coefficient matrix*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and an *augmented matrix*

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}.$$

The coefficient matrix is a *submatrix* of the augmented matrix. Matrix manipulations may be used to solve the given system. The advantages of the matrix technique of solution increase as more equations and unknowns are involved.

Two systems of linear equations are *equivalent* if they have the same solution. Certain operations, called *elementary equation operations*, can be applied to yield equivalent systems. They correspond to the elementary row operations on matrices. In particular they are:

B.1 (a) interchange of two equations,
(b) addition of a number multiple of one equation to another equation,
(c) multiplication of an equation by a number not zero.

The general procedure for solving any system of linear equations is to convert to an equivalent system which can be solved more easily, sometimes by inspection. Matrix methods provide a systematic way of doing this.

1. Given the system

$$\begin{aligned} x + 2y &= 5 \\ 3x - 4y &= -1 \end{aligned}$$

   (a) Write the augmented matrix,
   (b) apply elementary row operations to convert the coefficient sub-matrix to diagonal form,
   (c) from (b), write a system equivalent to the original, and give the solution for x and y.

Similar methods apply to systems of three equations in three unknowns.

2. Repeat Problem 1 for the system

$$\begin{aligned} 2x - y + 3z &= 4 \\ x + 3y - z &= 10 \\ 3x - y + 2z &= 5. \end{aligned}$$

3. Solve by matrix methods the following systems:

   (a) $\begin{aligned} 4x - y &= -11 \\ 3x + 5y &= 9, \end{aligned}$

   (b) $\begin{aligned} 2x - 4y + z &= 19 \\ 3x + 5y - z &= -6 \\ x + 6y + 3z &= 6, \end{aligned}$

   (c) $\begin{aligned} w + x + y + z &= 6 \\ w - x - y - z &= -2 \\ w + x + y - z &= 0 \\ w + x - y + z &= 4. \end{aligned}$

The equation $ax = b$, where a and b are real numbers, has the solution $x = b/a$ if $a \neq 0$. If $a = 0$ but $b \neq 0$, then there is no solution. If $a = 0$, $b = 0$, then $x = c$, where c is arbitrary, is the solution. In general, a system of equations has either a unique solution, no solution, or infinitely many solutions which can be described by one or more arbitrary constants. A system is *consistent* if it has at least one solution, and *inconsistent* otherwise.

4. Solve $2x + 3y = 6$ by letting $y = c$ (arbitrary) and solving for x in terms of c.
5. Solve

$$\begin{aligned} x + 2y - z &= 4 \\ y + 2z &= 6 \end{aligned}$$

   by setting $z = c$ (arbitrary), solving the second equation for y in terms of c, and then solving the first equation for x in terms of c.

6. Solve the following systems for x, y, z.

(a) $$\begin{aligned} x - y + 3z &= 8 \\ 2z &= 4, \end{aligned}$$

(b) $$\begin{aligned} x - 2y - 3z &= 5 \\ y + 2z &= 4 \\ 3z &= 6, \end{aligned}$$

(c) $x + y - 3z = 5,$

(d) $2x + 3y + z = 0.$

In each of the systems in Problems 4, 5, and 6 the augmented matrices are in echelon form. It is evident that any such system of equations can be solved easily by beginning at the bottom equation and moving upward. Thus a natural procedure for solving any system of linear equations is to convert the augmented matrix to echelon form and then solve its corresponding system. It should be noted that the solution of a system may assume many forms if arbitrary constants are involved, since there are many echelon forms of a matrix.

7. Solve the following systems by first converting the augmented matrix to echelon form.

(a) $$\begin{aligned} x + y - 2z &= 4 \\ 3x - y - z &= 2 \\ 5x + y - 5z &= 10, \end{aligned}$$

(b) $$\begin{aligned} x + y - 2z &= 4 \\ 3x - y - z &= 2, \end{aligned}$$

(c) $$\begin{aligned} 2x + y &= 10 \\ x - y &= 5 \\ 5x + 4y &= 25, \end{aligned}$$

(d) $$\begin{aligned} x - y &= 4 \\ x + y &= 6 \\ x - 3y &= 2, \end{aligned}$$

(e) $$\begin{aligned} 2x + y - z &= 4 \\ 3x - y + z &= 5 \\ x + 2y - 3z &= 2. \end{aligned}$$

From an echelon form of an augmented matrix we may determine certain features of a solution without actually finding the solution. For example, the process will always yield a solution except when some equation, in the echelon form, has all coefficients zero and a nonzero number on the right side of the equality. This means that the echelon-form matrix has some row with all entries zero except the last entry which is not zero. This conclusion can be stated in terms of rank.

B.2 A system of equations is consistent if and only if the row ranks of the coefficient and augmented matrices are equal.

We next investigate the number of arbitrary constants in an echelon-form system of consistent equations involving n unknowns. We need consider only the first r equations, where r is the rank of the augmented matrix, since the other equations involve only zeros. In successively solving the equations, arbitrary constants will frequently appear. The number of arbitrary constants contributed by each row is one less than the difference between the number of initial zeros in it and the number of initial zeros in the equation below. For example, if the second equation in a system has 3 initial zeros and the third equation has 5 initial zeros, then the second equation supplies $(5 - 3) - 1 = 1$ arbitrary constant. The total number of such excess zeros is $n - r$.

B.3 If a system of equations is consistent, then its solution involves $(n - r)$ arbitrary constants, where n is the number of unknowns and r is the rank of the coefficient (and augmented) matrix of the system.

8. The following echelon form matrices represent augmented matrices of systems of equations. Determine whether or not the system is consistent and, when consistent, give the number of arbitrary constants in the solution.

(a) $\begin{bmatrix} 1 & 3 & 5 & 2 & 0 \\ 0 & 0 & 3 & 7 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

(c) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix}$.

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9. Solve the following systems when possible.

(a) $\begin{aligned} 2x + 3y &= 5 \\ x - y &= 3, \end{aligned}$ (b) $\begin{aligned} x + 2y &= 6 \\ 2x + 4y &= 12, \end{aligned}$ (c) $\begin{aligned} x + 2y &= 6 \\ 2x + 4y &= 7, \end{aligned}$

(d) $\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y - z &= 3 \\ x + y - z &= 4, \end{aligned}$ (e) $\begin{aligned} x + y - 2z &= 5 \\ 3x + 4y + z &= 1 \\ 5x + 6y - 3z &= 11, \end{aligned}$

(f) $\begin{aligned} x + y + z &= 2 \\ 2x + 2y + 2z &= 4 \\ 3x + 3y + 3z &= 6, \end{aligned}$ (g) $\begin{aligned} x + y - z &= 3 \\ x + y + z &= 6 \\ 3x + 3y - z &= 12, \end{aligned}$

(h) $\begin{aligned} w + x - y + z &= 4 \\ w - x + y + z &= 6 \\ w + 3x - 3y + z &= 2 \\ 4w + 2x - 2y + 4z &= 18. \end{aligned}$

10. For the following systems determine consistency and give the number of arbitrary constants in the solution of those that are consistent.

(a) $\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 4y - z &= 2 \\ x + 6y + 2z &= 8 \\ x + 2y + 10z &= 13, \end{aligned}$ (b) $\begin{aligned} w - 2x - 3y + z &= 1 \\ w + x + y - z &= 3 \\ w - 3x - 5y + 3z &= -1 \\ 3w - 4x - 7y - 3z &= 3, \end{aligned}$

(c) $\begin{aligned} w + x - y - z &= 2 \\ w - x - 3y + z &= 4, \end{aligned}$ (d) $\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y - 5z &= 1, \end{aligned}$

(e) $\begin{aligned} x + y - z &= 3 \\ x + 2y + z &= 5 \\ 2x + y - 4z &= 4, \end{aligned}$ (f) $\begin{aligned} w + x - y + z &= 2 \\ w - x + 2y + z &= 3 \\ 2w + 6x - 8y + 2z &= 4. \end{aligned}$

C. *Span Set*

A tuple $\mathbf{u}$ is a *linear combination* of tuples $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ provided the equation

$$\mathbf{u} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n$$

has a solution for $x_1, x_2, \ldots, x_n$. This equation is an example of a *vector equation.* We can associate with each vector equation a system of linear equations by equating corresponding coordinates of the vectors. Therefore, the study of linear combinations of a given set of tuples becomes a matter of solving equations.

1. Given $\mathbf{u} = \langle 0, -12\rangle$, $\mathbf{v} = \langle 2, 1\rangle$, $\mathbf{w} = \langle 1, 3\rangle$,
   (a) find the system of linear equations associated with $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$,
   (b) show that $\mathbf{u}$ is a linear combination of $\{\mathbf{v}, \mathbf{w}\}$.
2. Given that $\mathbf{u} = \langle 1, 0, 3\rangle$ and $\mathbf{v} = \langle 2, 1, 5\rangle$, show
   (a) there exist numbers x, y such that $\langle 1, -1, 4\rangle = x\langle 1, 0, 3\rangle + y\langle 2, 1, 5\rangle$ and hence $\langle 1, -1, 4\rangle$ is a linear combination of $\{\mathbf{u}, \mathbf{v}\}$,
   (b) there do not exist numbers x, y such that $\langle 3, 1, 4\rangle = x\mathbf{u} + y\mathbf{v}$, and hence $\langle 3, 1, 4\rangle$ is not a linear combination of $\{\mathbf{u}, \mathbf{v}\}$.
3. Determine, in each case, whether or not $\mathbf{w}$ is a linear combination of $\{\mathbf{u}, \mathbf{v}\}$.
   (a) $\mathbf{u} = \langle 1, 0, -2\rangle$, $\mathbf{v} = \langle 3, 1, 7\rangle$, $\mathbf{w} = \langle 1, -1, -15\rangle$;
   (b) $\mathbf{u} = \langle 1, 1, 3\rangle$, $\mathbf{v} = \langle 1, 2, 4\rangle$, $\mathbf{w} = \langle 1, 3, 6\rangle$;
   (c) $\mathbf{u} = \langle 1, 1, 0, 2\rangle$, $\mathbf{v} = \langle 3, 0, 7, 5\rangle$, $\mathbf{w} = \langle 0, 3, -7, 1\rangle$.

The set of all linear combinations of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is called the *span set* of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. The span set of a given set of tuples is obtained by writing an arbitrary vector as a linear combination of the given set, and finding conditions for which the associated system is consistent.

4. Given $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle 2, 0, 4\rangle$, $\mathbf{w} = \langle a, b, c\rangle$, find an equation in a, b, c which implies that the vector equation $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$ has a solution for x, y.
5. Given $\mathbf{u} = \langle 1, 2, 0, 5\rangle$, $\mathbf{v} = \langle 3, 1, 0, 6\rangle$, $\mathbf{w} = \langle a, b, c, d\rangle$, find equations in a, b, c, d which imply that $\mathbf{w}$ is a linear combination of $\{\mathbf{u}, \mathbf{v}\}$.

If the equation $\langle a, b\rangle = x\mathbf{u} + y\mathbf{v}$ has a solution for all a, b, then the span set of $\{\mathbf{u}, \mathbf{v}\}$ is the set of all 2-tuples.

6. Describe by equation(s), or as the set of all 3-tuples, the condition which makes $\mathbf{w} = \langle a, b, c\rangle$ in the span set of
   (a) $\{\langle 2, 1, 3\rangle, \langle 1, -1, 3\rangle, \langle -1, -1, -1\rangle\}$,
   (b) $\{\langle 1, 3, 1\rangle, \langle 2, -1, -1\rangle\}$,
   (c) $\{\langle 1, 0, 2\rangle, \langle 2, 0, 4\rangle\}$,
   (d) $\{\langle 1, 2, 0\rangle, \langle 1, 1, 0\rangle, \langle 1, 0, 1\rangle\}$.

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7. Determine in each case if **w** is a linear combination of $\{\mathbf{u}, \mathbf{v}\}$.
   (a) $\mathbf{u} = \langle 1, 2, 3\rangle, \mathbf{v} = \langle 3, 1, 2\rangle, \mathbf{w} = \langle 5, 3, 7\rangle$;
   (b) $\mathbf{u} = \langle 1, 2, 5\rangle, \mathbf{v} = \langle 2, 4, -1\rangle, \mathbf{w} = \langle 4, 8, 9\rangle$;
   (c) $\mathbf{u} = \langle 1, 0, 3, 2\rangle, \mathbf{v} = \langle 2, 1, 5, 6\rangle, \mathbf{w} = \langle 5, 1, 14, 12\rangle$.
8. Describe the span set for each set of tuples.
   (a) $\{\langle 1, 0\}\rangle, \langle 1, 2\rangle, \langle 0, 1\rangle\}$,
   (b) $\{\langle 1, 2\rangle, \langle -2, -4\rangle\}$,
   (c) $\{\langle 1, 4, 2\rangle, \langle 2, 1, 6\rangle, \langle 5, 13, 12\rangle\}$,
   (d) $\{\langle 1, 0, 3\rangle, \langle 2, 5, 0\rangle, \langle 1, 8, 9\rangle\}$,
   (e) $\{\langle 1, 0, 2, 1\rangle, \langle 3, 1, 4, 0\rangle, \langle 0, -1, 2, 3\rangle\}$.

D. *Dependence and Independence*

A set, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, of tuples is (*linearly*) *dependent* if the zero vector is a nontrivial linear combination of the set. Thus it is dependent if $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$ for some $x_1, x_2, \ldots, x_n$ not all 0. This is the case if the corresponding system of equations has a solution other than the zero solution. Here $\mathbf{0}$ denotes $\langle 0, 0\rangle$, $\langle 0, 0, 0\rangle$, and so forth, according to the tuple size. A set which is not dependent is said to be (*linearly*) *independent.*

1. Given $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 4, 6\rangle$, convert the equation $x\mathbf{u} + y\mathbf{v} = \langle 0, 0\rangle$ to a system of equations and show there is a solution other than $x = 0$, $y = 0$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is dependent.
2. Given $\mathbf{u} = \langle 3, 1, 4\rangle$, $\mathbf{v} = \langle 2, 0, 5\rangle$, $\mathbf{w} = \langle 1, 1, -1\rangle$, convert the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \langle 0, 0, 0\rangle$ to a system of linear equations and show there is a solution other than $x = 0$, $y = 0$, $z = 0$. Hence, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is dependent.
3. Given $\mathbf{u} = \langle 1, 2, 0\rangle$, $\mathbf{v} = \langle 3, 1, -1\rangle$, $\mathbf{w} = \langle 0, 1, 5\rangle$, convert the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \langle 0, 0, 0\rangle$ to a system of equations and show there is no solution other than $x = 0$, $y = 0$, $z = 0$. Hence, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent.
4. Determine whether the following sets are dependent or independent.
   (a) $\{\langle 1, 3\rangle, \langle 2, 4\rangle, \langle 0, 1\rangle\}$,
   (b) $\{\langle 1, 0, 1\rangle, \langle 2, 0, 1\rangle, \langle 1, 0, 2\rangle\}$,
   (c) $\{\langle 1, 2, 5\rangle, \langle 3, 1, 4\rangle, \langle 0, 5, 7\rangle\}$,
   (d) $\{\langle 1, 0, 3, 5\rangle, \langle 2, 1, 7, 4\rangle, \langle 2, -1, 5, 16\rangle\}$,
   (e) $\{\langle 1, 1, 0, 1\rangle, \langle 2, 1, 0, 2\rangle, \langle 1, 3, 0, 1\rangle, \langle 1, 1, 1, 1\rangle\}$.
5. Determine values for a that will make the following sets dependent.
   (a) $\{\langle 1, a\rangle, \langle 3, 4\rangle\}$,
   (b) $\{\langle 1, 1, 7\rangle, \langle 3, 2, 5\rangle, \langle 0, a, 2\rangle\}$.

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6. Determine whether the following sets are dependent or independent.
   (a) $\{\langle 1, 0\rangle, \langle 3, 2\rangle, \langle -1, 4\rangle\}$,
   (b) $\{\langle 1, 2\rangle, \langle 2, 4\rangle\}$,
   (c) $\{\langle 1, 3\rangle, \langle 2, 5\rangle\}$,
   (d) $\{\langle 1, 0, 3\rangle, \langle 2, 1, 5\rangle, \langle 3, 8, 0\rangle\}$,
   (e) $\{\langle 1, 2, -1\rangle, \langle 2, 1, 6\rangle, \langle 1, 5, -9\rangle\}$,
   (f) $\{\langle 1, 0, 3, 2\rangle, \langle 2, 1, 1, 4\rangle, \langle 3, 2, 0, 5\rangle, \langle 0, -1, 4, 1\rangle\}$,
   (g) $\{\langle 1, 1, 0, 0\rangle, \langle 0, 1, 0, 1\rangle, \langle 1, 0, 1, 1\rangle, \langle 0, 0, 1, 0\rangle\}$.
7. Find values for a that will make the following sets dependent.
   (a) $\{\langle 1, a\rangle, \langle 2, 10\rangle\}$,
   (b) $\{\langle 1, 0, 2\rangle, \langle 3, a, 2\rangle, \langle 2, 1, 5\rangle\}$,
   (c) $\{\langle 1, 0, 1, 0\rangle, \langle 0, a, 0, 1\rangle, \langle 1, 1, 2, 1\rangle, \langle 0, 0, 1, 2\rangle\}$.

E. *Bases of* $\mathbf{R}^n$

The symbol $\mathbf{R}^n$ will denote the set of all n-tuples. A set of n-tuples is a basis of $\mathbf{R}^n$ if it is independent and its span set is $\mathbf{R}^n$. We shall first study bases of $\mathbf{R}^2$. If $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$, then $\{\mathbf{u}, \mathbf{v}\}$ is independent if the vector equation $x\mathbf{u} + y\mathbf{v} = \langle 0, 0\rangle$ has only $x = 0$, $y = 0$ as a solution. The associated system of equations is

$$\begin{aligned} a_1 x + b_1 y &= 0 \\ a_2 x + b_2 y &= 0. \end{aligned}$$

Taking the case $a_1 \neq 0$, we convert to echelon form and get the equivalent system,

$$\begin{aligned} a_1 x + \qquad\qquad b_1 y &= 0 \\ (a_1 b_2 - a_2 b_1) y &= 0, \end{aligned}$$

which has a nonzero solution if and only if $a_1 b_2 - a_2 b_1 \neq 0$. On the other hand, the span set of $\{\mathbf{u}, \mathbf{v}\}$ is $\mathbf{R}^2$, provided $x\mathbf{u} + y\mathbf{v} = \langle c_1, c_2\rangle$ has a solution for arbitrary c_1, c_2. The associated system is

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2, \end{aligned}$$

which has echelon form

$$\begin{aligned} a_1 x + \qquad\qquad b_1 y &= c_1 \\ (a_1 b_2 - a_2 b_1) y &= a_1 c_2 - a_2 c_1. \end{aligned}$$

The condition that the system necessarily has a solution is again $a_1 b_2 - a_2 b_1 \neq 0$, the same as the condition for independence. This leads to the following conclusion.

E.1 The following are equivalent.
(a) $\{\mathbf{u}, \mathbf{v}\}$ is a basis of $\mathbf{R}^2$,
(b) $\{\mathbf{u}, \mathbf{v}\}$ is independent,
(c) the span set of $\{\mathbf{u}, \mathbf{v}\}$ is $\mathbf{R}^2$.

A similar analysis shows that the set of 3-tuples $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbf{R}^3$ exactly when $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent. The result in E.1 may be generalized to $\mathbf{R}^n$.

1. Determine whether or not each of the following sets is a basis for the indicated $\mathbf{R}^n$.
   (a) $\{\langle 1, 2\rangle, \langle 3, 5\rangle\}$, $\mathbf{R}^2$;
   (b) $\langle\{1, 2\rangle, \langle 3, 6\rangle\}$, $\mathbf{R}^2$;
   (c) $\{\langle 1, 0, 3\rangle, \langle 2, 1, 5\rangle, \langle -1, 2, 4\rangle\}$, $\mathbf{R}^3$;
   (d) $\{\langle 1, 1, 7\rangle, \langle 2, 0, 3\rangle, \langle 1, 3, 18\rangle\}$, $\mathbf{R}^3$;
   (e) $\{\langle 1, 0, 1, 2\rangle, \langle 3, 1, 4, 5\rangle, \langle 2, 0, 7, 6\rangle, \langle 2, 1, -2, 1\rangle\}$, $\mathbf{R}^4$;
   (f) $\{\langle 1, 1, 1, 2\rangle, \langle 2, 1, 2, 3\rangle, \langle 3, 0, 3, 4\rangle, \langle 4, -1, 4, 6\rangle\}$, $\mathbf{R}^4$.

By looking at systems of equations one can also see that a set of n-tuples containing less than n elements cannot have all of $\mathbf{R}^n$ as its span set and a set of n-tuples having more than n elements cannot be independent.

2. Show, with a suitable system of equations, that the following sets are not a basis of the indicated $\mathbf{R}^n$.
   (a) $\{\langle 1, 3\rangle\}$, $\mathbf{R}^2$;
   (b) $\{\langle 1, 1\rangle, \langle 2, 0\rangle, \langle 3, -4\rangle\}$, $\mathbf{R}^2$;
   (c) $\{\langle 1, 0, 2\rangle, \langle 1, 4, 0\rangle, \langle 2, 1, -3\rangle, \langle 3, 1, 2\rangle\}$, $\mathbf{R}^3$;
   (d) $\{\langle 1, 1, 2\rangle, \langle 2, 0, 1\rangle\}$, $\mathbf{R}^3$.

Review

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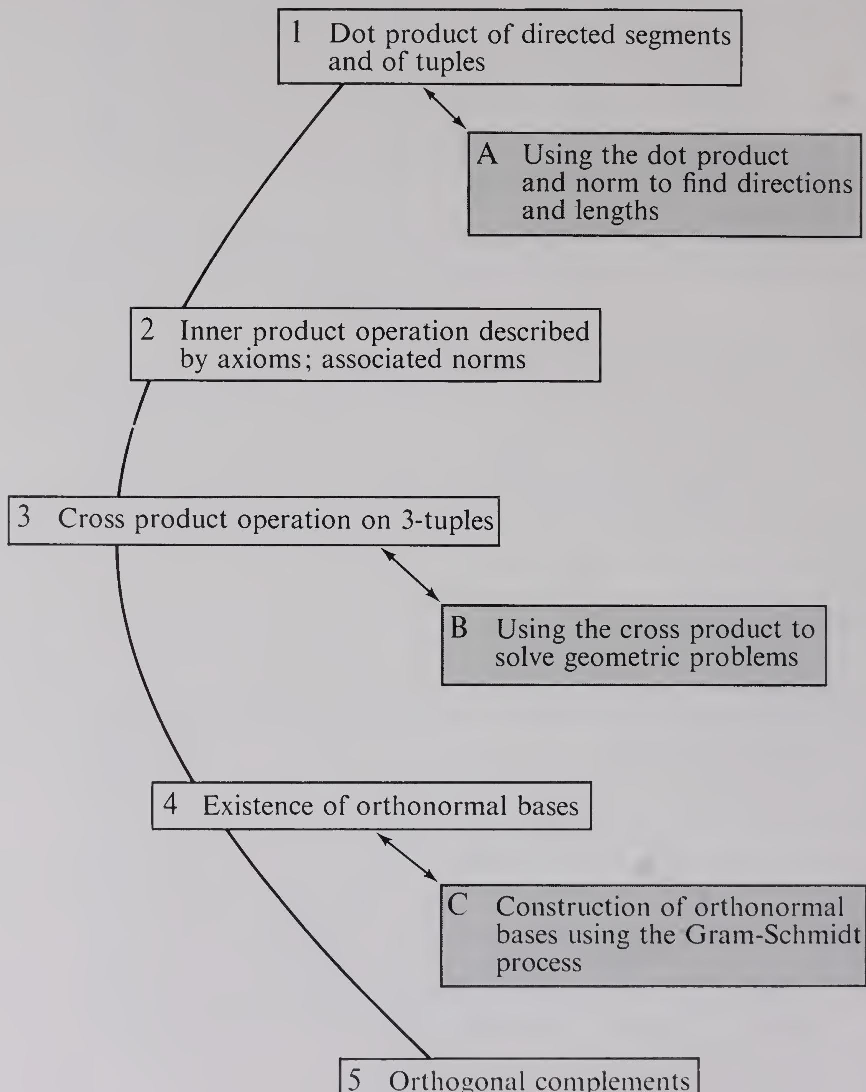
3. Determine whether or not each of the following sets is a basis of the indicated $\mathbf{R}^n$.
   (a) $\{\langle 3, 1\rangle\}$, $\mathbf{R}^2$;
   (b) $\{\langle 1, 0\rangle, \langle 2, 0\rangle\}$, $\mathbf{R}^2$;
   (c) $\{\langle 1, 0\rangle, \langle 2, 3\rangle\}$, $\mathbf{R}^2$;
   (d) $\{\langle 1, 0\rangle, \langle 2, 1\rangle, \langle 1, 4\rangle\}$, $\mathbf{R}^2$;
   (e) $\{\langle 1, 0, 1\rangle, \langle 2, 3, 4\rangle\}$, $\mathbf{R}^3$;
   (f) $\{\langle 1, 1, 0\rangle, \langle 3, 1, 4\rangle, \langle 1, 0, 5\rangle\}$, $\mathbf{R}^3$;
   (g) $\{\langle 1, 0, 2\rangle, \langle 2, 5, 7\rangle, \langle 1, -5, -1\rangle\}$, $\mathbf{R}^3$;
   (h) $\{\langle 1, 0, 0, 1\rangle, \langle 2, 1, 0, 3\rangle, \langle 3, 2, 1, 4\rangle, \langle 6, 3, 1, 8\rangle$, $\mathbf{R}^4$;
   (i) $\{\langle 1, 1, 0, 1\rangle, \langle 2, 0, 1, 3\rangle, \langle 1, 0, 2, 5\rangle, \langle 3, 1, 4, 7\rangle\}$, $\mathbf{R}^4$.


1 Dot product of directed segments and of tuples
A Using the dot product and norm to find directions and lengths
2 Inner product operation described by axioms; associated norms
3 Cross product operation on 3-tuples
B Using the cross product to solve geometric problems
4 Existence of orthonormal bases
C Construction of orthonormal bases using the Gram-Schmidt process
5 Orthogonal complements

III

Inner Product Spaces

Because it does not provide a means for describing length, area, and angles, as well as certain relationships among physical quantities, the vector-space structure provided by the addition and scalar multiplication operations is inadequate for our analysis of space. We shall therefore introduce two basic product operations on vectors. One of these, called *inner product* or *dot product*, provides a measurement for lengths and angles. The other operation, called the *cross product*, is limited to the study of three-dimensional space. It may be used to describe certain areas and volumes, among other things.

1. Dot Product of Directed Segments and Tuples

From physics, the *work* done by a constant force $\mathbf{F}$ acting on an object, moving it through a displacement $\mathbf{d}$, is given by

$$W = F_1 \, |\mathbf{d}|,$$

where F_1 is the component of $\mathbf{F}$ in the direction of $\mathbf{d}$ and $|\mathbf{d}|$ is the length of $\mathbf{d}$. If

θ denotes the angle between the directions of $\mathbf{F}$ and $\mathbf{d}$ (see Figure 3.1), then the component of $\mathbf{F}$ in the direction of $\mathbf{d}$ is

$$F_1 = |\mathbf{F}| \cos \theta,$$

where $|\mathbf{F}|$ is the magnitude of $\mathbf{F}$. By substitution, we get

$$W = |\mathbf{F}|\ |\mathbf{d}| \cos \theta.$$

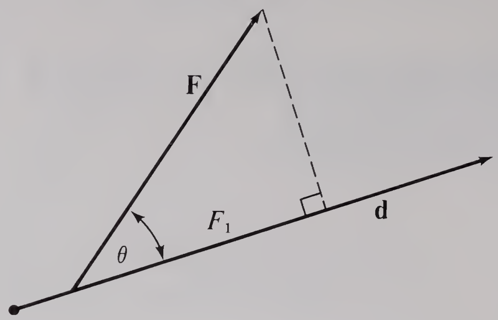


Figure 3.1

The physicist likes to express this work as a certain product of the force $\mathbf{F}$ and the displacement $\mathbf{d}$. This suggests our next definition.

Definition of Dot Product of Segments

The dot product *of* $\overline{OP}$ *and* $\overline{OQ}$ *is*

$$\overline{OP} \cdot \overline{OQ} = |\overline{OP}|\ |\overline{OQ}| \cos \theta,$$

where θ *is the angle between* $\overline{OP}$ *and* $\overline{OQ}$, $0 \leq \theta \leq \pi$.

From $\cos 0 = 1$, $\cos \pi = -1$, $\cos \pi/2 = 0$, the following proposition is easily seen.

Proposition 1.1

(a) If $\overline{OP}$, $\overline{OQ}$ have the same direction, then $\overline{OP} \cdot \overline{OQ} = |\overline{OP}|\ |\overline{OQ}|$.
(b) If $\overline{OP}$, $\overline{OQ}$ have opposite directions, then $\overline{OP} \cdot \overline{OQ} = -|\overline{OP}|\ |\overline{OQ}|$.
(c) If $\overline{OP}$, $\overline{OQ}$ are perpendicular, then $\overline{OP} \cdot \overline{OQ} = 0$.

The angle θ between $\overline{OP}$ and $\overline{OQ}$ is given by

$$\theta = \text{arc cos} \frac{\overline{OP} \cdot \overline{OQ}}{|\overline{OP}|\ |\overline{OQ}|}.$$

Thus, knowledge of the dot product of segments gives information about angular relationships. We next seek the corresponding property for tuples. Let $P=(a_1, a_2)$, $Q=(b_1, b_2)$ be arbitrary points noncollinear with the origin in the Cartesian plane. If θ is the angle between $\overline{OP}$ and $\overline{OQ}$, then by the law of cosines (see Figure 3.2),

$$|\overline{OP}|\,|\overline{OQ}| \cos\theta = \tfrac{1}{2}(|\overline{OP}|^2 + |\overline{OQ}|^2 - |\overline{PQ}|^2).$$

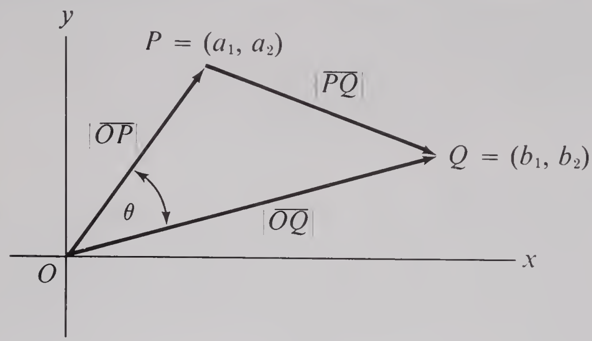


Figure 3.2

By the distance formula for the plane,

$$|\overline{OP}|^2 = a_1{}^2 + a_2{}^2, \qquad |\overline{OQ}|^2 = b_1{}^2 + b_2{}^2,$$
$$|\overline{PQ}|^2 = (b_1 - a_1)^2 + (b^2 - a_2)^2.$$

Substitution gives:

$$\begin{aligned} \overline{OP} \cdot \overline{OQ} &= |\overline{OP}|\,|\overline{OQ}| \cos\theta \\ &= \tfrac{1}{2}[(a_1{}^2 + a_2{}^2) + (b_1{}^2 + b_2{}^2) - (b_1 - a_1)^2 - (b_2 - a_2)^2] \\ &= a_1 b_1 + a_2 b_2 . \end{aligned}$$

A similar analysis in space gives:

$$\overline{OP} \cdot \overline{OQ} = a_1 b_1 + a_2 b_2 + a_3 b_3 ,$$

where $P=(a_1, a_2, a_3)$, $Q=(b_1, b_2, b_3)$. This suggests the following definition.

Definition of Dot Product of n-Tuples

$$\langle a_1, a_2, \ldots, a_n \rangle \cdot \langle b_1, b_2, \ldots, b_n \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

By this definition, the dot product of n-tuples and of directed segments correspond for $n=2$ and $n=3$. We next seek, for n-tuples, an analogue for length and distance. *If* $P=(a_1, a_2)$, then the length of $\overline{OP}$ is

$$|\overline{OP}| = \sqrt{a_1{}^2 + a_2{}^2}\,;$$

if $P=(a_1, a_2, a_3)$, then the length of $\overline{OP}$ is

$$|\overline{OP}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Therefore, the *norm* of a 2-tuple or 3-tuple will correspond to length according to the next definition.

Definition of Norm of an *n*-Tuple

The norm *of* $\mathbf{u} = \langle a_1, a_2, \ldots, a_n\rangle$ *is*

$$|\mathbf{u}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

From the equality $\overline{PQ} = \overline{OQ} - \overline{OP}$, it follows that the distance between P and Q is given by $|\overline{OQ} - \overline{OP}|$. Thus it is natural to define the *distance between tuples* $\mathbf{u}$ and $\mathbf{v}$ as $|\mathbf{v} - \mathbf{u}|$. The norm is an extension of the absolute value concept since $|b - a|$ represents the distance between a and b on the number line.

Questions

1. The vector-space term corresponding to length is ______.
2. Work is given by the ______ of a force and a displacement vector.
3. The dot product of two segments is 0 if they are ______.
4. The dot product of 2-tuples is a ______.
   (a) 2-tuple, (b) 4-tuple,
   (c) directed segment, (d) number.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. If a force of magnitude 50 is applied at an angle of $\pi/3$ to the direction of motion of an object, find the work done as the object is moved through a distance of 10 units.

Proofs

1. Prove the Cauchy-Schwarz inequality for 2-tuples:

$$|\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle| \leq |\langle a_1, a_2\rangle|\,|\langle b_1, b_2\rangle|.$$

2. Abstract Inner Product and Norm

The dot-product operation on segments and tuples can be characterized by properties in the same manner that the addition and scalar multiplication were characterized in Chapter I. Thus an abstract operation, called an *inner product*, will be defined by a list of properties, and it will be seen that the dot product studied in the previous section satisfies each of these properties.

Definition of Inner Product

An operation $(\cdot)$ *on a vector space* $\mathbf{V}$ *is an* inner product *if it assigns a real number to every ordered pair of vectors in* $\mathbf{V}$ *so that, for all* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *in* $\mathbf{V}$ *and all numbers* a, b, c, *the following properties are satisfied:*

(a) $\mathbf{u} \cdot \mathbf{u} \geq 0$; $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$,

(b) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,

(c) $(a\mathbf{u} + b\mathbf{v}) \cdot (c\mathbf{w}) = (ac)(\mathbf{u} \cdot \mathbf{w}) + (bc)(\mathbf{v} \cdot \mathbf{w})$.

The properties in this definition have the following corresponding names and abbreviations.

(a)	*positive definite,*	pos def $(\cdot)$;
(b)	*commutative,*	comm $(\cdot)$;
(c)	*associative distributive,*	as dist $(\cdot)$.

It can be verified that the dot product operation on segments and tuples satisfies the defining properties of the inner product. For instance, the positive definite property for 2-tuples follows from the number property.

$$a_1{}^2 + a_2{}^2 \geq 0, \quad \text{and} \quad a_1{}^2 + a_2{}^2 = 0 \quad \text{if and only if} \quad a_1 = a_2 = 0.$$

The commutative property is shown below.

$$\begin{aligned} \langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle &= a_1 b_1 + a_2 b_2 && \text{def } (\cdot) \text{ tup} \\ &= b_1 a_1 + b_2 a_2 && ab = ba \\ &= \langle b_1, b_2\rangle \cdot \langle a_1, a_2\rangle && \text{def } (\cdot) \text{ tup} \end{aligned}$$

Although the proof of the associative-distributive property requires more detail, it is straightforward.

There are many instances of inner product other than the dot product. For example, it may be verified that

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = 2a_1 b_1 + 3a_2 b_2$$

also satisfies the defining properties of an inner product. The dot product, however, is the only inner product of interest to us on the spaces of tuples and segments, and it will always be implied unless otherwise indicated.

An important inner product on the set of integrable functions on [0, 1] is given by:

$$f \cdot g = \int_0^1 fg\, dx .$$

As a consequence of properties of the integral, this also satisfies the inner product definition.

Example 2.1 If $f = x - 1$, $g = x^2$, then

$$f \cdot g = \int_0^1 (x-1)x^2\, dx = -\tfrac{1}{12} .$$

There is associated with each inner product on **V** a real function on **V** called the *norm* and denoted $|\ \ |$. This is an abstraction of the norm of a tuple, therefore, and it also represents an extension of the absolute-value of real numbers. The equality

$$|\langle a_1, a_2\rangle| = \sqrt{a_1{}^2 + a_2{}^2} = \sqrt{\langle a_1, a_2\rangle \cdot \langle a_1, a_2\rangle}$$

suggests the next definition.

Definition of the Norm of an Inner Product

The norm $|\mathbf{u}|$ *of* $\mathbf{u}$ *in* $\mathbf{V}$ *relative to an inner product* $(\cdot)$ *on* $\mathbf{V}$ *is given by*

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} .$$

Example 2.2 (a) The norm of $f(x) = x$, $0 \le x \le 1$, is given by

$$|f| = [\textstyle\int_0^1 (x)(x)\, dx]^{1/2} = \dfrac{1}{\sqrt{3}} .$$

(b) Using the inner product $\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = 2a_1 b_1 + 3a_2 b_2$, the norm of $\langle 2, 1\rangle$ is

$$[(2)(2)(2) + (3)(1)(1)]^{1/2} = \sqrt{11} .$$

We shall next observe some properties of the inner product and norm. We can obtain some useful special cases of the associative-distributive property by choosing particular values for the coefficient vectors. For instance,

$$(0\mathbf{u} + 0\mathbf{v}) \cdot 1\mathbf{u} = 0(\mathbf{u} \cdot \mathbf{u}) + 0(\mathbf{v} \cdot \mathbf{u})$$

gives $\mathbf{0} \cdot \mathbf{u} = 0$ for each vector $\mathbf{u}$. This and other similarly derived properties are listed in the next result.

Proposition 2.1

(a) $\mathbf{0} \cdot \mathbf{u} = 0$,
(b) $(a\mathbf{u}) \cdot (b\mathbf{v} + c\mathbf{w}) = (ab)(\mathbf{u} \cdot \mathbf{v}) + (ac)(\mathbf{u} \cdot \mathbf{w})$,
(c) $(a\mathbf{u}) \cdot (b\mathbf{v}) = (ab)(\mathbf{u} \cdot \mathbf{v})$,
(d) $\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v})$.

We now look at properties of the norm.

Proposition 2.2

(a) $|\mathbf{u}| \geq 0$; $|\mathbf{u}| = 0$ if and only if $\mathbf{u} = \mathbf{0}$;
(b) $|a\mathbf{u}| = |a|\,|\mathbf{u}|$.

Property (a) is called the *positive-definite* property of the norm; it is an immediate consequence of the corresponding property for the inner product. A proof of (b) is shown below.

$$\begin{aligned} |a\mathbf{u}| &= [(a\mathbf{u}) \cdot (a\mathbf{u})]^{1/2} && \text{def norm} \\ &= [(aa)(\mathbf{u} \cdot \mathbf{u})]^{1/2} && \text{Prop 2.1(c)} \\ &= (a^2)^{1/2}(\mathbf{u} \cdot \mathbf{u})^{1/2} && \sqrt{ab} = \sqrt{a}\,\sqrt{b} \\ &= |a|\,|\mathbf{u}|. && \text{def norm} \end{aligned}$$

Two important inequalities will be considered next. They are extensions of the absolute-value properties:

$$|ab| = |a|\,|b| \text{ and } |a + b| \leq |a| + |b|.$$

From $\overline{OP} \cdot \overline{OQ} = |\overline{OP}|\,|\overline{OQ}| \cos\theta$ and $|\cos\theta| \leq 1$, we see that

$$|\overline{OP} \cdot \overline{OQ}| \leq |\overline{OP}|\,|\overline{OQ}|.$$

Furthermore, equality holds only if $|\cos\theta| = 1$, which is the case when $\overline{OP}$ and $\overline{OQ}$ are collinear. The abstract formulation of this inequality can now be stated.

Cauchy-Schwarz Inequality

(a) $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$,
(b) $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$ if and only if $\{\mathbf{u}, \mathbf{v}\}$ is dependent.

Thus the equality $|ab| = |a|\,|b|$ becomes an inequality in the extension. The *triangle inequality* $|a+b| \le |a| + |b|$ extends, however, without change. If $\overline{OP}$ and $\overline{OQ}$ are not collinear, then $\overline{OP} + \overline{OQ}$ is the diagonal of a parallelogram with sides $|\overline{OP}|$, $|\overline{OQ}|$. Hence, $|\overline{OP} + \overline{OQ}|$ is the length of the side of a triangle having $|\overline{OP}|$ and $|\overline{OQ}|$ as lengths of the other two sides (see Figure 3.3). By elementary geometry,

$$|\overline{OP} + \overline{OQ}| \le |\overline{OP}| + |\overline{OQ}|.$$

The abstract formulation of this property is given next.

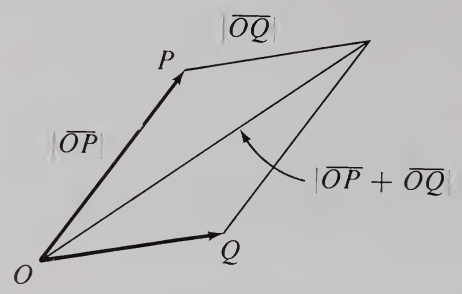


Figure 3.3

Minkowski's (Triangle) Inequality

$$|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|.$$

The Cauchy-Schwarz and Minkowski inequalities play a useful role in many areas of mathematics (see Proofs, exercises 5 and 6).

Example 2.3 (a) In $\mathbf{R}^2$ the Cauchy-Schwarz inequality is

$$\sqrt{a_1b_1 + a_2b_2} \le \sqrt{a_1^{\,2} + a_2^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2}}.$$

(b) For integrable functions on $[0, 1]$ the Cauchy-Schwarz inequality is

$$\left|\int_0^1 fg\,dx\right| \le \left[\left(\int_0^1 f^2\,dx\right)\left(\int_0^1 g^2\,dx\right)\right]^{1/2}.$$

Questions

1. The inner product is an abstraction of the ______ operation on directed segments.
2. The inequality $\langle 2, 1\rangle \cdot \langle 2, 1\rangle > 0$ exemplifies the ______ property of the inner product.
3. The ______ is an extension of the absolute value concept.
4. The triangle inequality means geometrically that ______.
   (a) the sum of the angles of a triangle is not more than π,
   (b) the shortest distance between two points is along a straight line,
   (c) the shortest distance from a point to a line is along a perpendicular.

Exercises

1. Find $f \cdot g$ if
   (a) $f(x) = x, g(x) = x^2, 0 \le x \le 1$;
   (b) $f(x) = e^x, g(x) = e^{2x}, 0 \le x \le 1$.
2. Find $|f|$ in Exercises 1(a) and 1(b).

Proofs

1. Prove $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.
2. Prove $||\mathbf{u}| - |\mathbf{v}|| \le |\mathbf{u} - \mathbf{v}|$.
3. Prove that $\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = 2a_1b_1 + 3a_2b_2$ satisfies the properties of an inner product.
4. Prove $(a\mathbf{u}) \cdot (b\mathbf{v}) = (ab)(\mathbf{u} \cdot \mathbf{v})$ from the definition of inner product.
5. Prove the Cauchy-Schwarz inequality. (*Hint*: For the case $\mathbf{u} \ne \mathbf{0}$ let $\mathbf{w} = |\mathbf{u}|^2\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u}$ and simplify the inequality $\mathbf{w} \cdot \mathbf{w} \ge 0$.)
6. Prove the Minkowski inequality. (*Hint*: Show

$$|\mathbf{u} + \mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2 + 2(\mathbf{u} \cdot \mathbf{v} - |\mathbf{u}|\,|\mathbf{v}|)$$

and apply the Cauchy-Schwarz inequality.)

3. *Cross Product*

In this section we shall study an operation, called the *cross product*, whose definition and significance as a vector-space operation is limited

to three-dimensional spaces. It has many uses in mathematics and physics, such as in finding perpendicular segments, evaluating areas, and defining moments.

Definition of Cross Product

The cross product of two 3-tuples is given by

$$\langle a, b, c\rangle \times \langle a', b', c'\rangle = \langle bc' - b'c, a'c - ac', ab' - a'b\rangle .$$

The cross product may be neatly expressed using the **i**, **j**, **k** symbolism and the determinant concept. The defining formula may then be written

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ a' & b' & c' \end{bmatrix},$$

where the determinant is evaluated by expansion along the top row. For those unfamiliar with 3×3 matrix determinants, this expansion gives

$$\det \begin{bmatrix} b & c \\ b' & c' \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} a & c \\ a' & c' \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \mathbf{k},$$

where

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12} .$$

Example 3.1 $\langle 2, 3, 6\rangle \times \langle 1, 0, -7\rangle$ is given by

$$\det \begin{bmatrix} 3 & 6 \\ 0 & -7 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} 2 & 6 \\ 1 & -7 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \mathbf{k}$$
$$= -21\mathbf{i} + 20\mathbf{j} - 3\mathbf{k} = \langle -21, 20, -3\rangle.$$

The development of many properties of the cross product is a matter of algebraic manipulation using the definition. Our most useful property is the first result.

Proposition 3.1 $(\mathbf{u} \times \mathbf{v})\ \ \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0.$

For the proof let $\mathbf{u} = \langle a, b, c\rangle$ and $\mathbf{v} = \langle a', b', c'\rangle$. Then,

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= (bc' - b'c)a + (a'c - ac')b + (ab' - a'b)c \\ &= 0\end{aligned}$$

as a result of the definitions of dot product and cross product, combined with elementary algebra. Similarly, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.

Since a zero value of the dot product of two directed segments corresponds to their perpendicularity, if $\mathbf{OR} = \mathbf{OP} \times \mathbf{OQ}$, then $\overline{OR}$ is perpendicular to both $\overline{OP}$ and $\overline{OQ}$ by Proposition 3.1. The next result says that the cross product of a 3-tuple with itself gives the zero vector. The proof may be seen immediately from the definition of cross product.

Proposition 3.2 $\quad \mathbf{u} \times \mathbf{u} = \mathbf{0}$.

The cross product operation is neither commutative nor associative. Nevertheless, it does behave well with the other vector-space operations, as evidenced by the following rules which can be proved by applying elementary algebra to the cross-product definition (see Proofs, exercise 1(a), (b)).

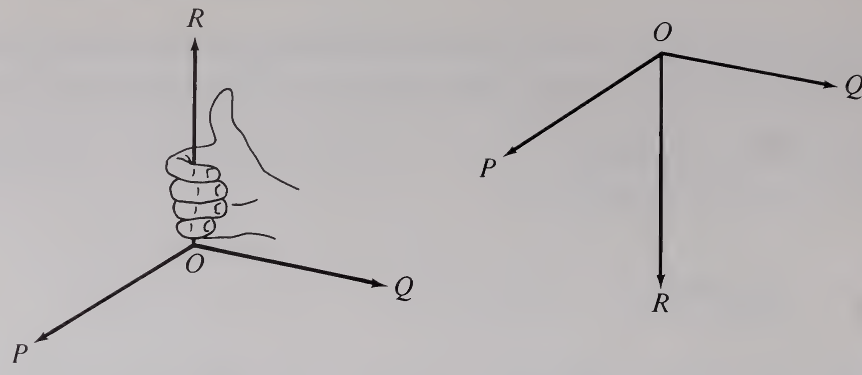
Proposition 3.3

(a) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$, (b) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,
(c) $\mathbf{u} \times c\mathbf{v} = c\mathbf{u} \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v})$, (d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
(e) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$,
(f) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

If $\overline{OP} = c\overline{OQ}$, then

$$\mathbf{OP} \times \mathbf{OQ} = (c\mathbf{OQ}) \times (\mathbf{OQ}) = c(\mathbf{OQ} \times \mathbf{OQ}) = c\mathbf{0} = \mathbf{0}$$

by Propositions 3.3(c) and 3.2. Therefore, collinear segments give a zero cross product. If $\overline{OP}$ and $\overline{OQ}$ are not collinear, then, as already noted, $\mathbf{OR} = \mathbf{OP} \times \mathbf{OQ}$ gives $\overline{OR}$ perpendicular to the plane determined by $\overline{OP}$, $\overline{OQ}$. It is of interest to know on which side of this plane $\overline{OR}$ lies. We formulate this problem using right- and left-hand systems. The ordered set of vectors $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ are said to form a right-hand system provided that when the fingers of the right hand are coiled in the direction from $\overline{OP}$ to $\overline{OQ}$ through the smaller of the angles between $\overline{OP}$ and $\overline{OQ}$, then the right thumb is extended in the direction of $\overline{OR}$ (see Figure 3.4). Otherwise, $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ are said to form a left-hand system. It can be shown that if $\mathbf{OR} = \mathbf{OP} \times \mathbf{OQ}$, then $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ form a right-hand system. The property $\mathbf{OP} \times \mathbf{OQ} = -\mathbf{OQ} \times \mathbf{OP}$ from Proposition 3.3(b) therefore implies that if $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ form a right-hand system, then $\overline{OQ}$, $\overline{OP}$, $\overline{OR}$ form a left-hand system. Thus it is clear that whether a set of three perpendicular vectors forms a right-hand or left-hand system depends on the selected order of the vectors. This may be verified for the various standard basis vectors from the following elementary consequences of the cross-product definition.



$\overline{OP}, \overline{OQ}, \overline{OR}$;
Right-hand system

$\overline{OP}, \overline{OQ}, \overline{OR}$;
Left-hand system

Figure 3.4

Proposition 3.4

(a) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, (b) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, (c) $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$,
(d) $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, (e) $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, (f) $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

For instance, if the fingers of the right hand are coiled from **i** to **j**, then the right thumb points in the direction of **k** (see Figure 3.5).

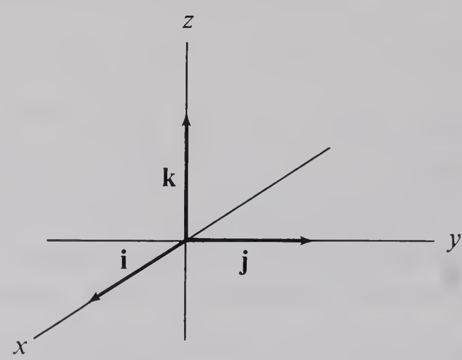


Figure 3.5

Having determined the direction of $\overline{OR}$, where $\mathbf{OR} = \mathbf{OP} \times \mathbf{OQ}$, we next study its magnitude. Again using algebraic manipulations the following proposition may be verified (see Proofs, exercise 1(c)).

Proposition 3.5 $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u} \cdot \mathbf{v}|^2$.

This bears an interesting relation to the Cauchy-Schwarz inequality, since it says that for 3-tuples the inequality is measured by the norm of the cross product. From Proposition 3.5 we shall now obtain a formula relating the cross

product to the area of a parallelogram. In Cartesian space, let θ denote the angle between $\overline{OP}$ and $\overline{OQ}$; then

$$\begin{aligned}|\mathbf{OP} \times \mathbf{OQ}|^2 &= |\mathbf{OP}|^2|\mathbf{OQ}|^2 - |\mathbf{OP} \cdot \mathbf{OQ}|^2 \\ &= |\overline{OP}|^2|\overline{OQ}|^2 - |\overline{OP} \cdot \overline{OQ}|^2 \\ &= |\overline{OP}|^2|\overline{OQ}|^2 - |\overline{OP}|^2|\overline{OQ}|^2 \cos^2\theta \\ &= |\overline{OP}|^2|\overline{OQ}|^2 \sin^2\theta.\end{aligned}$$

Since the parallelogram with sides $\overline{OP}$, $\overline{OQ}$ has $|\overline{OP}|$ as a base and $|\overline{OQ}| \sin\theta$ as an altitude (see Figure 3.6), the following relationship may be concluded.

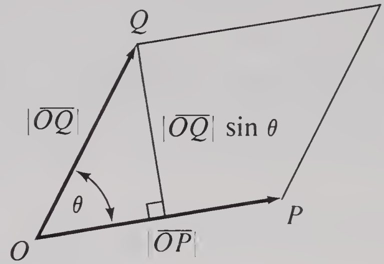


Figure 3.6

Proposition 3.6 The area of the parallelogram with sides $\overline{OP}$, $\overline{OQ}$ is $|\mathbf{OP} \times \mathbf{OQ}|$.

Example 3.2

(a) If $P = (2, 1. 3)$ and $Q = (4, 0, -1)$, then the area of the parallelogram with sides $\overline{OP}$, $\overline{OQ}$ is

$$|\langle 2, 1, 3\rangle \times \langle 4, 0, -1\rangle| = |\langle -1, 14, -4\rangle| = \sqrt{213}.$$

(b) Let $P = (a, b)$ and $Q = (c, d)$ be given points in the Cartesian plane. The area A of the parallelogram containing P, Q, and $(0, 0)$ is the same as the area of the parallelogram containing $(a, b, 0)$, $(c, d, 0)$, and $(0, 0, 0)$. By Proposition 3.6,

$$A = |\langle a, b, 0\rangle \times \langle c, d, 0\rangle| = |\langle 0, 0, ad - bc\rangle| = |ad - bc|.$$

Questions

1. The cross product of two 3-tuples is a ______.
   (a) number,
   (b) 3-tuple,
   (c) 6-tuple.

2. If $\mathbf{OR} = \mathbf{OP} \times \mathbf{OQ}$, then $\overline{OR}$ is ______.
   (a) perpendicular to both $\overline{OP}$ and $\overline{OQ}$,
   (b) parallel to $\overline{OP}$,
   (c) the zero segment.
3. If $\mathbf{OP} \times \mathbf{OQ} = \mathbf{OO}$ then ______.
   (a) $\overline{OP}$ and $\overline{OQ}$ are perpendicular,
   (b) $\overline{OP}$ and $\overline{OQ}$ are parallel,
   (c) either $\overline{OP} = \overline{OO}$ or $\overline{OQ} = \overline{OO}$.
4. The cross product operation is ______.
   (a) commutative,
   (b) associative,
   (c) distributive with respect to addition.

Problems

1. Do Problem Set B at the end of the chapter.

Proofs

1. Prove the properties: (a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, (b) $\mathbf{u} \times c\mathbf{v} = c(\mathbf{u} \times \mathbf{v})$, (c) $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u} \cdot \mathbf{v}|^2$.

4. Gram-Schmidt Theorem

A vector space with an inner product is called an *inner-product space*. We have already seen that $\overline{OP}$ and $\overline{OQ}$ are perpendicular if and only if $\overline{OP} \cdot \overline{OQ} = 0$. The vector space term which corresponds to perpendicular is *orthogonal.*

Definition of Orthogonal

Two vectors $\mathbf{u}$, $\mathbf{v}$ *in an inner-product space are* orthogonal *if and only if* $\mathbf{u} \cdot \mathbf{v} = 0$.

A subset of an inner-product space is orthogonal if every pair of vectors in the set is orthogonal. An orthogonal set may contain $\mathbf{0}$, although our interest

usually lies with those orthogonal sets which do not include $\mathbf{0}$. The main goal of this section is to show that every finite-dimensional inner-product space has for a basis an orthogonal set. To this end we define a *Euclidean vector space* as a finite-dimensional, inner-product space, and an *orthogonal basis* as an orthogonal set which is also a basis. The concept of "Euclidean vector space," which depicts an abstract system, should not be confused with the prior use of the term Euclidean in connection with the plane and space which are studied in high school geometry.

If $\overline{OP}$, $\overline{OQ}$ are any two perpendicular, nonzero segments in the plane, then $\mathbf{B} = \{\overline{OP}, \overline{OQ}\}$ is an orthogonal basis of the vector space of directed segments in the plane. Similarly, any mutually perpendicular set of three nonzero segments in space is an orthogonal basis. It is easily verified that the standard basis of $\mathbf{R}^n$ is also orthogonal. These observations are special cases of a result, to be proved later in this section, which says that every Euclidean vector space has an orthogonal basis. We first consider a preliminary proposition.

Proposition 4.1 Every finite orthogonal set not containing the zero vector is independent.

A proof will be made for an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. We assume

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

and we intend to show that

$$c_1 = c_2 = c_3 = 0.$$

The orthogonality condition implies

$$0 = \mathbf{v}_1 \cdot \mathbf{0} = \mathbf{v}_1 \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + 0 + 0,$$

from which $c_1 = 0$ by the positive-definite property of the inner product, since $\mathbf{v}_1 \neq \mathbf{0}$. In a similar manner, it may be seen that $c_2 = c_3 = 0$, and the desired independence is established.

If $\mathbf{V}$ denotes an n-dimensional Euclidean vector space, then any independent set of n vectors in $\mathbf{V}$ forms a basis of $\mathbf{V}$ by Proposition 4.3 in Chapter 2. It therefore follows from Proposition 4.1 that any orthogonal set of n nonzero vectors is a basis of $\mathbf{V}$. We shall now study a procedure for obtaining an orthogonal set of n vectors from a given basis of $\mathbf{V}$. The method is called the *Gram-Schmidt process*. The idea will first be demonstrated by a geometric example. Let $\{\overline{OP}, \overline{OQ}, \overline{OR}\}$ be a given basis of the directed segments in 3-dimensional space. The orthogonal set to be constructed will be denoted $\{\overline{OP'}, \overline{OQ'}, \overline{OR'}\}$. The steps in the construction are as follows.

(a) Choose $\overline{OP'}$ to be any nonzero segment collinear with $\overline{OP}$.
(b) Choose $\overline{OQ'}$ to be any nonzero segment perpendicular to $\overline{OP}$ and in the plane containing $\overline{OP}$, $\overline{OQ}$.

(c) Choose $\overline{OR'}$ to be any nonzero segment perpendicular to both $\overline{OP}$, $\overline{OQ}$ and in the set generated by $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$, which in this case is all of the underlying 3-dimensional space.

It is geometrically evident that this process will yield an orthogonal basis. The important feature, however, is that it has an algebraic analogue which works for any Euclidean vector space. It may be noted that $\overline{OQ'}$, $\overline{OR'}$ are each chosen to satisfy two conditions and that each of these can be described by an equation or equations. For instance, the condition that $\overline{OQ'}$ is perpendicular to $\overline{OP}$ is described by

$$\overline{OQ'} \cdot \overline{OP} = 0.$$

The condition $\overline{OQ'}$ is in the plane containing $\overline{OP}$, $\overline{OQ}$ is described by

$$\overline{OQ'} = a\overline{OP} + b\overline{OQ}$$

for some numbers a, b. Thus, finding orthogonal bases by this process becomes a matter of solving equations. Two examples will illustrate.

Example 4.1 We seek an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ from the basis $\{\langle 1, 1\rangle, \langle 2, 3\rangle\}$ of $\mathbf{R}^2$. If $\mathbf{u}_1 = \langle 1, 1\rangle$, then $\mathbf{u}_2$ must satisfy the following conditions.

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_1 &= 0, \\ \mathbf{u}_2 &= a\langle 1, 1\rangle + b\langle 2, 3\rangle \text{ for some } a, b. \end{aligned}$$

Combining these equations gives

$$0 = [a\langle 1, 1\rangle + b\langle 2, 3\rangle] \cdot \langle 1, 1\rangle = 2a + 5b,$$

which is satisfied by $a = 5$, $b = -2$. Therefore,

$$\mathbf{u}_2 = 5\langle 1, 1\rangle - 2\langle 2, 3\rangle = \langle 1, -1\rangle$$

completes a solution. It should be observed that the solution is not unique, since there are infinitely many values for a, b.

Example 4.2 We seek an orthogonal basis $\{f_0, f_1, f_2\}$ from the basis $\{1, x, x^2\}$ of functions of degree less than or equal to 2 on $[0, 1]$. If we let $f_0 = 1$, then f_1 satisfies the conditions,

$$\begin{aligned} f_1 \cdot f_0 &= 0, \\ f_1 &= a1 + bx \text{ for some } a, b. \end{aligned}$$

Combining these two equations gives

$$0 = \int_0^1 (a + bx)(1)\, dx = a + \frac{b}{2},$$

which is satisfied by $a = 1$, $b = -2$. Hence, we set $f_1 = 1 - 2x$. The function f_2 must satisfy the following equations.

$$f_2 \cdot f_0 = 0,$$
$$f_2 \cdot f_1 = 0,$$
$$f_2 = a'(1) + b'x + c'x^2 \text{ for some } a', b', c'.$$

Combining these gives two equations,

$$0 = \int_0^1 (a' + b'x + c'x^2)(1)\, dx = a' + \frac{b'}{2} + \frac{c'}{3},$$

and

$$0 = \int_0^1 (a' + b'x + c'x^2)(1 - 2x)\, dx = -\frac{b'}{6} - \frac{c'}{6},$$

which are satisfied by $a' = 1$, $b' = -6$, $c' = 6$. Thus, $f_2 = 1 - 6x + 6x^2$ completes a solution.

An *orthonormal basis* is an orthogonal basis in which every vector has a norm of one. An orthonormal basis can be obtained from an orthogonal basis by multiplying each vector by the reciprocal of its norm. Thus if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis, then $\{\mathbf{u}_1/|\mathbf{u}_1|, \mathbf{u}_2/|\mathbf{u}_2|, \mathbf{u}_3/|\mathbf{u}_3|\}$ is an orthonormal basis. We are now ready for our fundamental result.

Gram-Schmidt Theorem

Every Euclidean vector space ($\neq \{\mathbf{0}\}$) *has an orthonormal basis.*

For the case dim $\mathbf{V} = 3$, we shall give formulas which yield an orthonormal basis from a given basis. These formulas follow the Gram-Schmidt process and include solutions to the equation systems we have seen produced by the process. For concrete problems, however, it is usually simpler to solve the equations rather than apply the formulas. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a given basis of $\mathbf{V}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ the orthonormal basis to be obtained. The steps of the proof are shown.

(a) $\mathbf{v}_1 = \dfrac{\mathbf{u}_1}{|\mathbf{u}_1|}$;

(b) $\mathbf{v}_2' = \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{v}_1)\mathbf{v}_1, \qquad \mathbf{v}_2 = \dfrac{\mathbf{v}_2'}{|\mathbf{v}_2'|}$;

(c) $\mathbf{v}_3' = \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{v}_1)\mathbf{v}_1 - (\mathbf{u}_3 \cdot \mathbf{v}_2)\mathbf{v}_2, \quad \mathbf{v}_3 = \dfrac{\mathbf{v}_3'}{|\mathbf{v}_3'|}$.

Further conclusions may be successively verified.

(a′) $|\mathbf{v}_1| = 1$,

(b′) $\mathbf{v}_2' \cdot \mathbf{v}_1 = 0$; $\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$; $|\mathbf{v}_2| = 1$,

(c′) $\mathbf{v}_3' \cdot \mathbf{v}_1 = 0$; $\mathbf{v}_3 \cdot \mathbf{v}_1 = 0$; $\mathbf{v}_3' \cdot \mathbf{v}_2 = 0$; $\mathbf{v}_3 \cdot \mathbf{v}_2 = 0$; $|\mathbf{v}_3| = 1$.

Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set.

Example 4.3 Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ having the basis

$$\langle\{1, 0, 0, 0\rangle, \langle 1, 1, 0, 0\rangle, \langle 0, 1, 0, 2\rangle\}.$$

An orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is obtained for $\mathbf{S}$ by using the proof formulas as shown.

(a) $\mathbf{v}_1 = \langle 1, 0, 0, 0\rangle$;

(b) $\mathbf{v}_2' = \langle 1, 1, 0, 0\rangle - (\langle 1, 1, 0, 0\rangle \cdot \langle 1, 0, 0, 0\rangle)\langle 1, 0, 0, 0\rangle$
$= \langle 0, 1, 0, 0\rangle, \quad \mathbf{v}_2 = \langle 0, 1, 0, 0\rangle$;

(c) $\mathbf{v}_3' = \langle 0, 1, 0, 2\rangle - (\langle 0, 1, 0, 2\rangle \cdot \langle 1, 0, 0, 0\rangle)\langle 1, 0, 0, 0\rangle$
$-(\langle 0, 1, 0, 2\rangle \cdot \langle 0, 1, 0, 0\rangle)\langle 0, 1, 0, 0\rangle$
$$= \langle 0, 0, 0, 2\rangle, \quad \mathbf{v}_3 = \frac{\langle 0, 0, 0, 2\rangle}{2} = \langle 0, 0, 0, 1\rangle.$$

Questions

1. The vector-space term corresponding to perpendicular is ________.
2. A finite dimensional inner-product space is called a ________.
3. A set of vectors such that each has unit norm and any two are orthogonal is a(n) ________ set.
4. The procedure for finding an orthonormal basis from a given basis is called ________.
5. An orthogonal set is dependent if and only if ________.
6. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis of a vector space, then an orthonormal basis is ________.

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Find a value for a that makes the following pairs of vectors orthogonal.
   (a) $\{\langle 1, 3\rangle, \langle a, 2\rangle\}$, (b) $\{\langle 1, 2, 5\rangle, \langle 2, a-1, 3\rangle\}$,
   (c) $\{f, g\}$ where $f(x) = x$, $g(x) = ax^2 - 3x$ for $0 \le x \le 1$.

2. Find a vector which has a norm of 1 and is orthogonal to each of the following tuples or functions.
   (a) $\langle 1, 2\rangle$, (b) $\langle 2, 0, -3\rangle$ and $\langle 1, 0, 1\rangle$,
   (c) f where $f(x) = x$ and $\mathbf{V}$ is the collection of polynomials of degree less than or equal to 2 on $[0, 1]$.

3. Find an orthonormal basis of each of the sets below.
   (a) Sp $\{\langle 0, 1, 0, 0\rangle, \langle 1, 2, 0, 0\rangle, \langle 0, 1, 0, 1\rangle\}$,
   (b) Sp $\{f, g\}$ where $f(x) = 1 + x^2$, $g(x) = x$, $0 \le x \le 1$,
   (c) the polynomials of degree less than or equal to 3 on $[0, 1]$.

Proofs

1. For the case dim $\mathbf{V} = 4$, give the construction steps for finding an orthonormal basis, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, from a given basis, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

2. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set, then

$$\left|\sum_{i=1}^{n} \mathbf{u}_i\right|^2 = \sum_{i=1}^{n} |\mathbf{u}_i|^2.$$

5. *Orthogonal Complement*

If $\mathbf{T}$ is a nonempty subset of a Euclidean vector space $\mathbf{V}$, then the *orthogonal complement* of $\mathbf{T}$, denoted $\mathbf{T}^\perp$, is the set of vectors in $\mathbf{V}$ which are orthogonal to every vector in $\mathbf{T}$. Thus, $\mathbf{v}$ is in $\mathbf{T}^\perp$ if and only if $\mathbf{v} \cdot \mathbf{u} = 0$ for every $\mathbf{u}$ in $\mathbf{T}$. In this section we shall study properties of $\mathbf{T}^\perp$. Since $\mathbf{0} \cdot \mathbf{u} = 0$ for all $\mathbf{u}$ in $\mathbf{V}$, it is clear that $\mathbf{0}$ is in $\mathbf{T}^\perp$ for every $\mathbf{T}$. If $\mathbf{T} = \mathbf{V}$, then $\mathbf{T}^\perp$ contains only $\mathbf{0}$ by the positive-definite property of the inner product. At the other extreme, if $\mathbf{T} = \{\mathbf{0}\}$, then $\mathbf{T}^\perp = \mathbf{V}$, since $\mathbf{0}$ is orthogonal to every vector in $\mathbf{V}$.

Of more interest are the orthogonal complements of finite sets of directed segments. In the Euclidean plane, if $P \neq O$, then the orthogonal complement of $\{\overline{OP}\}$ is the set of all $\overline{OR}$, where R is on the line through O and perpendicular to $\overline{OP}$. In 3-dimensional space, if $P \neq O$, then $\{\overline{OP}\}^\perp$ consists of all $\overline{OR}$ where R is in

the plane through O and perpendicular to $\overline{OP}$. If $\overline{OP}$ and $\overline{OQ}$ are noncollinear, then $\{\overline{OP}, \overline{OQ}\}^{\perp}$ contains all $\overline{OR}$ where R is on the line through O and perpendicular to the plane containing $\overline{OP}$, $\overline{OQ}$.

It may be observed in all cases considered thus far that the orthogonal complement is a subspace. We shall now show this holds in general.

Proposition 5.1 $\quad \mathbf{T}^{\perp}$ is a subspace.

It must be shown that $\mathbf{T}^{\perp}$ is closed under addition and scalar multiplication. Assume $\mathbf{u}$, $\mathbf{v}$ are in $\mathbf{T}^{\perp}$. If $\mathbf{w}$ is an arbitrary vector in $\mathbf{T}$, then

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} = 0 + 0 = 0,$$

and, therefore, $\mathbf{u} + \mathbf{v}$ is also in $\mathbf{T}^{\perp}$. This proves addition is closed. The chain of equalities

$$\mathbf{w} \cdot (c\mathbf{u}) = c(\mathbf{w} \cdot \mathbf{u}) = 0$$

shows that scalar multiplication is closed, and this completes the proof.

If $\mathbf{S}$ is a subspace of $\mathbf{V}$, then so is $\mathbf{S}^{\perp}$ by Proposition 5.1. By the positive-definite property of the inner product, only the zero vector can lie in both $\mathbf{S}$ and $\mathbf{S}^{\perp}$. We shall now show that $\mathbf{S}$ and $\mathbf{S}^{\perp}$ "split" $\mathbf{V}$. This language is made more precise by the next proposition.

Proposition 5.2 $\quad$ If $\mathbf{S}$ is a subspace of a Euclidean vector space $\mathbf{V}$ and if $\mathbf{w}$ is in $\mathbf{V}$, then there exist unique vectors, $\mathbf{u}$ in $\mathbf{S}$ and $\mathbf{v}$ in $\mathbf{S}^{\perp}$, such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

The vectors $\mathbf{u}$, $\mathbf{v}$ in Proposition 5.2 are respectively called the *components* of $\mathbf{w}$ *parallel* and *perpendicular* to $\mathbf{S}$. These are illustrated in Figure 3.7 for the case in which $\mathbf{S}$ is a 1-dimensional subspace of $\mathbf{R}^2$. Another example, in $\mathbf{R}^3$, will precede a consideration of the proof.

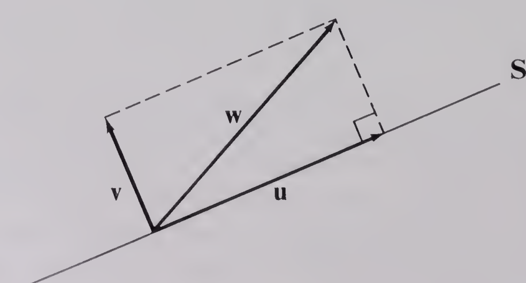


Figure 3.7

Example 5.1 Let $\mathbf{S} = \mathrm{Sp}\{\langle 1,0,1\rangle, \langle 0,2,1\rangle\}$. By the Gram-Schmidt process, an orthogonal basis of S is $\{\langle 1, 0, 1\rangle, \langle 1, -4, -1\rangle\}$. From $\langle 1, 0, 1\rangle \times \langle 1, -4, -1\rangle = \langle 4, 2, -4\rangle$ it is seen, by the perpendicular property of the cross product and geometric considerations, that $\mathbf{S}^\perp = \mathrm{Sp}\{\langle 4, 2, -4\rangle\}$. If $\mathbf{w} = \langle 7, -11, -3\rangle$, then solving the equation

$$\langle 7, -11, -3\rangle = a\langle 1, 0, 1\rangle + b\langle 1, -4, -1\rangle + c\langle 4, 2, -4\rangle$$

gives $a = 2$, $b = 3$, $c = 1/2$, from which

$$\mathbf{u} = 2\langle 1, 0, 1\rangle + 3\langle 1, -4, -1\rangle = \langle 5, -12, -1\rangle,$$
$$\mathbf{v} = \tfrac{1}{2}\langle 4, 2, -4\rangle = \langle 2, 1, -2\rangle$$

are the respective components of $\mathbf{w}$ parallel and perpendicular to S.

We shall now prove the existence portion of Proposition 5.2 for the case where $\dim \mathbf{V} = 5$ and $\dim \mathbf{S} = 3$. There exists by the Gram-Schmidt theorem an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of S. By the results of Chapter I related to the exchange theorem, this basis can be extended to a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_4, \mathbf{v}_5\}$ of V (see Proofs, exercise 1). If the Gram-Schmidt process is applied to this basis in order of subscripts, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ remain unchanged and an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5,\}$ of V results. It follows from the orthonormal property that $\mathbf{u}_4, \mathbf{u}_5$ are in $\mathbf{S}^\perp$, since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are in S. If $\mathbf{w}$ is an arbitrary vector in V, then

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5$$

for some numbers c_1, c_2, c_3, c_4, c_5. Letting

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3,$$
$$\mathbf{v} = c_4\mathbf{u}_4 + c_5\mathbf{u}_5,$$

then $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are in S and $\mathbf{S}^\perp$, respectively, as desired. For the proof of uniqueness for this proposition see Proofs, exercise 2.

Proposition 5.3 If S is a subspace of a Euclidean vector space V then

$$\dim \mathbf{S} + \dim \mathbf{S}^\perp = \dim \mathbf{V}.$$

For a proof of Proposition 5.3, let $\mathbf{B}_1, \mathbf{B}_2$ be respective orthonormal bases of S and $\mathbf{S}^\perp$ having n_1 and n_2 vectors. The union B of $\mathbf{B}_1, \mathbf{B}_2$ is then orthonormal and, hence, independent. Also $\mathrm{Sp}\,\mathbf{B} = \mathbf{V}$ by Proposition 5.2, which implies that every vector in V is a sum of a linear combination of $\mathbf{B}_1$ and a linear combination of $\mathbf{B}_2$. Therefore, B is a basis of V having $n_1 + n_2 (= \dim \mathbf{S} + \dim \mathbf{S}^\perp)$ elements. This completes the proof.

The validity of Proposition 5.3 is easily seen for subspaces of $\mathbf{R}^3$, where there are the following possibilities.

Graph of S	dim S	Graph of $S^\perp$	dim $S^\perp$	dim S + dim $S^\perp$
point (origin)	0	all of space	3	3
line, **L**	1	plane perpendicular to **L**	2	3
plane, **P**	2	line perpendicular to **P**	1	3
all of space	3	point (origin)	0	3

Questions

1. A vector **v** is in the orthogonal complement of **T** provided ______.
   (a) **v** is orthogonal to each vector in the complement of **V**,
   (b) **v** is orthogonal to each vector in **T**,
   (c) **v** is orthogonal to at least one vector in **T**.

2. If $\mathbf{T} = \mathrm{Sp}\{\langle 1, 1\rangle\}$, then the graph of $\mathbf{T}^\perp$ in the xy plane is ______.
   (a) the entire plane,
   (b) the origin,
   (c) the line $y = x$,
   (d) the line $y = -x$.

3. If S is a subspace of a Euclidean vector space **V**, then every vector in **V** is ______.
   (a) in either S or $S^\perp$,
   (b) the sum of vectors in S and $S^\perp$,
   (c) the cross product of vectors in S and $S^\perp$.

4. If S is a subspace of **V**, dim S = 4, and dim **V** = 12, then dim $S^\perp$ = ______.

Exercises

1. Describe, geometrically, the graph of the orthogonal complement of each of the following sets.
   (a) $\{\langle 1, 2\rangle, \langle 0, 1\rangle\}$, (b) $\{\langle 0, 0\rangle\}$,
   (c) $\{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, (d) $\{\langle 1, 3\rangle\}$,
   (e) $\{\langle 1, 2, -1\rangle\}$, (f) $\{\langle 1, 0, 3\rangle, \langle 2, -1, 0\rangle\}$.

2. Write **w** in each case as the sum of a vector **u** in S and a vector **v** in $S^\perp$.
   (a) $\mathbf{w} = \langle 5, 3\rangle$, $S = \mathrm{Sp}\{2, 1\rangle\}$,
   (b) $\mathbf{w} = \langle 1, 0, 2\rangle$, $S = \mathrm{Sp}\,\{\langle 1, 0, 1\rangle\}$,
   (c) $\mathbf{w} = \langle 2, 0, 5\rangle$, $S = \mathrm{Sp}\,\{\langle 0, 1, 1\rangle, \langle 1, 1, 0\rangle\}$.

Proofs

1. Prove that if $\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set in $\mathbf{R}^3$, then there is an orthonormal basis of $\mathbf{R}^3$ which contains $\mathbf{B}$.
2. Let $\mathbf{S}$ be a subspace of $\mathbf{V}$. Prove that if $\mathbf{w} = \mathbf{u}_1 + \mathbf{v}_1 = \mathbf{u}_2 + \mathbf{v}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in $\mathbf{S}$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbf{S}^\perp$, then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$. (*Hint*: $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_1$ is in both $\mathbf{S}$ and $\mathbf{S}^\perp$.)

Problems

A. *Dot Product and Norm*

The dot product for 2-tuples and 3-tuples is given respectively by the following rules.

A.1 (a) $\langle a, b\rangle \cdot \langle a', b'\rangle = aa' + bb'$,
(b) $\langle a, b, c\rangle \cdot \langle a', b', c'\rangle = aa' + bb' + cc'$.

There exists the obvious extension to larger size tuples. Using the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ symbolism, A.1 can be rewritten.

A.2 (a) $(a\mathbf{i} + b\mathbf{j}) \cdot (a'\mathbf{i} + b'\mathbf{j}) = aa' + bb'$,
(b) $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}) = aa' + bb' + cc'$.

1. Evaluate:

(a) $\langle 2, 3\rangle \cdot \langle 1, 5\rangle$, (b) $\langle 3, 1\rangle \cdot \langle -2, 6\rangle$,
(c) $\langle 3, 6, 0\rangle \cdot \langle 2, 1, 7\rangle$, (d) $(\mathbf{i} + 2\mathbf{j}) \cdot (3\mathbf{i} - 5\mathbf{j})$,
(e) $(3\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k})$, (f) $\langle 1, 2, 0, 3\rangle \cdot \langle 2, -1, 4, 6\rangle$,
(g) $\langle 1, 1, 2, 7\rangle \cdot \langle 3, 0, 1, 5\rangle$, (h) $\langle 1, 1, 0, 6, 2\rangle \cdot \langle 2, 1, 1, 5, 4\rangle$.

The *norm* of a tuple, $\mathbf{u}$, is defined by

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

For 2-tuples and 3-tuples this leads to the following equalities.

A.3 (a) $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$,
(b) $|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$.

2. Evaluate:
   (a) $|\langle 3, 5\rangle|$, (b) $|\langle 2, -1\rangle|$, (c) $|\langle 3, 2, 4\rangle|$,
   (d) $|2\mathbf{i} - \mathbf{j}|$, (e) $|\mathbf{i} + 3\mathbf{j} - \mathbf{k}|$, (f) $|\langle 1, 0, 3, -2\rangle|$,
   (g) $|\langle 1, 2, 4, -1\rangle|$, (h) $|\langle 1, 2, 3, -1, 4\rangle|$.

The *distance*, $|\overline{PQ}|$, between points P and Q in the Cartesian plane or space is given by the following equation.

A.4 $|\overline{PQ}| = |\mathbf{PQ}|$.

This is also called the *length* of $\overline{PQ}$. The right side of A.4 is evaluated according to A.3. If $R = (x, y)$ is not the origin, then the *direction angle* of $\mathbf{OR}$ is the angle, θ, measured in radians counter-clockwise from the positive x axis to $\overline{OR}$, with $0 \le \theta < 2\pi$ (see Figure 3.8). The direction angle of any directed segment $\overline{PQ}$ ($P \ne Q$) in the Cartesian plane is defined to be the direction angle of $\mathbf{PQ}$. From trigonometry (see Figure 3.9) the direction angle of $\overline{PQ}$ for $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\arctan \frac{y_2 - y_1}{x_2 - x_1}.$$

3. Find the length and direction angle of $\overline{PQ}$ in each case.
   (a) $P = (1, 4)$, $Q = (3, -2)$; (b) $P = (2, 7)$, $Q = (5, -3)$.

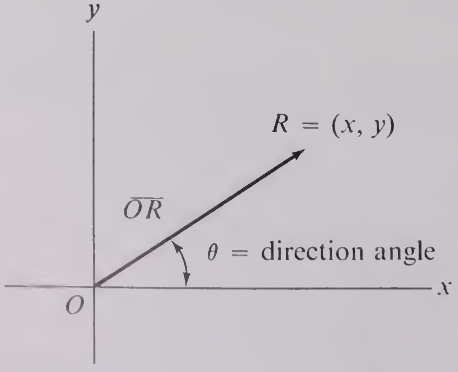


Figure 3.8

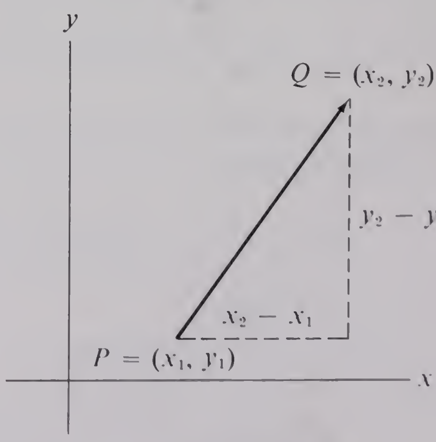


Figure 3.9

4. Find **PQ** if $\overline{PQ}$ has length 6 and direction angle $\pi/4$.

5. Find Q if $P = (3, 0)$ and $\overline{PQ}$ has length 2 and direction angle $3\pi/4$.

The direction of a directed segment $\overline{PQ}$ in Cartesian space is described by three numbers, called the *direction cosines*, which measure the relative rates of change in the x, y, and z directions when proceeding from P to Q. From Figure 3.10 we see that if $\mathbf{PQ} = \langle a, b, c\rangle$ then the cosines of the angles between $\overline{PQ}$ and the x, y, and z axis-directions are, respectively, $\dfrac{a}{|\overline{PQ}|}, \dfrac{b}{|\overline{PQ}|}, \dfrac{c}{|\overline{PQ}|}$. Thus,

A.5 If $\mathbf{PQ} = \langle a, b, c\rangle$, then the *direction cosines* of PQ are $\dfrac{a}{|\overline{PQ}|}, \dfrac{b}{|\overline{PQ}|}$, and $\dfrac{c}{|\overline{PQ}|}$.

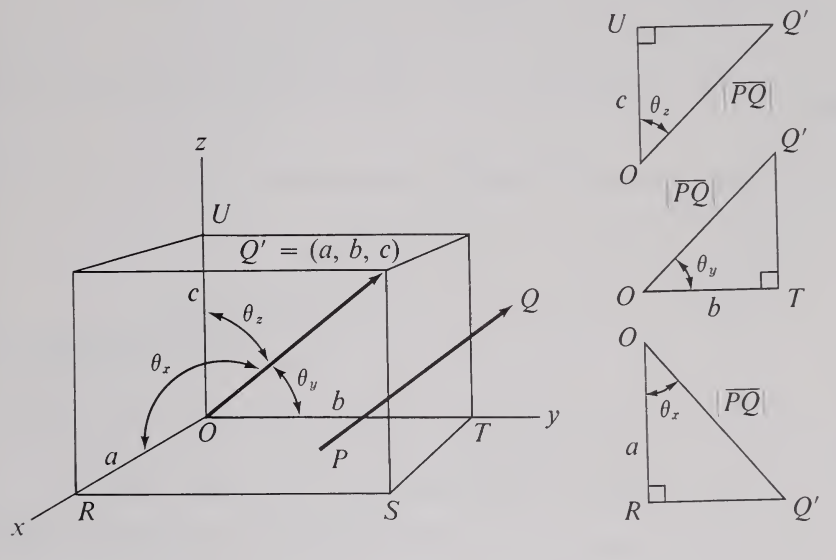


Figure 3.10

Since $|\overline{PQ}|^2 = a^2 + b^2 + c^2$, it follows from A.5 that the sum of the squares of the direction cosines is equal to 1.

6. Find the direction cosines of $\overline{PQ}$ for each of the following pairs of points.

   (a) $P = (2, 1, 5)$, $Q = (3, -4, 6)$,
   (b) $P = (1, 1, 2)$, $Q = (3, 0, -2)$.

7. Find **PQ**, given the conditions below.

   (a) $\overline{PQ}$ has length 4 and direction cosines $\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}$.

   (b) $\overline{PQ}$ has length 6 and direction cosines $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$.

8. Given $P = (2, 1, 5)$ and the conditions in (a) and (b), find Q.

   (a) $\overline{PQ}$ has length 2 and direction cosines $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.
   (b) $\overline{PQ}$ has length 5 and direction cosines $\frac{3}{5}, 0, -\frac{4}{5}$.

In the Cartesian plane or space the angle θ between $\overline{PQ}$ and $\overline{PR}$ is given by

A.6 $$\cos\theta = \frac{\mathbf{PQ}\cdot\mathbf{PR}}{|\mathbf{PQ}|\,|\mathbf{PR}|}.$$

9. Find $\cos\theta$, if θ is the angle between $\overline{PQ}$ and $\overline{PR}$, given the points below.

   (a) $P = (2, 1)$, $Q = (3, 2)$, $R = (1, 4)$,
   (b) $P = (1, 0)$, $Q = (2, 5)$, $R = (7, 3)$,
   (c) $P = (1, 1, 2)$, $Q = (2, 3, 0)$, $R = (6, -1, 2)$.

From $\cos \pi/2 = 0$, it is implied by A.6 that $\overline{PQ}$ and $\overline{PR}$ are perpendicular provided $\mathbf{PQ}\cdot\mathbf{PR} = 0$.

10. Find a so that $\overline{PQ}$ and $\overline{PR}$ are perpendicular.

    (a) $P = (2, 0)$, $Q = (1, 5)$, $R = (a, 4)$;
    (b) $P = (0, a)$, $Q = (2, 3)$, $R = (-1, 4)$.

11. Given $P = (7,1)$ and $Q = (4, 5)$, find a point R on the

    (a) x axis (b) y axis so that $\overline{PQ}$ and $\overline{PR}$ are perpendicular.

12. Find an equation of the line through P and perpendicular to $\overline{PQ}$.

    (a) $P = (2, 3)$, $Q = (1, 8)$; (b) $P = (1, 1)$, $Q = (2, -5)$.
    (*Hint*: Let $R = (x, y)$ be on the line and apply $\mathbf{PQ}\cdot\mathbf{PR} = 0$.)

13. Find an equation of the plane through P and perpendicular to $\overline{PQ}$.

    (a) $P = (1, 1, 4)$, $Q = (0, 6, 3)$;
    (b) $P = (2, 1, 1)$, $Q = (1, -7, 3)$.

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14. Evaluate:

    (a) $\langle 3, 4\rangle\cdot\langle 1, 7\rangle$, (b) $\langle 1, 1, 5\rangle\cdot\langle 0, 2, 6\rangle$,
    (c) $(\mathbf{i} - 3\mathbf{j})\cdot(2\mathbf{i} + 4\mathbf{j})$, (d) $(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot(3\mathbf{i} - \mathbf{j} + 7\mathbf{k})$,
    (e) $\langle 1, 0, 2, 6\rangle\cdot\langle 3, 1, 4, -2\rangle$, (f) $|\langle 2, 3\rangle|$,
    (g) $|\langle 1, 2, 6\rangle|$, (h) $|\mathbf{i} - 3\mathbf{j}|$,
    (i) $|3\mathbf{i} + \mathbf{j} - 2\mathbf{k}|$, (j) $|\langle 1, 0, 3, 5, 8\rangle|$.

15. Find the length and direction angle or direction cosines of $\overline{PQ}$.

    (a) $P = (1, 3)$, $Q = (7, -4)$; (b) $P = (2, 2)$, $Q = (1, 5)$;
    (c) $P = (1, 1, 2)$, $Q = (3, 0, -4)$; (d) $P = (1, 0, 3)$, $Q = (1, 7, 2)$.

16. Find **PQ** given the following conditions.

    (a) $\overline{PQ}$ has length 3 and direction angle $3\pi/4$.
    (b) $\overline{PQ}$ has length 9 and direction cosines $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

17. Find P from the following information.

    (a) $Q = (1, 3)$ and $\overline{PQ}$ has length 4 and direction angle $\pi/3$.
    (b) $Q = (1, 0, 2)$ and $\overline{PQ}$ has length 10 and direction cosines $0, \frac{4}{5}, \frac{3}{5}$.

18. Find $\cos\theta$ if θ is the angle between $\overline{PQ}$ and $\overline{PR}$ given the points below.

    (a) $P = (1, 0)$, $Q = (3, 5)$, $R = (2, 7)$;
    (b) $P = (1, 1, 2)$, $Q = (2, 1, 6)$, $R = (3, 0, 5)$.

19. Given $P = (2, 1)$, $Q = (3, -4)$ find

    (a) the point R having the form $(4, a)$ so that $\overline{PQ}$ and $\overline{PR}$ are perpendicular; also
    (b) the point R on the x axis so that $\overline{PQ}$ and $\overline{PR}$ are perpendicular, and
    (c) the equation of the line through Q and perpendicular to $\overline{PQ}$

B. *Cross Product*

The *cross product* of two 3-tuples is represented by

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k})$$

and defined to be the 3-tuple shown in B.1.

B.1 $$\det\begin{bmatrix} b & c \\ b' & c' \end{bmatrix}\mathbf{i} - \det\begin{bmatrix} a & c \\ a' & c' \end{bmatrix}\mathbf{j} + \det\begin{bmatrix} a & b \\ a' & b' \end{bmatrix}\mathbf{k}.$$

The determinant of a 2×2 matrix is here given by the formula

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The coefficient matrix of **i** may be found from the matrix

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ a' & b' & c' \end{bmatrix}$$

by deleting the row and column containing **i**. The coefficient matrices for **j** and **k** are found similarly.

1. Form the appropriate matrix and evaluate the cross product.

    (a) $(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$,
    (b) $\langle 3, 1, 4\rangle \times \langle 2, -5, 1\rangle$.

If $\overline{PQ}$ and $\overline{PR}$ are not collinear and $\mathbf{PS} = \mathbf{PQ} \times \mathbf{PR}$, then $\overline{PS}$ is perpendicular to both $\overline{PQ}$ and $\overline{PR}$, and hence, to the plane containing $\overline{PQ}$ and $\overline{PR}$. Moreover, if $\overline{PT}$ is perpendicular to both $\overline{PQ}$ and $\overline{PR}$, then $\mathbf{PT} = r(\mathbf{PQ} \times \mathbf{PR})$ for some number r.

2. Find a and b so that $\overline{PS}$ is perpendicular to both $\overline{PQ}$ and $\overline{PR}$.

   (a) $P = (1, 0, 3)$, $Q = (2, 1, 4)$, $R = (4, 2, 5)$, $S = (a, b, 4)$;
   (b) $P = (5, 1, 2)$, $Q = (3, 0, 1)$, $R = (2, 0, 4)$, $S = (2, a, b)$.

3. Find all points S such that $|\overline{PS}| = 1$ and $\overline{PS}$ is perpendicular to the plane containing P, Q, R.

   (a) $P = (1, 0, 2)$, $Q = (2, -1, 1)$, $R = (3, 0, 2)$;
   (b) $P = (2, 0, 1)$, $Q = (1, 1, 0)$, $R = (4, 2, 0)$.

4. Describe in the form $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ the line through P and perpendicular to the plane containing P, Q, R as given.

   (a) $P = (1, 0, 3)$, $Q = (2, -1, 1)$, $R = (4, 1, 6)$;
   (b) $P = (2, -1, 0)$, $Q = (0, 4, 2)$, $R = (1, 5, 7)$.

   (*Hint*: Let $S = (x, y, z)$ be on the line and apply $\mathbf{PS} = r(\mathbf{PQ} \times \mathbf{PR})$ to get three equations in x, y, z, r. Eliminate r to get the answer.)

If $\overline{PQ}$ and $\overline{RS}$ are parallel or collinear, then $\mathbf{PQ} \times \mathbf{RS} = \mathbf{0}$.

5. Describe by two equations the line containing the following points.

   (a) $P = (1, 0, 6)$, $Q = (2, 1, 3)$,
   (b) $P = (4, 2, 1)$, $Q = (0, 5, 3)$.

   (*Hint*: Let $R = (x, y, z)$ be on the line and solve $\mathbf{PQ} \times \mathbf{PR} = \mathbf{0}$. Any two of the resulting equations suffice, since each can be obtained from the other two.)

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6. Evaluate

   (a) $(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$,
   (b) $\langle 3, 2, 0 \rangle \times \langle 1, 5, -6 \rangle$.

7. Given $P = (1, 1, 3)$, $Q = (2, 0, 1)$, $R = (2, -1, 0)$,

   (a) find $S = (a, 2, b)$ so that $\overline{PS}$ is perpendicular to the plane containing P, Q, R;
   (b) find all points S so that $|\overline{PS}| = 1$ and $\overline{PS}$ is perpendicular to $\overline{PQ}$ and $\overline{PR}$;
   (c) find the line through P and perpendicular to the plane containing P, Q, R;
   (d) find the line through P and Q.

C. *Gram-Schmidt Process*

Two tuples, $\mathbf{u}$ and $\mathbf{v}$, are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$. A set of tuples is orthogonal if any two tuples selected from the set are orthogonal. The set is *orthonormal* if it is orthogonal and every tuple in the set has a norm of 1. Every vector space of tuples contains an orthonormal basis which can be found from a given basis by a technique called the *Gram-Schmidt process.* This is illustrated by the problems that follow.

1. Let $\mathbf{u}_1 = \langle 3, 0, 1\rangle$ and $\mathbf{u}_2 = \langle 2, 1, 1\rangle$. Find an orthonormal basis of the span set of $\{\mathbf{u}_1, \mathbf{u}_2\}$ as follows:

   (a) set $\mathbf{v}_1' = \mathbf{u}_1$,
   (b) let $\mathbf{v}_2' = a\mathbf{u}_1 + b\mathbf{u}_2$ and find a value of a and b so that $\mathbf{v}_2' \cdot \mathbf{v}_1' = 0$,
   (c) set $\mathbf{v}_1 = \dfrac{\mathbf{v}_1'}{|\mathbf{v}_1'|}$ and $\mathbf{v}_2 = \dfrac{\mathbf{v}_2'}{|\mathbf{v}_2'|}$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the desired orthonormal basis.

2. Find an orthonormal basis of the span set of $\{\langle 2, 0, -1\rangle, \langle 1, 1, 3\rangle\}$.
3. Let $\mathbf{u}_1 = \langle 1, 0, 0, 1\rangle$, $\mathbf{u}_2 = \langle 0, 1, 0, 2\rangle$, and $\mathbf{u}_3 = \langle 1, 1, 2, 1\rangle$. Find an orthonormal basis of the span set of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ as follows:

   (a), (b) proceed as in (a) and (b) of Problem 1.
   (c) Let $\mathbf{v}_3' = c\mathbf{u}_1 + d\mathbf{u}_2 + e\mathbf{u}_3$ and find a value of c, d, e so that $\mathbf{v}_3' \cdot \mathbf{v}_1' = \mathbf{v}_3' \cdot \mathbf{v}_2' = 0$.
   (d) Set $\mathbf{v}_1 = \dfrac{\mathbf{v}_1'}{|\mathbf{v}_1'|}$, $\mathbf{v}_2 = \dfrac{\mathbf{v}_2'}{|\mathbf{v}_2'|}$, and $\mathbf{v}_3 = \dfrac{\mathbf{v}_3'}{|\mathbf{v}_3'|}$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the desired orthonormal basis.

4. Find an orthonormal basis of the span set of
$$\{\langle 1, 0, 2, 3\rangle, \langle -1, 0, 0, 1\rangle, \langle 0, 1, 1, 1\rangle\}.$$

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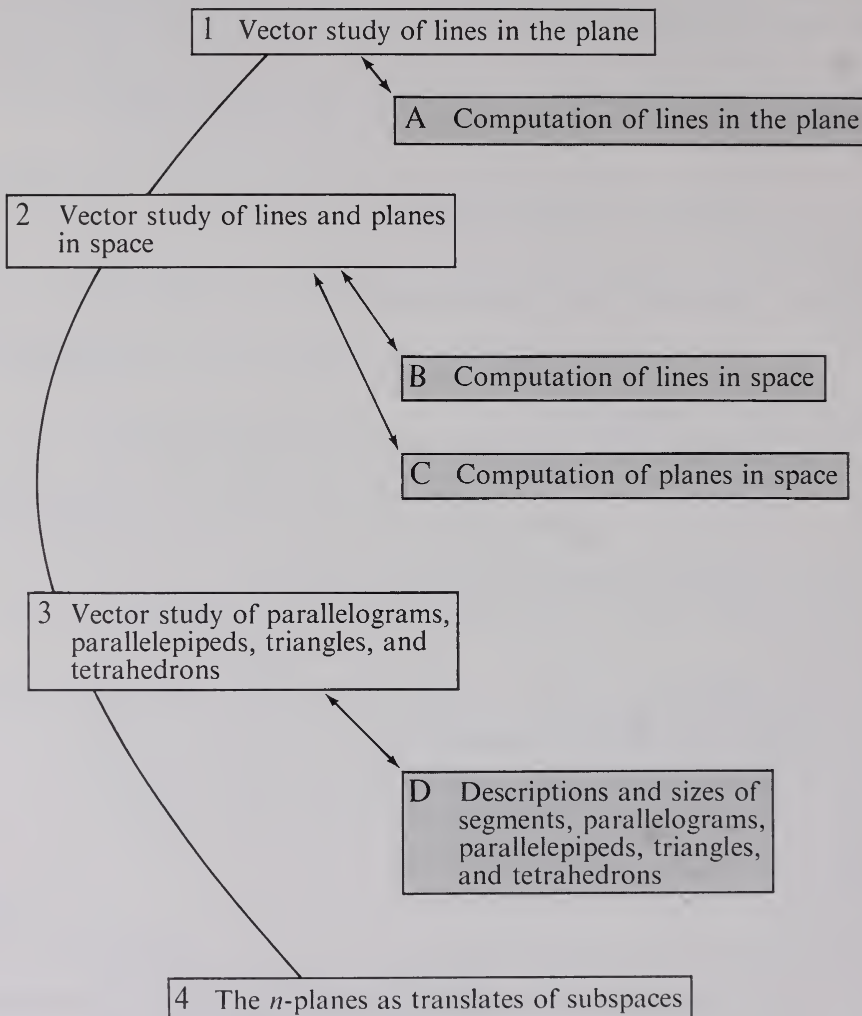
5. Find an orthonormal basis of the span set of each of the sets below:

   (a) $\{\langle 1, 0, 1\rangle, \langle 2, 1, 5\rangle\}$,
   (b) $\{\langle 1, 0, 1, 3\rangle, \langle 1, 1, 0, 2\rangle, \langle 3, 1, 1, -1\rangle\}$.


1 Vector study of lines in the plane
A Computation of lines in the plane
2 Vector study of lines and planes in space
B Computation of lines in space
C Computation of planes in space
3 Vector study of parallelograms, parallelepipeds, triangles, and tetrahedrons
D Descriptions and sizes of segments, parallelograms, parallelepipeds, triangles, and tetrahedrons
4 The n-planes as translates of subspaces

IV

Lines, Planes, and Parallelotopes

This chapter is concerned with important subsets of vector spaces. Thus far only the subspaces have been given attention. The graph of a proper subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$ is a line or plane through the origin. Any line or plane can be obtained from a parallel line or plane through the origin by a shifting, or *translation*. This geometric maneuver will be described by using the vector-space structure.

Lines and planes are unbounded sets. We shall also study subsets of $\mathbf{R}^2$ and $\mathbf{R}^3$ which have a graph of finite extent and can be described using the vector-space structure. These bounded sets fall into two categories:

(a) *parallelotopes*, whose graphs include line segments, parallelograms, and parallelepipeds;

(b) *simplices*, whose graphs include line segments, triangles, and tetrahedrons.

1. Lines in $\mathbf{R}^2$

In this section we shall study sets in $\mathbf{R}^2$ which have for their graph a line. Corresponding lines in the Euclidean plane will be considered first. The line through O and Q consists of all points R where $\overline{OR} = r\overline{OQ}$ for some number r. It is convenient to designate this line as $r\overline{OQ}$. If P is a point not on $r\overline{OQ}$, then any point R on the line through P and parallel to $r\overline{OQ}$, satisfies $\overline{OR} = \overline{OP} + r\overline{OQ}$ for some number r (see Figure 4.1). This line will be denoted

$$\overline{OP} + r\overline{OQ}.$$

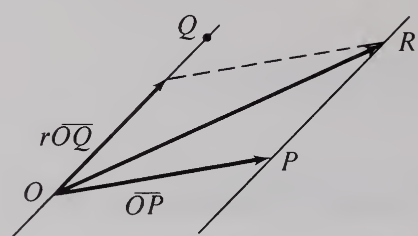


Figure 4.1

We call $\overline{OP}$ a *position vector* and $\overline{OQ}$ a *direction vector* of this line. It is geometrically evident that if P' is another point on $\overline{OP} + r\overline{OQ}$, then

$$\overline{OP} + r\overline{OQ} = \overline{OP'} + r\overline{OQ},$$

which is to say that any point on a line serves to describe a position vector of the line. The direction vector $\overline{OQ}$ may be replaced by any nonzero scalar multiple of $\overline{OQ}$, and the same line results. Also, it is geometrically evident that two lines having a direction vector in common are equal or parallel.

Example 1.1 The line $\overline{OP} + r\overline{OQ}$ contains the terminal point of $\overline{OP} + 3\overline{OQ}$, and therefore

$$\overline{OP} + r\overline{OQ} = (\overline{OP} + 3\overline{OQ}) + r\overline{OQ}.$$

This line may also be written $(\overline{OP} + 3\overline{OQ}) + r(2\overline{OQ})$ (see Figure 4.2).

Using the correspondence between directed segments and 2-tuples, we can observe that a line in the Cartesian plane is the graph of a set $\mathbf{L}$,

$$\mathbf{L} = \{\mathbf{u} + r\mathbf{v}\colon r \text{ is a real number}\},$$

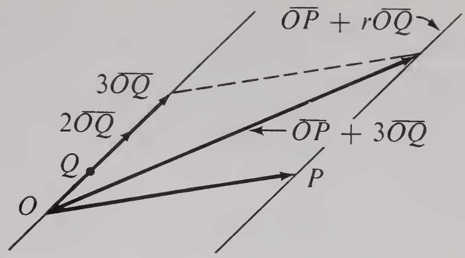


Figure 4.2

where $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors in $\mathbf{R}^2$. This set is abbreviated

$$\mathbf{L} = \mathbf{u} + r\mathbf{v},$$

and $\mathbf{u}$, $\mathbf{v}$ are respectively called a position vector and a direction vector of $\mathbf{L}$. We shall frequently call the vector set $\mathbf{L}$ a line, and identify it verbally with its graph. By our correspondence, $\mathbf{u}$ may be replaced by any vector in $\mathbf{L}$, and $\mathbf{v}$ by any nonzero scalar multiple of itself.

Example 1.2 The set $\mathbf{L} = \langle 1, 3\rangle + r\langle 2, -1\rangle$ has for its graph the line in Figure 4.3. From $\langle 1, 3\rangle + 4\langle 2, -1\rangle = \langle 9, -1\rangle$, we see that

$$\mathbf{L} = \langle 9, -1\rangle + r\langle 2, -1\rangle.$$

Replacing $\langle 2, -1\rangle$ by $2\langle 2, -1\rangle$ gives

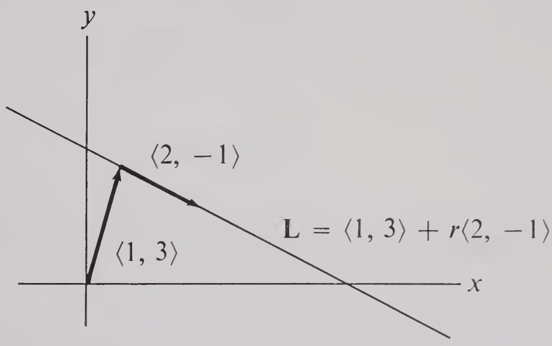


Figure 4.3

$$\mathbf{L} = \langle 9, -1\rangle + r\langle 4, -2\rangle$$

as another form for $\mathbf{L}$. The line through (2, 6) and parallel to $\mathbf{L}$ may be described by $\langle 2, 6\rangle + r\langle 2, -1\rangle$, since it has the same direction vectors as $\mathbf{L}$.

The lines $\overline{OP} + r\overline{OQ}$ and $\overline{OP'} + r\overline{OQ'}$ are perpendicular if their direction vectors are perpendicular. This is the case provided $\overline{OQ} \cdot \overline{OQ'} = 0$. Hence $\mathbf{u} + r\mathbf{v}$ and $\mathbf{u}' + r\mathbf{v}'$ in $\mathbf{R}^2$ are perpendicular if $\mathbf{v} \cdot \mathbf{v}' = 0$.

Example 1.3 We seek the line $\mathbf{L}$ in $\mathbf{R}^2$ which passes through (2, 3) and is perpendicular to the line $\langle 2, 6\rangle + r\langle 4, -1\rangle$. By the preceding discussion, $\mathbf{L} = \langle 2, 3\rangle + r\langle a, b\rangle$, where $\langle 4, -1\rangle \cdot \langle a, b\rangle = 0$. Any solution of $4a - b = 0$, such as $a = 1$, $b = 4$, gives a direction vector of $\mathbf{L}$. Hence,

$$\mathbf{L} = \langle 2, 3\rangle + r\langle 1, 4\rangle.$$

There is a unique line through two points P and Q in the Euclidean plane. From Figure 4.4, or by successively setting $r = 0$ and then $r = 1$, we see that the line

$$\overline{OP} + r(\overline{OQ} - \overline{OP})$$

contains both P and Q. Hence in $\mathbf{R}^2$, the vectors $\mathbf{u}$, $\mathbf{v}$ are contained in the set $\mathbf{L}$,

$$\mathbf{L} = \mathbf{u} + r(\mathbf{v} - \mathbf{u}).$$

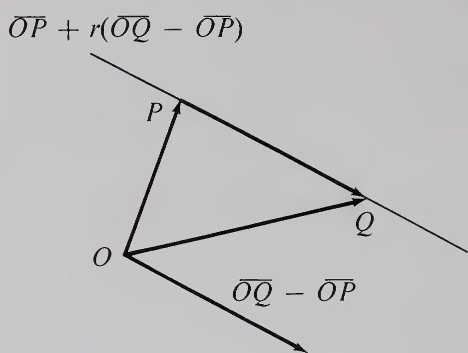


Figure 4.4

Example 1.4 The line through (2, 4) and (3, −2) is described by

$$\mathbf{L} = \langle 2, 4\rangle + r(\langle 3, -2\rangle - \langle 2, 4\rangle) = \langle 2, 4\rangle + r\langle 1, -6\rangle.$$

If the line $\overline{OP} + r\overline{OQ}$ does not pass through O, then it has a unique perpendicular segment $\overline{OR}$, where R is on the line (see Figure 4.5). Since R is on $\overline{OP} + r\overline{OQ}$, then

$$\overline{OR} = \overline{OP} + r_0\overline{OQ}$$

for some r_0. The perpendicularity condition implies that

$$\overline{OR} \cdot \overline{OQ} = 0.$$

Combining these equalities gives

$$(\overline{OP} + r_0\overline{OQ}) \cdot \overline{OQ} = 0,$$

from which

$$r_0 = -\frac{\overline{OP}\cdot\overline{OQ}}{|\overline{OQ}|^2}.$$

Substitution gives

$$\overline{OR} = \overline{OP} - \frac{\overline{OP}\cdot\overline{OQ}}{|\overline{OQ}|^2}\overline{OQ}.$$

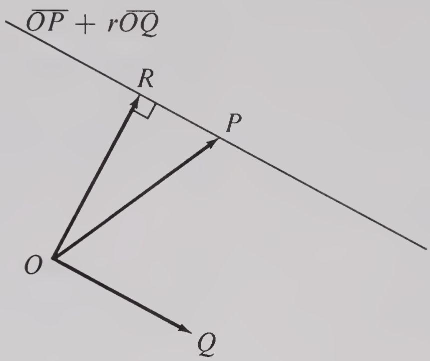


Figure 4.5

The segment $\overline{OR}$ has the shortest length of all segments from O to the line $\overline{OP} + r\overline{OQ}$, and $|\overline{OR}|$ is called the *distance* from O to $\overline{OP} + r\overline{OQ}$. Thus in $\mathbf{R}^2$, the point R on the graph of $\mathbf{L} = \mathbf{u} + r\mathbf{v}$ such that $\overline{OR}$ is perpendicular to the graph satisfies

$$\mathbf{OR} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

Also, $\mathbf{OR}$ has the least norm of any 2-tuple in $\mathbf{L}$.

Example 1.5 The point R on the line $\mathbf{L} = \langle 1, 2\rangle + r\langle 3, 4\rangle$ such that $\overline{OR}$ is perpendicular to $\mathbf{L}$ satisfies the equation

$$\mathbf{OR} = \langle 1, 2\rangle - \frac{\langle 1, 2\rangle\cdot\langle 3, 4\rangle}{25}\langle 3, 4\rangle = \frac{1}{25}\langle -8, 6\rangle.$$

The distance from O to $\mathbf{L}$ is

$$\left|\frac{1}{25}\langle -8, 6\rangle\right| = \frac{2}{5}.$$

We now consider alternative descriptions of $\overline{OP} + r\overline{OQ}$. If this line passes through O and $\overline{OR}$ is a nonzero segment perpendicular to the line, then $\{\overline{OR}\}^{\perp}$ consists of all segments with terminal point on $\overline{OP} + r\overline{OQ}$. We therefore denote

$\overline{OP} + r\overline{OQ}$ by $\{\overline{OR}\}^{\perp}$. If $\overline{OP} + r\overline{OQ}$ does not pass through the origin, let $\overline{OR}$ be perpendicular to this line, where R is on the line. Then

$$\overline{OP} + r\overline{OQ} = \overline{OR} + \{\overline{OR}\}^{\perp},$$

since $\overline{OR}$ is a position vector perpendicular to the direction vector of $\overline{OP} + r\overline{OQ}$.

If S is any point on $\overline{OR} + \{\overline{OR}\}^{\perp}$, then

$$\overline{OR} \cdot \overline{OS} = |\overline{OR}|\,|\overline{OS}| \cos \theta = |\overline{OR}|^2,$$

where θ is the angle between $\overline{OR}$ and $\overline{OS}$ (see Figure 4.6). Thus, for each nonzero vector $\overline{OR}$, the points S which satisfy $\overline{OR} \cdot \overline{OS} = |\overline{OR}|^2$ form the line $\overline{OR} + \{\overline{OR}\}^{\perp}$. Hence, in $\mathbf{R}^2$ if $\mathbf{w} = \mathbf{OR}$, then

$$\{\mathbf{v} \colon \mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2\}$$

has for its graph the line through R and perpendicular to $\overline{OR}$.

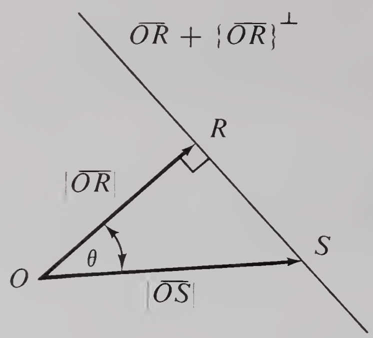


Figure 4.6

Example 1.6 The line which passes through $R = (2, 3)$ and is perpendicular to $\overline{OR}$ may be described as $\{\mathbf{v} \colon \langle 2, 3\rangle \cdot \mathbf{v} = 13\}$ (*note*: $13 = |\langle 2, 3\rangle|^2$). Letting (x, y) be an arbitrary point on this line, from $\langle 2, 3\rangle \cdot \langle x, y\rangle = 13$ we get $2x + 3y = 13$ as an equation of the line.

We next consider the line described by the equation

$$ax + by = c.$$

Multiplication of both sides of the equality by $c/(a^2 + b^2)$ results in the equation

$$\mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2$$

where

$$\mathbf{w} = \frac{c}{a^2 + b^2} \langle a, b\rangle \quad \text{and} \quad \mathbf{v} = \langle x, y\rangle.$$

Therefore, the point R on the line such that $\overline{OR}$ is perpendicular to the line satisfies the equation

$$\mathbf{OR} = \frac{c}{a^2 + b^2} \langle a, b\rangle.$$

The distance from the origin to the line is

$$|\mathbf{OR}| = \frac{|c|}{\sqrt{a^2 + b^2}}.$$

Example 1.7 Given the line $3x + 4y = 2$, the point R on the line such that $\overline{OR}$ is perpendicular to the line satisfies

$$\mathbf{OR} = \frac{2}{3^2 + 4^2} \langle 3, 4 \rangle.$$

Hence $R = (6/25, 8/25)$ and the distance from the origin to the line is $|OR| = 2/5$.

Questions

1. $\overline{OP} + r\overline{OQ} = \overline{OP} + r\overline{OQ'}$ provided that ______.
   (a) Q' is on $\overline{OP} + r\overline{OQ}$,
   (b) $\overline{OQ'}$ and $\overline{OQ}$ are collinear,
   (c) $\overline{OQ'}$ and $\overline{OQ}$ are perpendicular.
2. $\overline{OP} + r\overline{OQ} = \overline{OP'} + r\overline{OQ}$ provided ______.
   (a) P' is on $\overline{OP} + r\overline{OQ}$,
   (b) $\overline{OP}$ and $\overline{OP'}$ are parallel,
   (c) $\overline{OP}$ and $\overline{OP'}$ are perpendicular.
3. $\overline{OP} + r\overline{OQ}$ and $\overline{OP'} + r\overline{OQ'}$ are perpendicular if their ______ vectors are perpendicular.
4. The graphs of $\langle 1, 3\rangle + r\langle 2, 1\rangle$ and $\langle 1, 5\rangle + r\langle 2, 1\rangle$ are ______.
   (a) equal,
   (b) parallel,
   (c) perpendicular.
5. The lines $\langle 1, 3\rangle + r\langle 2, 1\rangle$ and $\langle 1, 3\rangle + r\langle 1, -2\rangle$ are ______.
   (a) equal,
   (b) parallel,
   (c) perpendicular.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. (a) Show geometrically that the distance from (x_o, y_o) to $ax + by = c$ is the same as the distance from the origin to $a(x + x_o) + b(y + y_o) = c$.
   (b) Prove that the distance from (x_o, y_o) to $ax + by = c$ is

$$\frac{|ax_o + by_o - c|}{\sqrt{a^2 + b^2}}.$$

2. Obtain a formula for the distance between the parallel lines $ax + by = c$ and $ax + by = c'$.

2. Lines and Planes in $\mathbf{R}^3$

Much of the geometrical study of lines in the Euclidean plane carries over into the study of lines in 3-dimensional space. A line in space may be described by

$$\overline{OP} + r\overline{OQ},$$

where $\overline{OP}$ is a position vector and $\overline{OQ}$ is a nonzero direction vector. Also $\overline{OP}$ may again be replaced by any $\overline{OP'}$, where P' is on the line, and $\overline{OQ}$ may be replaced by any nonzero scalar multiple of itself without changing the line. The lines

$$\overline{OP} + r\overline{OQ} \quad \text{and} \quad \overline{OP'} + r\overline{OQ}$$

are parallel or equal. If $\overline{OP} + r\overline{OQ}$ and $\overline{OP'} + r\overline{OQ'}$ intersect, then they are perpendicular provided $\overline{OQ} \cdot \overline{OQ'} = 0$. The line,

$$\overline{OP} + r(\overline{OQ} - \overline{OP})$$

contains P and Q. As before, the point R on $\overline{OP} + r\overline{OQ}$, such that $\overline{OR}$ is perpendicular to $\overline{OP} + r\overline{OQ}$, is given by

$$\overline{OR} = \overline{OP} - \frac{\overline{OP} \cdot \overline{OQ}}{|\overline{OQ}|^2} \overline{OQ},$$

assuming $\overline{OP} + r\overline{OQ}$ does not pass through O. Correspondences hold for $\mathbf{R}^3$ by replacing segments by their corresponding tuples.

The formulation of a plane involves two direction vectors. If $\overline{OQ}$ and $\overline{OR}$ are not collinear and P is arbitrary, then

$$\overline{OP} + r\overline{OQ} + s\overline{OR}$$

describes the plane consisting of all points S such that

$$\overline{OS} = \overline{OP} + r\overline{OQ} + s\overline{OR}$$

for some numbers r, s. We call $\overline{OP}$ a position vector and $\overline{OQ}$ and $\overline{OR}$ direction vectors of the plane (see Figure 4.7). The segment $\overline{OP}$ may be replaced by any $\overline{OP'}$, where P' is in the plane, and the same plane results. Also $\overline{OQ}$, $\overline{OR}$ may be replaced by any noncollinear $\overline{OQ'}$, $\overline{OR'}$, lying in the plane containing $\overline{OQ}$ and $\overline{OR}$, and the same plane is again obtained. The planes

$$\overline{OP} + r\overline{OQ} + s\overline{OR} \quad \text{and} \quad \overline{OP'} + r\overline{OQ} + s\overline{OR}$$

are equal or parallel. The usual correspondences with $\mathbf{R}^3$ again exist.

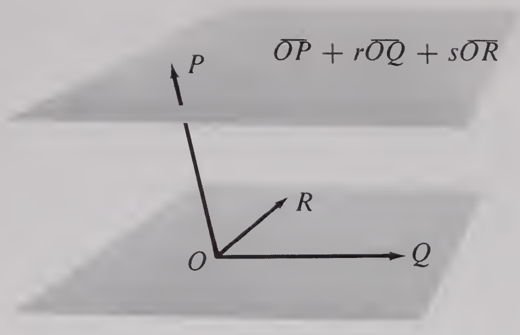


Figure 4.7

Example 2.1 The set $\langle 2, 1, 3\rangle + r\langle 3, 0, 2\rangle + s\langle 2, -1, 5\rangle$ describes a plane in $\mathbf{R}^3$. The plane which is parallel to it and passes through $(3, -1, 4)$ is

$$\langle 3, -1, 4\rangle + r\langle 3, 0, 2\rangle + s\langle 2, -1, 5\rangle.$$

The plane containing the noncollinear points P, Q, R is

$$\overline{OP} + r(\overline{OQ} - \overline{OP}) + s(\overline{OR} - \overline{OP}).$$

This may be seen by a geometric argument (see Figure 4.8), or by successively setting $r = 0$, $s = 0$; $r = 1$, $s = 0$; and $r = 0$, $s = 1$.

Example 2.2 The plane through the points

$$(1, 0, 3), (2, -1, 5), (3, 4, 6)$$

is given by the equation

$$\begin{aligned}&\langle 1, 0, 3\rangle + r(\langle 2, -1, 5\rangle - \langle 1, 0, 3\rangle)\\ &\qquad + s(\langle 3, 4, 6\rangle - \langle 1, 0, 3\rangle)\\ &= \langle 1, 0, 3\rangle + r\langle 1, -1, 2\rangle + s\langle 2, 4, 3\rangle.\end{aligned}$$

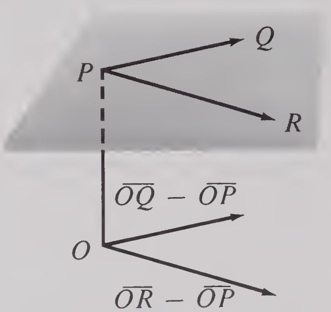


Figure 4.8

A line perpendicular to the plane

$$\overline{OP} + r\overline{OQ} + s\overline{OR}$$

must have its direction vector perpendicular to the direction vectors $\overline{OQ}$, $\overline{OR}$ of the plane. In $\mathbf{R}^3$ such a direction vector is given by the cross-product operation.

Example 2.3 The line through $(1, 4, 3)$ and perpendicular to the plane

$$\langle 2, 1, 5\rangle + r\langle 3, 2, -1\rangle + s\langle 1, 0, 5\rangle$$

is the line

$$\langle 1, 4, 3\rangle + r(\langle 3, 2, -1\rangle \times \langle 1, 0, 5\rangle) = \langle 1, 4, 3\rangle + r\langle 10, -16, -2\rangle.$$

If the point R is not on the line $\overline{OP} + r\overline{OQ}$, then the plane containing R and $\overline{OP} + r\overline{OQ}$ is

$$\overline{OP} + r\overline{OQ} + s(\overline{OR} - \overline{OP}).$$

This is seen by first setting $s = 0$, to show that $\overline{OP} + r\overline{OQ}$ is in the plane, and then setting $r = 0$, $s = 1$ to show that R is in the plane. Successively setting $r = 0$, and $s = 0$ shows that if $\overline{OQ}$ and $\overline{OR}$ are not collinear, then the plane which contains the lines $\overline{OP} + r\overline{OQ}$ and $\overline{OP} + s\overline{OR}$ is given by

$$\overline{OP} + r\overline{OQ} + s\overline{OR}.$$

Example 2.4 (a) The plane which contains the point (1, 4, 0) and the line $\langle 1, 3, 1\rangle + r\langle 2, 6, -3\rangle$, is described by the vector set

$$\langle 1, 3, 1\rangle + r\langle 2, 6, -3\rangle + s(\langle 1, 4, 0\rangle - \langle 1, 3, 1\rangle) = \langle 1, 3, 1\rangle + r\langle 2, 6, -3\rangle + s\langle 0, 1, -1\rangle.$$

(b) The lines

$$\langle 2, 2, 0\rangle + r\langle 3, -1, 4\rangle;\ \langle 2, 2, 0\rangle + s\langle 2, 5, 3\rangle$$

are contained in the plane

$$\langle 2, 2, 0\rangle + r\langle 3, -1, 4\rangle + s\langle 2, 5, 3\rangle.$$

If $R \neq O$, then $\overline{OS}$ is in $\{\overline{OR}\}^\perp$ provided S is in the plane passing through O and perpendicular to $\overline{OR}$. Thus

$$\overline{OR} + \{\overline{OR}\}^\perp$$

describes a plane through R and perpendicular to $\overline{OR}$ (see Figure 4.9). By an argument similar to that for lines in the Euclidean plane, S is in the plane $\overline{OR} + \{\overline{OR}\}^\perp$ if and only if $\overline{OR} \cdot \overline{OS} = |\overline{OR}|^2$. Hence, in $\mathbf{R}^3$, if $\mathbf{w} = \mathbf{OR}$, then

$$\{\mathbf{v}\colon \mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2\}$$

has for its graph the plane through R and perpendicular to $\overline{OR}$.

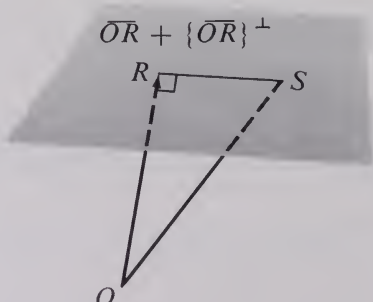


Figure 4.9

Example 2.5 The plane that passes through $R=(3, 2, -5)$ and is perpendicular to $\overline{OR}$ is the set

$$\{\mathbf{v}: \langle 3, 2, -5\rangle \cdot \mathbf{v} = 38\} \qquad (\text{note: } 38 = |\langle 3, 2, -5\rangle|^2).$$

Letting (x, y, z) be an arbitrary point in this plane, from

$$\langle 3, 2, -5\rangle \cdot \langle x, y, z\rangle = 38$$

we get

$$3x + 2y - 5z = 38$$

as an equation of the plane.

We next consider the plane described by the equation

$$ax + by + cz = d.$$

Multiplication of both sides of the equality by

$$\frac{d}{a^2 + b^2 + c^2}$$

results in the equation

$$\mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2,$$

where

$$\mathbf{v} = \langle x, y, z\rangle \quad \text{and} \quad \mathbf{w} = \frac{d}{a^2 + b^2 + c^2}\langle a, b, c\rangle.$$

Therefore, the point R in the plane such that $\overline{OR}$ is perpendicular to the plane satisfies the equation

$$\mathbf{OR} = \frac{d}{a^2 + b^2 + c^2}\langle a, b, c\rangle.$$

The distance from the origin to the plane is

$$|\mathbf{OR}| = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 2.6 Given the plane $x + 2y - 3z = 4$, the point R in the plane such that $\overline{OR}$ is perpendicular to the plane satisfies $\mathbf{OR} = 4/14$ $\langle 1, 2, -3\rangle$. Hence $R = (2/7, 4/7, -6/7)$ and the distance from the origin to the plane is $|\overline{OR}| = 4/\sqrt{14}$.

Questions

1. If $\overline{ST}$ is perpendicular to $\overline{OP} + r\overline{OQ} + s\overline{OR}$, then **ST** is orthogonal to _______ and _______.
2. $\overline{OP} + r\overline{OQ}$ is _______ $\overline{OP} + r\overline{OQ} + s\overline{OR}$.
   (a) equal to,
   (b) perpendicular to,
   (c) contained in.
3. $\langle 1, 0, 1\rangle + r\langle 2, -1, 3\rangle$ and $\langle 1, 0, 1\rangle + r\langle 4, -2, 6\rangle$ are _______.
   (a) equal,
   (b) parallel,
   (c) perpendicular.
4. $\langle 1, -1, 2\rangle + r\langle 1, 0, 3\rangle + s\langle 2, 1, 0\rangle$ and $\langle 3, -1, 4\rangle + r\langle 1, 0, 3\rangle +$ $+ s\langle 2, 1, 0\rangle$ are _______.
   (a) equal,
   (b) parallel,
   (c) perpendicular.

Problems

1. Do Problem Sets B and C at the end of the chapter.

Exercises

1. Find a relationship among a, b, and c so that $\langle a, b, c\rangle + r\langle 2, 1, 5\rangle$ and $\langle 3, 0, 6\rangle + r\langle -2, 7, 1\rangle$ intersect.
2. Given $\mathbf{L} = \langle 1, 0, 1\rangle + r\langle 2, -1, 1\rangle$ and $\mathbf{L}' = \langle 0, 1, 0\rangle + r\langle -1, 0, 1\rangle$.
   (a) Find the line which is perpendicular to both $\mathbf{L}$, $\mathbf{L}'$ and also intersects $\mathbf{L}$, $\mathbf{L}'$.
   (b) Find the points where $\mathbf{L}$ and $\mathbf{L}'$ are nearest.
   (c) Find the distance between $\mathbf{L}$ and $\mathbf{L}'$.
3. Find the plane which contains the intersecting lines

$$\langle 4, 6, -6\rangle + r\langle 1, 0, 3\rangle \quad \text{and} \quad \langle 3, 8, -7\rangle + r\langle 2, -1, 5\rangle.$$

Proofs

1. Show that the distance from (x_0, y_0, z_0) to $ax + by + cz = d$ is

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

2. Prove that the cosine of the angle between the intersecting planes

$$ax + + by\,cz = d \quad \text{and} \quad a'x + b'y + c'z = d'$$

is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|},$$

where $\mathbf{u} = \langle a, b, c\rangle$ and $\mathbf{v} = \langle a', b', c'\rangle$. (*Hint*: The angle of intersection is also the angle between intersecting lines perpendicular to the planes.)

3. Obtain a formula for the distance between the parallel planes

$$ax + by + cz = d \quad \text{and} \quad ax + by + cz = d'.$$

3. *Parallelograms, Parallelepipeds, Triangles, and Tetrahedrons*

In this section we shall consider sets in $\mathbf{R}^2$ and $\mathbf{R}^3$ that have bounded graphs and can be described by our vector-space structure. We again give first attention to the Euclidean plane and space, and, this time, it will be convenient to use geometric vectors. In the plane or in space the symbol

$$\overline{OP} + r\overline{PQ},$$

where r takes on all real-number values, will be used to describe the line through P in the direction of $\overline{PQ}$. If the values of r are restricted to the interval [0, 1], the result is a subset, the segment of this line having end points P and Q (see Figure 4.10). This segment will be represented by the expression

$$\overline{OP} + [r\overline{PQ}].$$

In the plane or in space the symbol

$$\overline{OP} + r\overline{PQ} + s\overline{QR}$$

describes the plane through P having $\overline{PQ}$ and $\overline{QR}$ as direction vectors. (If P, Q, and R are in a Euclidean plane, then $\overline{OP} + r\overline{PQ} + s\overline{QR}$ is the entire plane).

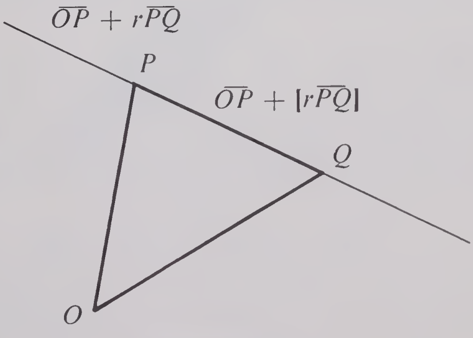


Figure 4.10

If the values of r and s are each restricted to the interval [0, 1], the result is a parallelogram having sides $\overline{PQ}$ and $\overline{QR}$. This parallelogram will be designated as

$$\overline{OP} + [r\overline{PQ} + s\overline{QR}].$$

Similarly, in space, the symbol

$$\overline{OP} + [r\overline{PQ} + s\overline{QR} + t\overline{RS}]$$

describes the set consisting of all the terminal points of segments having the form

$$\overline{OP} + r\overline{PQ} + s\overline{QR} + t\overline{RS},$$

where r, s and t are each in the interval [0, 1]. In this case, the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ is obtained. Correspondences in $\mathbf{R}^2$ and $\mathbf{R}^3$ are again found by replacing segments by corresponding tuples.

> **Example 3.1** The graphs of (a) $\langle 2, 1\rangle + [r\langle 3, -2\rangle]$, (b) $\langle 3, 0\rangle + [r\langle 1, 1\rangle + s\langle 3, -2\rangle]$, and (c) $\langle -2, 0, 0\rangle + [r\langle 2, 0, 4\rangle + s\langle 0, 3, -1\rangle]$ are shown in Figure 4.11.

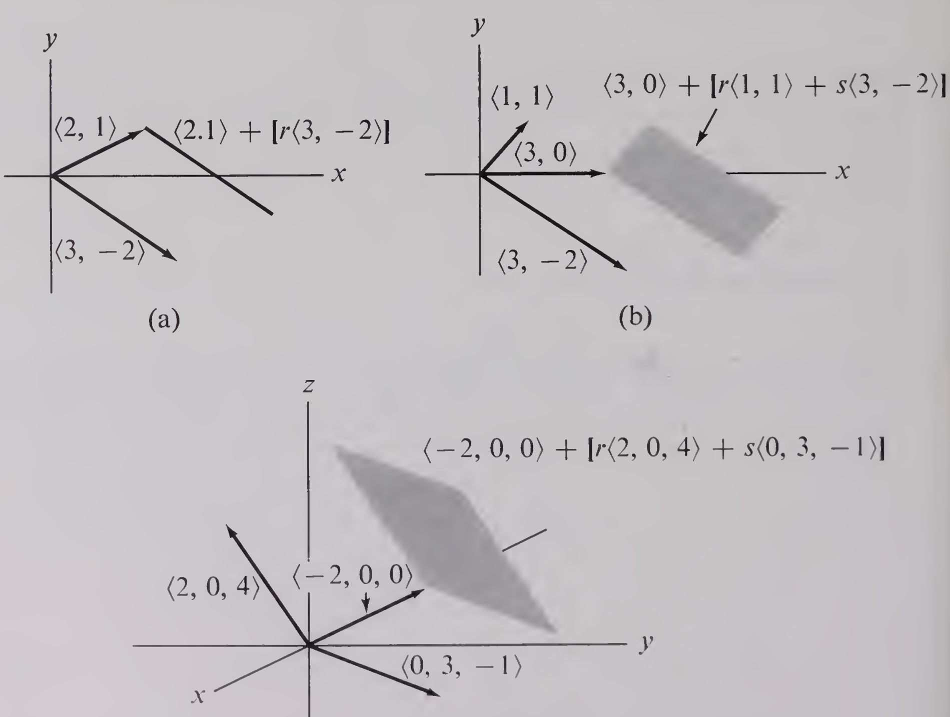


Figure 4.11

In the Euclidean plane or space the triangle with vertices P, Q, and R may also be described using vector operations. If

$$\overline{OQ'} = \overline{OP} + r_o\overline{PQ} \quad \text{and} \quad \overline{OR'} = \overline{OP} + r_o\overline{PR},$$

where $0 < r_o < 1$, then Q' and R' lie on the perimeter of this triangle (see Figure 4.12). An arbitrary point on the line segement with end points Q' and R' is given by the equation

$$\overline{OQ'} + s_0\,\overline{Q'R'} = \overline{OQ'} + s_0(\overline{OR'} - \overline{OQ'}),$$

where $0 \leq s_0 \leq 1$. Substitution for $\overline{OQ'}$ and $\overline{OR'}$ gives, after simplification,

$$\overline{OQ'} + s_o\,\overline{Q'R'} = \overline{OP} + r_o\overline{PQ} + r_o s_o\,\overline{QR}.$$

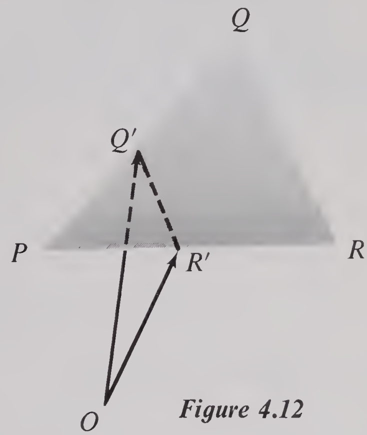


Figure 4.12

We may conclude that the triangle with vertices P, Q, and R consists of all terminal points of vectors of the form $\overline{OP} + r\overline{PQ} + rs\overline{QR}$, where r, s are in the interval [0, 1]. This triangle will hereafter be denoted as

$$\overline{OP} + [r\overline{PQ} + rs\overline{QR}].$$

The final set to be considered here is the tetrahedron with vertices P, Q, R, and S. Letting

$$\overline{OR'} = \overline{OP} + r_o\overline{PQ} + r_o s_o\,\overline{QR} \quad \text{and} \quad \overline{OS'} = \overline{OP} + r_o\overline{PQ} + r_o s_o\,\overline{QS},$$

where $0 < r_o, s_o < 1$, the points R' and S' are in the triangles with vertices P, Q, R and P, Q, S, respectively (see Figure 4.13). If Q' is on the segment with end points R', S', then

$$\overline{OQ'} = \overline{OR'} + t_o(\overline{OS'} - \overline{OR'})$$

for some t_o in [0, 1]. Substitution for $\overline{OR'}$ and $\overline{OS'}$ gives, after simplification,

$$\overline{OQ'} = \overline{OP} + r_o\overline{PQ} + r_o s_o\,\overline{QR} + r_o s_o t_o\,\overline{RS}.$$

Consequently, the tetrahedron with vertices P, Q, R, S is described by

$$\overline{OP} + [r\overline{PQ} + rs\,\overline{QR} + rst\overline{RS}],$$

using the previous interpretation of the bracket symbols. The formulations for triangles and tetrahedrons have the usual correspondences in $\mathbf{R}^2$, $\mathbf{R}^3$.

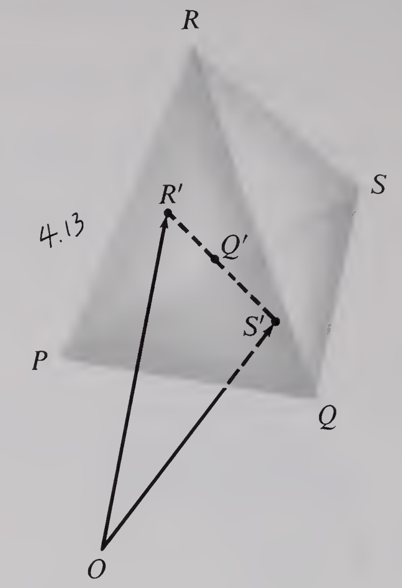


Figure 4.13

Example 3.2

(a) The triangle with vertices (1, 0, 3), (2, 1, 5), (6, 3, 0) is the graph of $\langle 1,0,3\rangle + [r\langle 2-1, 1-0, 5-3\rangle + rs\langle 6-2, 3-1, 0-5\rangle] = \langle 1, 0, 3\rangle + [r\langle 1, 1, 2\rangle + rs\langle 4, 2, -5\rangle]$.

(b) The tetrahedron with vertices (1, 0, 3), (2, 1, 5), (6, 3, 0), (7, 2, −1) is the graph of $\langle 1, 0, 3\rangle + [r\langle 1, 1, 2\rangle + rs\langle 4, 2, -5\rangle + rst\langle 1, -1, -1\rangle]$.

There is associated with each of the sets in this section a length, area, or volume. The length of $\overline{OP} + [r\overline{PQ}]$ is clearly $|\overline{PQ}|$. The area of the parallelogram $\overline{OP} + [r\overline{PQ} + s\overline{QR}]$ in space is $|\mathbf{PQ} \times \mathbf{QR}|$ from the previous study of the cross product. Since a triangle forms half a parallelogram, it is evident that the area of the triangle $\overline{OP} + [r\overline{PQ} + rs\overline{QR}]$ in space is $\frac{1}{2}|\mathbf{PQ} \times \mathbf{QR}|$. We next seek the volume of the parallelepiped $\overline{OP} + [r\overline{PQ} + s\overline{QR} + t\overline{RS}]$. By geometric considerations this volume is the product of the area of a base and an altitude.

Using $\overline{OP} + [r\overline{PQ} + s\overline{QR}]$ as a base, the desired volume is

$$|\mathbf{PQ} \times \mathbf{QR}|(|\mathbf{RS}|\,|\cos\theta|),$$

where θ is the angle between $\overline{RS}$ and a line through S perpendicular to the plane of the base (see Figure 4.14). Since the corresponding segment of $\mathbf{PQ} \times \mathbf{QR}$ has the direction of this line, it follows from the definition of dot product of segments that the volume may be written

$$|(\mathbf{PQ} \times \mathbf{QR}) \cdot \mathbf{RS}|.$$

4.14

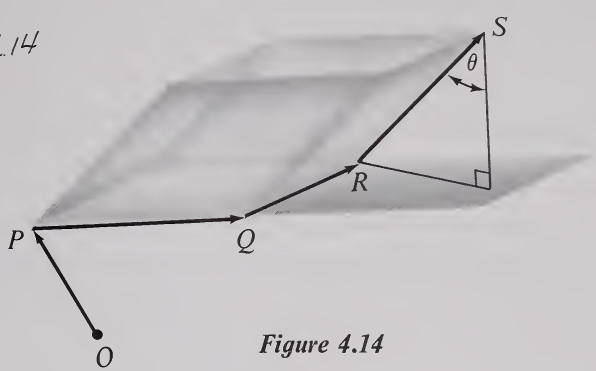


Figure 4.14

The volume formula for the tetrahedron

$$\overline{OP} + [r\overline{PQ} + rs\overline{QR} + rst\overline{RS}]$$

may be obtained from that of a parallelepiped by using a complicated geometric argument, or by techniques of integration; it is given by

$$\tfrac{1}{6}|(\mathbf{PQ} \times \mathbf{QR}) \cdot \mathbf{RS}|.$$

Example 3.3 Let $P = (1, 0, 3)$, $Q = (2, 1, 1)$, $R = (3, 5, 4)$, and $S = (6, 1, 0)$. Then,

$$\mathbf{PQ} \times \mathbf{QR} = \langle 1, 1, -2\rangle \times \langle 1, 4, 3\rangle = \langle 11, -5, 3\rangle$$

and

$$(\mathbf{PQ} \times \mathbf{QR}) \cdot \mathbf{RS} = \langle 11, -5, 3\rangle \cdot \langle 3, -4, -4\rangle = 41.$$

Hence, the area of the triangle with vertices P, Q, R is

$$\tfrac{1}{2}|\langle 11, -5, 3\rangle| = \tfrac{1}{2}\sqrt{155}.$$

The volume of the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ is 41, and the volume of the tetrahedron with vertices P, Q, R, S is 41/6.

Questions

1. The line segment with end points P and Q consists of the terminal points of $\overline{OP} + r\overline{PQ}$, where r lies between ______ and ______.
2. The three-dimensional analogue of a triangle is a ______.
3. The three-dimensional analogue of a parallelogram is a ______.
4. $\langle 1, 0\rangle + [r\langle 2, 3\rangle + rs\langle 3, -2\rangle]$ is a ______.
   (a) parallelogram,
   (b) line segment,
   (c) triangle.

Problems

1. Do Problem Set D at the end of the chapter.

Proofs

1. A set of vectors is *convex* if, for each pair of vectors $\mathbf{u}$, $\mathbf{v}$ in the set, each vector of the form $\mathbf{u} + r(\mathbf{v} - \mathbf{u})$, $0 \leq r \leq 1$, is also in the set. Prove that (a) $\mathbf{u}_0 + [r\mathbf{u}_1 + s\mathbf{u}_2]$ and (b) $\mathbf{u}_0 + [r\mathbf{u}_1 + rs\mathbf{u}_2]$ are convex.
2. (a) Show that $\overline{OP} + 1/2\ \overline{PQ}$ is the midpoint of $\overline{OP} + r[\overline{PQ}.]$
   (b) Show that the diagonal segments of $\overline{OP} + [r\overline{PQ} + s\overline{QR}]$ intersect at their midpoints. Use the corresponding set of 2-tuples for the proof.

4. n-Planes

In this section we shall extend the analysis of the previous sections to arbitrary Euclidean vector spaces. We shall show that the geometric derivations of previous sections can be extended using only the algebraic structure of an abstract, finite-dimensional, inner-product vector space.

Geometric lines or planes can be obtained from parallel lines or planes through the origin by a shifting operation. This shifting can be achieved algebraically by adding a suitable fixed vector to each vector in the line or plane through the origin. Lines or planes through the origin correspond to subspaces. This suggests the following definition.

Definition of n-Plane

An n-plane *in a Euclidean vector space is a set of the form* $\mathbf{P} = \mathbf{u} + \mathrm{S}$, *where* S *is an n-dimensional subspace and* $\mathbf{u} + \mathrm{S}$ *is the set obtained by adding* $\mathbf{u}$ *to each vector in* S.

In this definition $\mathbf{u}$ can be replaced by any vector $\mathbf{v}$ in $\mathbf{P}$, whereas $\mathbf{S}$ is unique in the expression for $\mathbf{P}$. This will now be stated formally.

Proposition 4.1

(a) If $\mathbf{P} = \mathbf{u} + \mathbf{S}$ and $\mathbf{v}$ is in $\mathbf{P}$, then $\mathbf{P} = \mathbf{v} + \mathbf{S}$.
(b) If $\mathbf{P} = \mathbf{u}_1 + \mathbf{S}_1 = \mathbf{u}_2 + \mathbf{S}_2$, then $\mathbf{S}_1 = \mathbf{S}_2$.

For the proof of (a) it will first be shown that an arbitrary vector $\mathbf{w}$ in $\mathbf{P}$ is also in $\mathbf{v} + \mathbf{S}$. By our assumptions, there exist vectors $\mathbf{v}_1$, $\mathbf{v}_2$ in $\mathbf{S}$ such that

$$\mathbf{w} = \mathbf{u} + \mathbf{v}_1, \; \mathbf{v} = \mathbf{u} + \mathbf{v}_2 .$$

Since $\mathbf{S}$ is a subspace, it may be deduced that $\mathbf{v}_1 - \mathbf{v}_2$ is in $\mathbf{S}$, and hence,

$$\mathbf{w} = \mathbf{v} + (\mathbf{v}_1 - \mathbf{v}_2)$$

is in $\mathbf{v} + \mathbf{S}$. This proves that $\mathbf{P}$ is a subset of $\mathbf{v} + \mathbf{S}$. To show the reverse inclusion we let $\mathbf{w}$ (arbitrary) be in $\mathbf{v} + \mathbf{S}$. Then there exist $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbf{S}$ such that

$$\mathbf{w} = \mathbf{v} + \mathbf{v}_1 \quad \text{and} \quad \mathbf{v} = \mathbf{u} + \mathbf{v}_2 .$$

Hence,

$$\mathbf{w} = \mathbf{u} + (\mathbf{v}_1 + \mathbf{v}_2)$$

is in $\mathbf{u} + \mathbf{S} = \mathbf{P}$. This shows that $\mathbf{v} + \mathbf{S}$ is a subset of $\mathbf{P}$ and concludes the proof of (a). For (b) see Proofs, exercise 1.

We next extend to n dimensions the property that every line or plane contains a point R where $\overline{OR}$ is perpendicular to the line or plane. The minimal length property of $\overline{OR}$ is also generalized.

Proposition 4.2 If $\mathbf{P} = \mathbf{u} + \mathbf{S}$ is an n-plane in a Euclidean vector space, then
(a) there is a unique $\mathbf{w}$ belonging to both $\mathbf{P}$ and $\mathbf{S}^\perp$,
(b) the vector $\mathbf{w}$ in (a) has the least norm of any vector in $\mathbf{P}$.

A proof of (a) will be made for the case $n = 3$. In this case $\mathbf{V}$ has an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, and it will be shown that the desired vector is

$$\mathbf{w} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{u} \cdot \mathbf{u}_2)\mathbf{u}_2 - (\mathbf{u} \cdot \mathbf{u}_3)\mathbf{u}_3 .$$

It is evident that $\mathbf{w}$ is in $\mathbf{P} = \mathbf{u} + \mathbf{S}$, since $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are in $\mathbf{S}$ and $\mathbf{S}$ is a subspace. To show that $\mathbf{w}$ is in $\mathbf{S}^\perp$ we first observe

$$\begin{aligned}\mathbf{w} \cdot \mathbf{u}_1 &= \mathbf{u} \cdot \mathbf{u}_1 - (\mathbf{u} \cdot \mathbf{u}_1)(\mathbf{u}_1 \cdot \mathbf{u}_1) - (\mathbf{u} \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{u}_1) - (\mathbf{u} \cdot \mathbf{u}_3)(\mathbf{u}_3 \cdot \mathbf{u}_1)\\ &= \mathbf{u} \cdot \mathbf{u}_1 - \mathbf{u} \cdot \mathbf{u}_1 - 0 - 0\\ &= 0\end{aligned}$$

by the orthonormal property of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. In a similar manner,

$$\mathbf{w} \cdot \mathbf{u}_2 = \mathbf{w} \cdot \mathbf{u}_3 = 0,$$

and, since every vector in $\mathbf{S}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, it easily follows that $\mathbf{w}$ is in $\mathbf{S}^\perp$. To prove that $\mathbf{w}$ is unique, suppose that another vector $\mathbf{w}_1$ is also in both $\mathbf{P}$ and $\mathbf{S}^\perp$. Then $\mathbf{w}_1 - \mathbf{w}$ is in both $\mathbf{S}$, $\mathbf{S}^\perp$ since $\mathbf{S}$, $\mathbf{S}^\perp$ are subspaces. Therefore $(\mathbf{w}_1 - \mathbf{w}) \cdot (\mathbf{w}_1 - \mathbf{w}) = 0$ and $\mathbf{w}_1 = \mathbf{w}$ by the positive-definite property of inner products. This concludes the proof of (a); for (b) see Proofs, exercise 2.

Example 4.1 Let $\mathbf{P} = \langle 1, 0, 3, 2\rangle + \text{Sp}\,\{\langle 2, 1, 0, 2\rangle, \langle -1, 2, 2, 0\rangle\}$. An orthonormal basis of $\text{Sp}\{\langle 2, 1, 0, 2\rangle, \langle -1, 2, 2, 0\rangle\}$ is

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\left\langle \frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}\right\rangle, \left\langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0\right\rangle\right\}.$$

Referring to the proof of Proposition 4.2 and letting $\mathbf{u} = \langle 1, 0, 3, 2\rangle$, we see the vector in $\mathbf{P}$ having the least norm is

$$\mathbf{w} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{u} \cdot \mathbf{u}_2)\mathbf{u}_2 = \tfrac{1}{9}\langle 2, -16, 17, 6\rangle.$$

The next proposition generalizes the property that $\overline{OP} + r(\overline{OQ} - \overline{OP})$ is the unique line through P, Q, and $\overline{OP} + r(\overline{OQ} - \overline{OP}) + s(\overline{OR} - \overline{OP})$ is the unique plane through noncollinear points P, Q, R.

Proposition 4.3 If $\mathbf{u}_o, \mathbf{u}_1, \ldots, \mathbf{u}_n$ are in $\mathbf{V}$ and $\{\mathbf{u}_1 - \mathbf{u}_o, \ldots, \mathbf{u}_n - \mathbf{u}_o\}$ is independent, then

$$\mathbf{P} = \mathbf{u}_o + \text{Sp}\{\mathbf{u}_1 - \mathbf{u}_o, \ldots, \mathbf{u}_n - \mathbf{u}_o\}$$

is the unique n-plane containing $\mathbf{u}_o, \mathbf{u}_1, \ldots, \mathbf{u}_n$.

A proof will be made for $n = 2$. From

$$\mathbf{u}_o = 1\mathbf{u}_o, \quad \mathbf{u}_1 = 1\mathbf{u}_o + 1(\mathbf{u}_1 - \mathbf{u}_o), \quad \text{and} \quad \mathbf{u}_2 = 1\mathbf{u}_o + 1(\mathbf{u}_2 - \mathbf{u}_o)$$

it is seen that $\mathbf{P}$ contains $\mathbf{u}_o, \mathbf{u}_1, \mathbf{u}_2$. It follows from the assumed independence condition that $\mathbf{P}$ is a 2-plane. If $\mathbf{P}'$ is also a 2-plane containing $\mathbf{u}_o, \mathbf{u}_1, \mathbf{u}_2$, then $\mathbf{P}' = \mathbf{u}_o + \mathbf{S}'$, where $\mathbf{S}'$ is a 2-dimensional subspace. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are in $\mathbf{P}'$, we see that $\mathbf{u}_1 - \mathbf{u}_o$, $\mathbf{u}_2 - \mathbf{u}_o$ are in $\mathbf{S}'$ and hence form a basis of $\mathbf{S}'$. Therefore $\mathbf{P}' = \mathbf{u}_o + \mathbf{S}' = \mathbf{P}$.

Example 4.2 The unique 2-plane containing

$$\langle 1, 0, 2, 3\rangle, \langle 4, -5, 1, 0\rangle, \langle 7, 2, -3, 6\rangle$$

is

$\langle 1, 0, 2, 3\rangle + \text{Sp}\,\{\langle 4, -5, 1, 0\rangle - \langle 1, 0, 2, 3\rangle, \langle 7, 2, -3, 6\rangle - \langle 1, 0, 2, 3\rangle\} = \langle 1, 0, 2, 3\rangle + \text{Sp}\,\{\langle 3, -5, -1, -3\rangle, \langle 6, 2, -5, 3\rangle\}$.

A *line* in $\mathbf{V}$ is, by definition, a 1-plane; a *hyperplane* is an $(n-1)$-plane where $n = \dim \mathbf{V}$. A line has the form $\mathbf{L} = \mathbf{u} + \mathrm{Sp}\,\{\mathbf{v}\}$. From Proposition 4.3 the unique line containing $\mathbf{u}_0$ and $\mathbf{u}_1$ is $\mathbf{u}_0 + \mathrm{Sp}\,\{\mathbf{u}_1 - \mathbf{u}_0\}$. The graph of a line in $\mathbf{R}^2$ and $\mathbf{R}^3$ is a (geometric) line. The graph of a hyperplane in $\mathbf{R}^2$ is a line and in $\mathbf{R}^3$ is a plane. We have already seen that a line in the Euclidean plane and a plane in Euclidean space can be written in the form $\overline{OR} + \{\overline{OR}\}^\perp$, provided it does not pass through O. This will now be generalized.

Proposition 4.4 $\mathbf{P}$ is a hyperplane which does not contain $\mathbf{0}$ if and only if there exists a nonzero $\mathbf{w}$ in $\mathbf{P}$ such that $\mathbf{P} = \mathbf{w} + \{\mathbf{w}\}^\perp$.

For a given hyperplane $\mathbf{P} = \mathbf{u} + \mathrm{Sp}\,\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$, where $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$ is orthonormal, the vector $\mathbf{w}$ such that $\mathbf{P} = \mathbf{w} + \{\mathbf{w}\}^\perp$ can be obtained using the obvious generalization,

$$\mathbf{w} = \mathbf{u} - \sum_{i=1}^{n-1} (\mathbf{u} \cdot \mathbf{u}_i)\mathbf{u}_i,$$

of the formula in the proof of Proposition 4.2.

Example 4.3 Let

$$\mathbf{u} = \langle 1, 1, 0, 2\rangle,$$
$$\mathbf{u}_1 = \tfrac{1}{5}\langle 3, 0, 4, 0\rangle,\ \mathbf{u}_2 = \tfrac{1}{5}\langle -4, 0, 3, 0\rangle,$$
$$\mathbf{u}_3 = \langle 0, 0, 0, 1\rangle,$$

and

$$\mathbf{P} = \mathbf{u}_0 + \mathrm{Sp}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}.$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal and from

$$\mathbf{w} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{u} \cdot \mathbf{u}_2)\mathbf{u}_2 - (\mathbf{u} \cdot \mathbf{u}_3)\mathbf{u}_3 = \langle 0, 1, 0, 0\rangle,$$

we have $\mathbf{P} = \langle 0, 1, 0, 0\rangle + \{\langle 0, 1, 0, 0\rangle\}^\perp$.

We next obtain a general property which includes as special cases the previous observations that a line in the Cartesian plane can be described by an equation of the form $ax + by = c$, and a plane in Cartesian space can be described by an equation $ax + by + cz = d$.

Proposition 4.5 $\mathbf{P}$ is a hyperplane in $\mathbf{V}$ if and only if there exists a nonzero vector $\mathbf{u}$ in $\mathbf{V}$ and a real number c such that $\mathbf{P} = \{\mathbf{v} : \mathbf{u} \cdot \mathbf{v} = c\}$.

The formulas for the perpendicular vectors

$$\frac{c}{\sqrt{a^2+b^2}}\langle a, b\rangle \quad \text{and} \quad \frac{d}{\sqrt{a^2+b^2+c^2}}\langle a, b, c\rangle$$

and their associated lengths are generalized in the following proposition.

Proposition 4.6 In Proposition 4.5, if **P** does not contain **O**, then the vector in **P** of least norm is

$$\frac{c}{|\mathbf{u}|^2}\mathbf{u}$$

and its norm is

$$\frac{|c|}{|\mathbf{u}|}.$$

Example 4.4 The equation $3w+2x-y+3z=8$ describes the hyperplane

$$\mathbf{P}=\{\langle w, x, y, z\rangle\colon \langle 3, 2, -1, 3\rangle \cdot \langle w, x, y, z,\rangle=8\}$$

in $\mathbf{R}^4$. The vector in **P** of least norm is

$$\frac{8}{|\langle 3, 2, -1, 3\rangle|^2}\langle 3, 2, -1, 3\rangle=\tfrac{8}{23}\langle 3, 2, -1, 3\rangle,$$

and its norm is $8/\sqrt{23}$.

There is also an extension of the parallelogram, parallelepiped, triangle, and tetrahedron sets. An *n-parallelotope* in **V** is a set of the form

$$\{\mathbf{u}_o+r_1\mathbf{u}_1+r_2\mathbf{u}_2+\cdots+r_n\mathbf{u}_n\colon 0\le r_1, r_2, \ldots, r_n\le 1\},$$

where $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an independent set in **V**. Thus the graph of a 1-parallelotope in $\mathbf{R}^2$ and $\mathbf{R}^3$ is a line segment, the graph of a 2-parallelotope is a parallelogram, and the graph of a 3-parallelotope in $\mathbf{R}^3$ is a parallelepiped. The graph of a 0-parallelotope is a point.

An *n-simplex* in **V** is a set of the form

$$\{\mathbf{u}_o+r_1\mathbf{u}_1+r_1r_2\mathbf{u}_2+\cdots+r_1r_2\cdots r_n\mathbf{u}_n\colon 0\le r_1, r_2, \ldots, r_n\le 1\},$$

where again $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is independent. Graphs of simplices of dimension 0, 1, 2, or 3 are points, line segments, triangles, and tetrahedrons, respectively.

Our formulas for the size of the parallelotope graphs in the Cartesian plane and space have in most cases involved the cross-product operation, which has no extension to $\mathbf{R}^n$. Therefore, these formulas have no evident generalization to $\mathbf{R}^n$. General formulas will be obtained later, however, using the concept of absolute determinant.

Questions

1. A translation of an n-dimensional subspace is called a(n) ______.
2. $\mathbf{P} = \mathbf{u}_o + \mathrm{Sp}\{\mathbf{u}_1, \mathbf{u}_2\}$ is a 2-plane provided ______.
3. A single vector constitutes an n-plane for $n =$ ______.
4. A 1-plane is also called a ______.
5. An n-plane in R^5 is a hyperplane, provided $n =$ ______.
6. Parallelograms and parallelepipeds are examples of an n- ______.
7. Triangles and tetrahedrons are examples of an n- ______.

Exercises

1. Determine an n for which the following is an n-plane:
   (a) $\overline{OP} + \mathrm{Sp}\{\overline{OO}\}$,
   (b) $\langle 1, 2\rangle + \mathrm{Sp}\{\langle 3, 4\rangle\}$,
   (c) $\langle 1, 0, 1\rangle + \mathrm{Sp}\{\langle 1, 1, 1\rangle, \langle 2, 0, 2\rangle\}$,
   (d) $\langle 2, 1, 0\rangle + \mathrm{Sp}\{\langle 2, 1, 0\rangle, \langle 1, -1, 2\rangle\}$,
   (e) $\langle 3, 1, 6, 2\rangle + \mathrm{Sp}\{\langle 1, 0, 2, 3\rangle, \langle 2, -1, 1, 4\rangle, \langle 3, -1, 3, 7\rangle, \langle 4, -1, 5, 10\rangle\}$.
2. Find the vector of least norm in each of the following sets.
   (a) $\langle 1, 1, 0, 1\rangle + \mathrm{Sp}\{\langle 2, 1, 5, 3\rangle\}$,
   (b) $\langle 1, 0, 3, 7\rangle + \mathrm{Sp}\{\langle 1, 0, 0, 1\rangle, \langle 0, 1, 1, 0\rangle\}$,
   (c) $\langle 1, 0, 2, 6\rangle + \mathrm{Sp}\{\langle 1, 0, 0, 0\rangle, \langle 0, 1, 0, 0\rangle, \langle 0, 0, 1, 1\rangle\}$.
3. Find the unique 2-plane containing
$$\langle 1, 1, 0, 2\rangle, \quad \langle 3, 0, 1, 4\rangle, \quad \text{and} \quad \langle 2, -1, 3, 2\rangle.$$
4. Write the hyperplane $w - x + y + z = 4$ in $\mathbf{R}^4$ in the forms
   (a) $\{\mathbf{v} : \mathbf{u} \cdot \mathbf{v} = c\}$, (b) $\mathbf{w} + \{\mathbf{w}\}^\perp$.
5. Write, in the form $aw + bx + cy + dz = e$, the hyperplane
$$\langle 1, 0, 2, 3\rangle + \mathrm{Sp}\{\langle 1, 0, 0, 1\rangle, \langle 0, -1, 0, 0\rangle, \langle 0, 0, 1, 0\rangle\}$$
in $\mathbf{R}^4$.

Proofs

1. Prove that if $\mathbf{u}_1 + \mathrm{S}_1 = \mathbf{u}_2 + \mathrm{S}_2$, where S_1 and S_2 are subspaces, then $\mathrm{S}_1 = \mathrm{S}_2$. (*Hint*: First show $\mathbf{u}_2 + \mathrm{S}_2 = \mathbf{u}_1 + \mathrm{S}_2$.)
2. Prove that if $\mathbf{w} \neq \mathbf{u}$ and $\mathbf{w}$ is in both $\mathbf{P} = \mathbf{u} + \mathrm{S}$ and $\mathrm{S}^\perp$, then $|\mathbf{w}| < |\mathbf{u}|$. (*Hint*: Let $\mathbf{u} = \mathbf{w} + \mathbf{v}$, where $\mathbf{v}$ is in S, and show $\mathbf{u} \cdot \mathbf{u} - \mathbf{w} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} > 0$.)
3. Prove that if $\mathbf{w} \neq \mathbf{0}$, then $\mathbf{w} + \{\mathbf{w}\}^\perp = \{\mathbf{v} : \mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2\}$. (*Hint*: For each $\mathbf{v}$ there exists a number a and a vector $\mathbf{u}$ in $[\mathrm{Sp}\{\mathbf{w}\}]^\perp = \{\mathbf{w}\}^\perp$ such that $\mathbf{v} = a\mathbf{w} + \mathbf{u}$. Show that if $\mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|^2$, then $a = 1$.)

4. A suitable definition for the distance from $\mathbf{u}_o$ to $\mathbf{P} = \mathbf{u} + \mathbf{S}$ is the least norm of vectors in $\mathbf{P}' = (\mathbf{u} - \mathbf{u}_o) + \mathbf{S}$.
   (a) Show this geometrically in the Cartesian plane.
   (b) Prove the distance from $\mathbf{u}_o$ to $\mathbf{w} + \{\mathbf{w}\}^\perp$ is

$$\frac{|\mathbf{w} \cdot (\mathbf{w} - \mathbf{u}_o)|}{|\mathbf{w}|}.$$

$\left(\textit{Hint}: \text{Show } (\mathbf{w} - \mathbf{u}_o) + \{\mathbf{w}\}^\perp = \mathbf{w}' + \{\mathbf{w}'\}^\perp, \text{ where } \mathbf{w}' = \dfrac{\mathbf{w} \cdot (\mathbf{w} - \mathbf{u}_o)}{|\mathbf{w}|^2}\,\mathbf{w}.\right)$

   (c) Prove that the distance from $\mathbf{u}_o$ to $\{\mathbf{v} : \mathbf{u} \cdot \mathbf{v} = c\}$ is

$$\frac{|\mathbf{u} \cdot \mathbf{u}_o - c|}{|\mathbf{u}|}.$$

(*Hint*: $\{\mathbf{v} : \mathbf{u} \cdot \mathbf{v} = c\} = \mathbf{w} + \{\mathbf{w}\}^\perp$, where $\mathbf{w} = \dfrac{c}{|\mathbf{u}|^2}\,\mathbf{u}$. Apply the conclusion of (b).)
   (d) From (c) obtain a formula for the distance from $\langle b_1, b_2, \ldots, b_n \rangle$ to the hyperplane $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$.

Problems

A. *Lines in the Cartesian Plane*

In the Cartesian plane let O denote the origin and P, Q arbitrary distinct points, $Q \neq O$. The line through P in the direction of $\overline{OQ}$ is described by the *vector equation*

A.1 $\quad x\mathbf{i} + y\mathbf{j} = \mathbf{OP} + r\mathbf{OQ}$.

Letting $\mathbf{OP} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{OQ} = b_1\mathbf{i} + b_2\mathbf{j}$, we may write A.1 as

$$x\mathbf{i} + y\mathbf{j} = (a_1\mathbf{i} + a_2\mathbf{j}) + r(b_1\mathbf{i} + b_2\mathbf{j}).$$

Equating the coefficients of **i** and **j** gives the *parametric equations*

A.2 $\quad x = a_1 + rb_1,$
$\quad\quad y = a_2 + rb_2.$

The equations in A.2 can be written as a single matrix equation,

A.3 $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + r\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$

It is understood that the product of a number and a matrix is obtained by multiplying each entry of the matrix by the number. Since $Q \neq O$, either $b_1 \neq 0$ or $b_2 \neq 0$. If $b_1 \neq 0$, then

$$r = \frac{x - a_1}{b_1}$$

in A.2; and letting

$$m = \frac{b_2}{b_1} \quad \text{and} \quad b = \frac{a_2 b_1 - a_1 b_2}{b_1}$$

we obtain, by substitution in the expression for y, the *slope intercept equation*

A.4 $\quad y = mx + b.$

By elementary algebra, A.4 may be converted to a *standard equation*,

A.5 $\quad ax + by = c.$

A standard form can also be found when $b_1 = 0$ by first letting $r = (y - a_2)/b_2$ and then substituting this in the expression for x.

1. Given the line $x\mathbf{i} + y\mathbf{j} = (2\mathbf{i} + 3\mathbf{j}) + r(\mathbf{i} - \mathbf{j})$, find
   (a) the parametric equations,
   (b) a matrix equation,
   (c) the slope intercept form of the line, and
   (d) a standard form of that line.

2. Given the line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + r\begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

   find
   (a) the parametric equations,
   (b) a vector equation.

The various equation forms of a given line are not unique, with the exception of the slope-intercept form. In vector equation form, P can be replaced by any point on the line and **OQ** by any nonzero scalar multiple of itself. The standard equation is unique except for a multiplicative constant of the equation.

3. Given the line with standard equation $3x - y = 6$, find
   (a) the slope-intercept equation,
   (b) the parametric equations using $x = 0 + 1r$ as one equation,
   (c) a matrix equation from (b),
   (d) a vector equation from (b).

For simplicity, the line $x\mathbf{i} + y\mathbf{j} = \mathbf{OP} + r\mathbf{OQ}$ will often be written $\mathbf{OP} + r\mathbf{OQ}$. Some fundamental properties using this notation are enumerated below.

A.6 (a) $\mathbf{OP} + r\mathbf{OQ}$ and $\mathbf{OP}' + r\mathbf{OQ}$ are equal or parallel.
(b) If $\mathbf{OQ} \cdot \mathbf{OQ}' = 0$, then $\mathbf{OP} + r\mathbf{OQ}$ and $\mathbf{OP}' + r\mathbf{OQ}'$ are perpendicular.
(c) the line through P and Q is $\mathbf{OP} + r(\mathbf{OQ} - \mathbf{OP})$.

4. Find a vector equation of the line passing through
   (a) (3, 5) and parallel to $(\mathbf{i} + 4\mathbf{j}) + r(2\mathbf{i} - \mathbf{j})$,
   (b) (2, 7) and parallel to $(7\mathbf{i} + 3\mathbf{j}) + r(4\mathbf{i} + 5\mathbf{j})$,
   (c) $(-1, 4)$ and perpendicular to $(2\mathbf{i} + 3\mathbf{j}) + r(\mathbf{i} - \mathbf{j})$,
   (d) $(4, -6)$ and perpendicular to $(2\mathbf{i} + 3\mathbf{j}) + r(\mathbf{i} - \mathbf{j})$.
5. Find the line through the given pairs of points.
   (a) $(2, 7)$ and $(4, -3)$, (b) $(5, 1)$ and $(6, 0)$.

A standard equation of the line through P and Q may be found by substituting the coordinates of P, Q for x, y in the equation $ax + by = c$ and solving the resulting two equations for a, b, c (there are infinitely many answers).

6. Find a standard equation of the lines in 5(a) and 5(b).

The line $\mathbf{OP} + r\mathbf{OQ}$ contains a unique point R such that $\overline{OR}$ is perpendicular to the line. R is given by the expression,

A.7 $$\mathbf{OR} = \mathbf{OP} - \frac{\mathbf{OP} \cdot \mathbf{OQ}}{|\mathbf{OQ}|^2}\mathbf{OQ}.$$

The *distance* from the origin to $\mathbf{OP} + r\mathbf{OQ}$ is $|\mathbf{OR}|$ where $\mathbf{OR}$ is determined by A.7. An alternate formula for $\mathbf{OR}$ may be obtained from a standard form of the line, $ax + by = c$. It is

A.8 $$\mathbf{OR} = \frac{c}{a^2 + b^2}(a\mathbf{i} + b\mathbf{j}).$$

7. Find the point R on each of the following lines so that $\mathbf{OR}$ is perpendicular to the line.
   (a) $(2\mathbf{i} + 3\mathbf{j}) + r(\mathbf{i} - \mathbf{j})$, (b) $(\mathbf{i} - \mathbf{j}) + r(4\mathbf{i} + 3\mathbf{j})$,
   (c) $2x + 4y - 7 = 0$.
8. Find the distance from the origin to the lines in 7(a), (b), and (c).

The distance from S to $\mathbf{OP} + r\mathbf{OQ}$ is the same as the distance from the origin to

$$(\mathbf{OP} - \mathbf{OS}) + r\mathbf{OQ} = \mathbf{SP} + r\mathbf{OQ}.$$

(See Figure 4.15.)

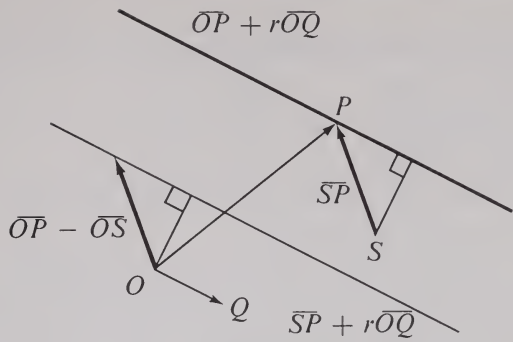


Figure 4.15

9. Find the distance from
   (a) (1, 3) to $(\mathbf{i} - \mathbf{j}) + r(2\mathbf{i} + 3\mathbf{j})$,
   (b) (2, 5) to $(3\mathbf{i} - 4\mathbf{j}) + r(5\mathbf{i} - \mathbf{j})$.

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10. Find parametric equations, a matrix equation, a slope intercept form, and a standard form for each case below.
    (a) $(\mathbf{i} - 4\mathbf{j}) + r(\mathbf{i} + 2\mathbf{j})$, (b) $(3\mathbf{i} + 5\mathbf{j}) + r(2\mathbf{i} - \mathbf{j})$.
11. Find a matrix and vector equation, given the following equations:
    (a) $x = 4 + 2r$, $y = 3 - r$, (c) $4x + y = 8$,
    (b) $x = 2 + 3r$, $y = 1 + 5r$, (d) $x + 3y = 5$.
12. For each of the lines in 11, find the point on the line nearest the origin, and the distance from the origin.
13. Find the distance from (2, 6) to $(\mathbf{i} - \mathbf{j}) + r(2\mathbf{i} + 3\mathbf{j})$.

B. *Lines in Cartesian Space*

Letting O denote the origin, the *vector equation* of the line in Cartesian space through P and in the direction of $\overline{OQ}$ is

B.1 $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{OP} + r\mathbf{OQ}$.

Setting $\mathbf{OP} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{OQ} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then B.1 may be written as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + r(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}).$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ gives the *parametric equations*,

B.2
$$x = a_1 + rb_1,$$
$$y = a_2 + rb_2,$$
$$z = a_3 + rb_3.$$

From B.2 we obtain the *matrix equation*,

B.3 $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + r \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The line $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{OP} + r\mathbf{OQ}$ will be abbreviated $\mathbf{OP} + r\mathbf{OQ}$. Fundamental properties of lines are enumerated in B.4.

B.4 (a) $\mathbf{OP} + r\mathbf{OQ}$ and $\mathbf{OP}' + r\mathbf{OQ}$ are equal or parallel.
(b) If $\mathbf{OQ} \cdot \mathbf{OQ}' = 0$, then $\mathbf{OP} + r\mathbf{OQ}$ and $\mathbf{OP} + r\mathbf{OQ}'$ are perpendicular.
(c) The line through P and Q is $\mathbf{OP} + r(\mathbf{OQ} - \mathbf{OP})$.

The line $\mathbf{OP} + r\mathbf{OQ}$ contains a unique point R such that $\overline{OR}$ is perpendicular to the line, and R is given by

B.5 $$\mathbf{OR} = \mathbf{OP} - \frac{\mathbf{OP} \cdot \mathbf{OQ}}{|\mathbf{OQ}|^2}\,\mathbf{OQ}.$$

The *distance* from the origin to the line $\mathbf{OP} + r\mathbf{OQ}$ is $|\mathbf{OR}|$.
The *direction cosines* of $\mathbf{OP} + r(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$ are:

B.6 $$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

1. Given the line $(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + r(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, find
   (a) the direction cosines,
   (b) parametric equations,
   (c) a matrix equation,
   (d) the point on the line nearest the origin,
   (e) the distance from the origin to the line,
   (f) the parallel line through the point (1, 0, 4).
2. Find a vector equation of the line,
   (a) whose parametric equations are $x = 2 + 3r$, $y = 3 - r$, and $z = 2 + r$;
   (b) through $(1, 2, -3)$ and $(3, 1, 4)$; and finally,
   (c) through (5, 1, 7) and having direction cosines $1/3$, $2/3$, $-2/3$.

Two lines in space may or may not intersect. If they do intersect, then the point of intersection can be determined by solving systems of linear equations.

3. Find the point of intersection (a, b, c) of the two lines

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix} + r' \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

   using the following procedure.
   (a) Let r_o and r_o' be, respectively, the values of r and r' corresponding to the point of intersection, and find equations relating a, b, c, r_o, and r_o'.
   (b) From (a) write, in matrix form, a system of three equations in two unknowns, r_o and r_o', and solve for r_o or r_o'.
   (c) Substitute from (b) into (a).

4. Find the point of intersection of the lines

$$\begin{bmatrix} 4 \\ -5 \\ -1 \end{bmatrix} + r\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + r'\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

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5. Find direction cosines and a matrix equation for the given line.
   (a) $(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + r(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$,
   (b) $(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + r(\mathbf{i} + \mathbf{j} - 4\mathbf{k})$.
6. For the lines in 5(a) and (b), find the point nearest the origin and the distance to the origin.
7. Find a vector equation of the line described.
   (a) The line through $(2, -1, 3)$ and parallel to

$$(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + r(3\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

   (b) The line with parametric equations $x = 4$, $y = 3 + 2r$, $z = -r$.
   (c) The line through $(2, 1, 4)$ having direction cosines $-1/3$, $-2/3$, $2/3$.
8. Find the point of intersection of the two lines shown below.

$$\begin{bmatrix} 2 \\ 1 \\ -11 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 7 \\ 2 \\ -2 \end{bmatrix} + r'\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

C. *Planes in Cartesian Space*

If $\overline{OQ}$ and $\overline{OR}$ are noncollinear, then the *vector equation* of the plane through P and having direction vectors $\overline{OQ}$, $\overline{OR}$ is

C.1 $\quad x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{OP} + r\mathbf{OQ} + s\mathbf{OR}.$

If

$$\mathbf{OP} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \mathbf{OQ} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

and

$$\mathbf{OR} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},$$

then *parametric equations* of this plane are

C.2
$$\begin{aligned} x &= a_1 + rb_1 + sc_1, \\ y &= a_2 + rb_2 + sc_2, \\ z &= a_3 + rb_3 + sc_3. \end{aligned}$$

A *matrix equation* of this plane is,

C.3 $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + r\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + s\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The parameters r and s may be eliminated from the parametric equations above to give a *standard equation*, shown in C.4.

C.4 $ax + by + cz = d$.

Fundamental properties of this plane are shown below.

C.5 (a) $\mathbf{OP} + r\mathbf{OQ} + s\mathbf{OR}$ and $\mathbf{OP}' + r\mathbf{OQ} + s\mathbf{OR}$ are equal or parallel.
(b) The line through S and perpendicular to $\mathbf{OP} + r\mathbf{OQ} + s\mathbf{OR}$ is $\mathbf{OS} + r(\mathbf{OQ} \times \mathbf{OR})$.
(c) The plane through noncollinear points P, Q, and R is $\mathbf{OP} + r(\mathbf{OQ} - \mathbf{OP}) + s(\mathbf{OR} - \mathbf{OP})$.

1. Given the plane $(\mathbf{i} - \mathbf{j} + \mathbf{k}) + r(\mathbf{i} + \mathbf{j}) + s(\mathbf{j} - \mathbf{k})$, find
   (a) parametric equations,
   (b) a matrix equation,
   (c) a standard equation,
   (d) the parallel plane through the point (1, 4, 3), and
   (e) the perpendicular line through (1, 4, 3).

2. Find the plane through the points given below:
   (a) $(1, -1, 2), (3, 1, 0), (2, -1, 4)$;
   (b) $(5, 2, 7), (1, 0, 8), (6, 1, 3)$.

A standard equation of a line through three given points can be found directly by substituting the coordinates of the points into $ax + by + cz = d$ and solving the resulting three equations for a, b, c, and d.

3. Find directly a standard equation for the plane through the points in Problem 2(a) and (b).

If S is in the plane $ax + by + cz = d$ and $\overline{OS}$ is perpendicular to the plane, then we have the equation

C.6 $$\mathbf{OS} = \frac{d}{a^2 + b^2 + c^2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).$$

Then S is the point in the plane nearest the origin, and the distance from the origin to the plane is

$$|\mathbf{OS}| = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

4. Given the plane $(\mathbf{i} - \mathbf{j} + \mathbf{k}) + r(\mathbf{i} + 2\mathbf{j}) + s(\mathbf{j} - 3\mathbf{k})$;
   (a) find the point in the plane nearest the origin, and
   (b) find the distance from the plane to the origin.

Further properties of lines and planes are:

C.7 (a) If R is not on the line $\mathbf{OP} + r\mathbf{OQ}$, then the plane containing R and $\mathbf{OP} + r\mathbf{OQ}$ is

$$\mathbf{OP} + r\mathbf{OQ} + s(\mathbf{OR} - \mathbf{OP}).$$

(b) The plane containing the lines $\mathbf{OP} + r\mathbf{OQ}$ and $\mathbf{OP} + r\mathbf{OQ}'$ is $\mathbf{OP} + r\mathbf{OQ} + s\mathbf{OQ}'$.

5. Find the plane containing the line $(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + r(2\mathbf{i} - \mathbf{j} - \mathbf{k})$ and the point $(1, 0, -2)$.
6. Find the plane containing the lines

$$(\mathbf{i} + \mathbf{k}) + r(3\mathbf{i} - \mathbf{j}) \quad \text{and} \quad (\mathbf{i} + \mathbf{k}) + r(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}).$$

The intersection of two nonparallel planes is a line. An equation of the line can be found from two points lying in the intersection of the planes.

7. Find a vector equation of the intersection of the lines $4x - y + 3z = 6$ and $2x - y + z = 8$ as follows:
   (a) Set $z = 0$ and solve the resulting two equations for x and y.
   (b) Set $z = 1$ and solve the resulting two equations for x and y.
   (c) Find the line containing the points found in (a) and (b).
8. Find a vector equation of the intersection of

$$x - y + z = 4 \quad \text{and} \quad x + y - z = 6.$$

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9. Given the plane $(\mathbf{i} - 2\mathbf{k}) + r(\mathbf{i} + 3\mathbf{j}) + s(2\mathbf{j} - \mathbf{k})$, find
   (a) a matrix equation,
   (b) a standard equation,
   (c) the parallel plane through $(4, 0, 7)$,
   (d) the perpendicular line through $(1, 3, 2)$,
   (e) the point in the plane nearest the origin, and
   (f) the distance from the plane to the origin.
10. Find a vector equation of each given plane.
    (a) $3x + 4y - z + 5 = 0$,
    (b) the plane through $(1, 0, 1)$, $(2, -1, 0)$, $(1, 1, -1)$,
    (c) the plane through $(2, 0, -1)$ and containing $\mathbf{i} + r(\mathbf{j} - \mathbf{k})$,
    (d) the plane containing $(\mathbf{i} + \mathbf{k}) + r(\mathbf{j} + \mathbf{k})$ and $(\mathbf{i} + \mathbf{k}) + r(3\mathbf{i} - \mathbf{j})$.
11. Find, in standard form, the plane through $(1, -1, 2)$, $(1, 0, 3)$, $(2, -1, 1)$.
12. Find a vector equation of the intersection of $3x + y + z = 8$ and $x - y - 2z = 4$.

D. *Segments, Parallelograms, Triangles*

In the Cartesian plane or space the line segment with end points P and Q is the graph of the set of all tuples of the form $\mathbf{OP} + r\mathbf{PQ}$, $0 \le r \le 1$. This set is denoted

$$\mathbf{OP} + [r\mathbf{PQ}].$$

1. Describe, as a set, the line segment with end points:
   (a) $(1, 2), (3, -4)$; (b) $(2, 5), (7, -3)$;
   (c) $(3, 1, 4,), (6, 0, 2)$; (d) $(7, 0, -2), (1, 1, 4)$.

The parallelogram having sides $\overline{PQ}$ and $\overline{QR}$ is the graph of the set of all tuples of the form $\mathbf{OP} + r\mathbf{PQ} + s\mathbf{QR}$, $0 \le r, s \le 1$. This set is denoted

$$\mathbf{OP} + [r\mathbf{PQ} + s\mathbf{QR}].$$

Other sets in the Cartesian plane or space have similar descriptions. In each case, the values of the parameters will lie between 0 and 1. They may be listed as follows:

Line segment with end points P, Q	$\mathbf{OP} + [r\mathbf{PQ}]$,
Parallelogram with sides $\overline{PQ}$, $\overline{QR}$	$\mathbf{OP} + [r\mathbf{PQ} + s\mathbf{QR}]$,
Triangles with vertices P, Q, R	$\mathbf{OP} + [r\mathbf{PQ} + rs\mathbf{QR}]$,
Parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$	$\mathbf{OP} + [r\mathbf{PQ} + s\mathbf{QR} + t\mathbf{RS}]$,
Tetrahedron with vertices P, Q, R, S	$\mathbf{OP} + [r\mathbf{PQ} + rs\mathbf{QR} + rst\mathbf{RS}]$.

2. Describe as a set the parallelogram with sides $\overline{PQ}$, $\overline{QR}$ and the triangle with vertices P, Q, and R, given the points below.
   (a) $P = (1, 2)$, $Q = (3, 0)$, $R = (5, -3)$,
   (b) $P = (4, -1)$, $Q = (2, 5)$, $R = (7, 3)$,
   (c) $P = (1, 1, 2)$, $Q = (3, 0, 6)$, $R = (7, -1, 2)$.
3. Describe as a set the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and the tetrahedron with vertices P, Q, R, and S, given the following information.
   (a) $P = (2, 1,6)$, $Q = (3, 4, -2)$, $R = (7, 5, -1)$, $S = (4, 0, 6)$;
   (b) $P = (3, 0, 7)$, $Q = (2, -5, 1)$, $R = (6, -3, 4)$, $S = (2, 7, -4)$.

Associated with each of the previous sets is a length, area or volume. The following formulas are valid.

Set	Size
$\mathbf{OP} + [r\mathbf{PQ}]$	Length $= \lvert\mathbf{PQ}\rvert$
$\mathbf{OP} + [r\mathbf{PQ} + s\mathbf{QR}]$ (in space)	Area $= \lvert\mathbf{PQ} \times \mathbf{QR}\rvert$
$\mathbf{OP} + [r\mathbf{PQ} + rs\mathbf{QR}]$ (in space)	Area $= \frac{1}{2}\lvert\mathbf{PQ} \times \mathbf{QR}\rvert$
$\mathbf{OP} + [r\mathbf{PQ} + s\mathbf{QR} + t\mathbf{RS}]$	Volume $= \lvert(\mathbf{PQ} \times \mathbf{QR}) \cdot \mathbf{RS}\rvert$
$\mathbf{OP} + [r\mathbf{PQ} + rs\mathbf{QR} + rst\mathbf{RS}]$	Volume $= \frac{1}{6}\lvert(\mathbf{PQ} \times \mathbf{QR}) \cdot \mathbf{RS}\rvert$

4. Find the length, area, or volume, as applicable.
   (a) $\langle 3, 2\rangle + [r\langle 2, -4\rangle]$,
   (b) $\langle 3, 0, 1\rangle + [r\langle 4, 2, 6\rangle]$,
   (c) $\langle 1, 0, 2\rangle + [r\langle 2, 0, 1\rangle + s\langle 0, 2, 3\rangle]$,
   (d) $\langle 1, 1, 3\rangle + [r\langle 1, 0, 2\rangle + rs\langle -1, 1, 5\rangle]$,
   (e) $\langle 1, 0, 5\rangle + [r\langle 1, -1, 2\rangle + s\langle 3, 0, 5\rangle + t\langle 1, -1, 4\rangle]$,
   (f) $\langle 1, 1, 3\rangle + [r\langle 1, 0, 2\rangle + rs\langle 1, -1, 5\rangle + rst\langle 2, 0, 7\rangle]$.

The area of a parallelogram or triangle in the Cartesian plane may be obtained from the corresponding space formula by identifying the Cartesian plane with the *xy* plane in space. This is accomplished by adding 0 as a third coordinate to each point.

5. Find the area of each set below.
   (a) $\langle 3, 0\rangle + [r\langle 1, 1\rangle + s\langle 2, -1\rangle]$,
   (b) $\langle 3, 1\rangle + [r\langle 2, 3\rangle + s\langle 1, 0\rangle]$,
   (c) $\langle 1, 0\rangle + [r\langle 1, 0\rangle + rs\langle 2, 7\rangle]$,
   (d) $\langle 3, 5\rangle + [r\langle 2, -3\rangle + rs\langle 6, 5\rangle]$.

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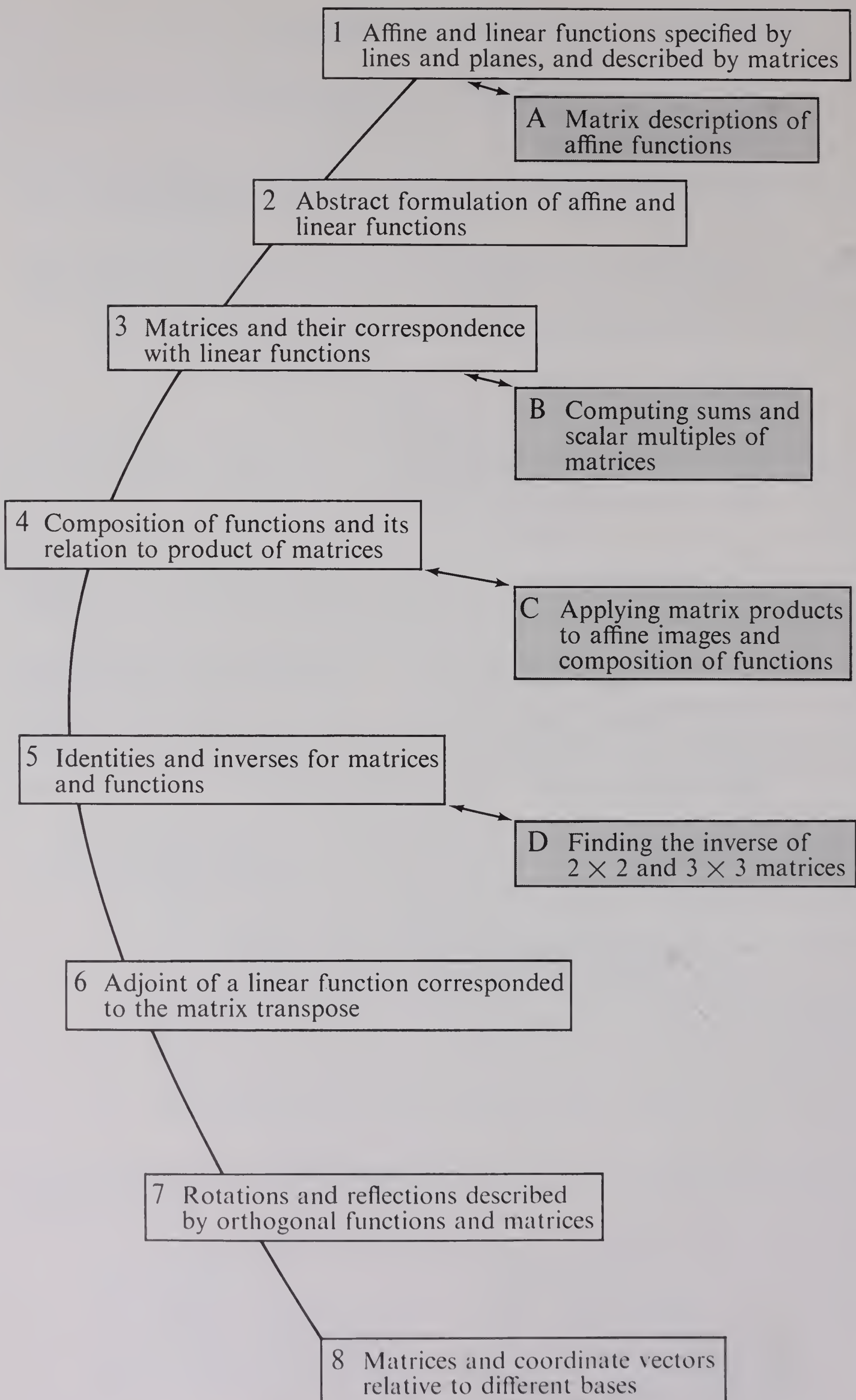
6. Given $P = (1, 0, 6)$ and $Q = (2, -1, 3)$, describe $\overline{PQ}$ as a set and find its length.
7. Given $P = (1, 7)$, $Q = (2, -4)$, and $R = (3, 5)$, describe as a set and find the area of the following figures.
   (a) the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$,
   (b) the triangle with vertices P, Q, R.
8. Repeat 7 given that $P = (1, 0, 3)$, $Q = (2, 8, -4)$, and $R = (1, 6, 3)$.
9. Given $P = (1, 0, 5)$, $Q = (2, -1, 4)$, $R = (6, 6, 1)$, and $S = (7, 0, 3)$, describe as a set and find the volume of:
   (a) the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$,
   (b) the tetrahedron with vertices P, Q, R, and S.


1 Affine and linear functions specified by lines and planes, and described by matrices
A Matrix descriptions of affine functions
2 Abstract formulation of affine and linear functions
3 Matrices and their correspondence with linear functions
B Computing sums and scalar multiples of matrices
4 Composition of functions and its relation to product of matrices
C Applying matrix products to affine images and composition of functions
5 Identities and inverses for matrices and functions
D Finding the inverse of 2 × 2 and 3 × 3 matrices
6 Adjoint of a linear function corresponded to the matrix transpose
7 Rotations and reflections described by orthogonal functions and matrices
8 Matrices and coordinate vectors relative to different bases

V

Linear Functions and Matrices

Most applications of vector-space theory involve more than one vector space. For example, in the study of an object moving through space and acted upon by a gravitational force field, the assumed position vectors and force vectors range over distinct vector spaces. These vector spaces are mathematically related by means of functions. Such functions, which correspond vectors to vectors, are called *vector functions*; these are studied in the calculus of several variables. We shall be interested here only in the subclass of vector functions called *affine functions*. In fact, much of our attention will be focused on a smaller subclass called *linear functions*. Only the simplest physical situations produce vector functions which are affine; however, a knowledge of many vector functions occurring in physics and engineering can be increased by use of related affine functions called *affine approximations*.

Affine functions from one vector space to another can be described by the basic vector space operations of addition and scalar multiplication. The functions usually found in elementary calculus are described by rules, often expressed

by equations, which specify an assigned number to each number in the domain of the function. Comparable descriptions for affine and linear functions require several equations, since the range space is a vector space rather than the set of real numbers. Such functions can be more compactly described by arrays of numbers called *matrices*. The first systematic study of matrices is usually credited to the Englishman Arthur Cayley (1821–1895) in 1858. Operations on matrices, which conform to operations on functions, make possible the use of matrices as a valuable computational tool in problems involving affine functions. Systems of matrices also play an important theoretic role in modern mathematics.

1. *Affine and Linear Functions* ($\mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3$)

In elementary algebra and analytic geometry a real function of one variable is called *linear* if its graph is a straight line. Within the terminology of vector space study, however, such a function is said to be *affine*; it is linear only if the graph passes through the origin. Linear functions relate to subspaces, while affine functions relate to n-planes in vector space study. A precise meaning of these relationships will be revealed in the next two chapters.

An arbitrary vector in the line $\mathbf{u}+r\mathbf{v}$ in $\mathbf{R}^2$ has the form

$$\langle x, y\rangle = \mathbf{u}+r\mathbf{v}$$

for some number r. If we let $\mathbf{u}=\langle b_1, b_2\rangle$ and $\mathbf{v}=\langle a_{11}, a_{21}\rangle$, this equation becomes

$$\langle x, y\rangle = \langle b_1, b_2\rangle + r\langle a_{11}, a_{21}\rangle.$$

Equating the coordinates gives

$$x=b_1+ra_{11} \quad \text{and} \quad y=b_2+ra_{21}.$$

These two equations describe a function,

$$g(r)=\langle x, y\rangle=\langle b_1+ra_{11}, b_2+ra_{21}\rangle,$$

from $\mathbf{Re}$ to $\mathbf{R}^2$. The function g is called an *affine* function; if, in addition, $b_1=b_2=0$, then g is said to be *linear*. The graph of $\mathbf{u}+r\mathbf{v}$ is called the *image graph* of g (see Figure 5.1). It should be observed that infinitely many affine functions correspond to each line in $\mathbf{R}^2$; hence the correspondence between such affine functions and the image is not one-to-one.

In a similar manner, the line in $\mathbf{R}^3$,

$$\langle b_1, b_2, b_3\rangle + r\langle a_{11}, a_{21}, a_{31}\rangle,$$

corresponds to an affine function,

$$g(r)=\langle x, y, z\rangle,$$

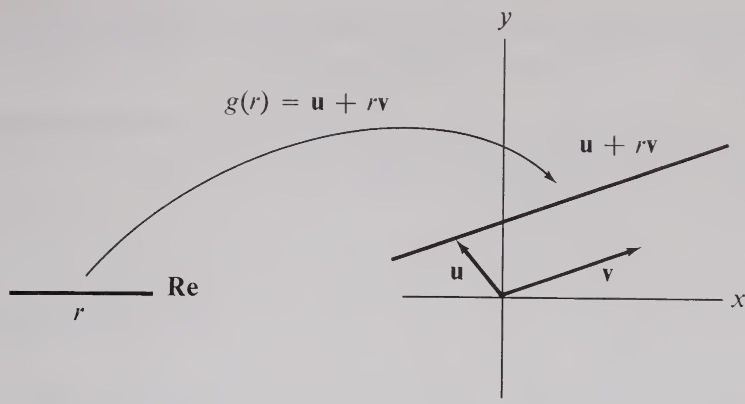


Figure 5.1

where

$$
\begin{aligned}
x &= b_1 + ra_{11},\\
y &= b_2 + ra_{21},\\
z &= b_3 + ra_{31}.
\end{aligned}
$$

Also the plane,

$$\langle b_1, b_2, b_3\rangle + r\langle a_{11}, a_{21}, a_{31}\rangle + s\langle a_{12}, a_{22}, a_{32}\rangle,$$

corresponds to an affine function,

$$g(\langle r, s\rangle) = \langle x, y, z\rangle,$$

where

$$
\begin{aligned}
x &= b_1 + ra_{11} + sa_{12},\\
y &= b_2 + ra_{21} + sa_{22},\\
z &= b_3 + ra_{31} + sa_{32}.
\end{aligned}
$$

The notation $g(r, s)$ will often be used instead of $g(\langle r, s\rangle)$. Again, g is called *linear* if each of the b_i's is 0. The graph of the line or plane vector set is again the *image graph* of g.

The study of affine functions need not relate to lines and planes. An affine function g from $\mathbf{R}^n$ to $\mathbf{R}^m$ has the form

$$g(x_1, x_2, \ldots, x_n) = \langle y_1, y_2, \ldots, y_m\rangle,$$

where

$$
\begin{aligned}
y_1 &= b_1 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\\
y_2 &= b_2 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\\
&\ \ \vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
y_m &= b_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n.
\end{aligned}
$$

The function g is *linear* if $b_1 = b_2 = \cdots = b_m = 0$. For dimension three or less, we shall use r, s, t to denote domain variables and x, y, z to denote range variables.

It is convenient to describe affine and linear functions by rectangular arrays of numbers called *matrices* (see Problem Set A in Chapter II). The general affine function is then written

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

The reason for this particular representation will be seen more clearly when the operations on matrices are defined in the next section. The matrix of b_i's is called the *constant matrix* of g, and consists only of 0's if g is linear. The matrix of a_{ij}'s is called the *linear matrix* of g.

Example 1.1 The affine function

$$g(r, s) = \langle 2r - s, 1 + 3r + s, 2 + 5r + 7s\rangle$$

is described by the equation system

$$\begin{aligned} x &= \phantom{1 + {}} 2r - s \\ y &= 1 + 3r + s \\ z &= 2 + 5r + 7s \end{aligned}$$

and has matrix representation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

The constant and linear matrices are, respectively,

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 5 & 7 \end{bmatrix}.$$

The entries in these matrices may be determined from the vectors assigned by g to the origin and standard basis vectors. For example, let g be an affine function from $\mathbf{R}^2$ to $\mathbf{R}^2$ with

$$g(0, 0) = \langle c_1, c_2\rangle; \qquad g(1, 0) = \langle c_{11}, c_{21}\rangle; \qquad g(0, 1) = \langle c_{12}, c_{22}\rangle.$$

If $g(\langle r, s\rangle) = \langle x, y\rangle$ is given by the equations

$$\begin{aligned} x &= b_1 + a_{11}r + a_{12}s \\ y &= b_2 + a_{21}r + a_{22}s, \end{aligned}$$

then substitution of $\langle r, s\rangle = \langle 0, 0\rangle$ and $\langle x, y\rangle = \langle c_1, c_2\rangle$ gives

$$b_1 = c_1, \, b_2 = c_2 .$$

Substitution of $\langle r, s\rangle = \langle 1, 0\rangle$ and $\langle x, y\rangle = \langle c_{11}, c_{21}\rangle$ gives

$$a_{11} = c_{11} - c_1, \quad \text{and} \quad a_{21} = c_{21} - c_2 .$$

Substitution of $\langle r, s\rangle = \langle 0, 1\rangle$ and $\langle x, y\rangle = \langle c_{12}, c_{22}\rangle$ gives

$$a_{12} = c_{12} - c_1, \, a_{22} = c_{22} - c_2 .$$

The matrix representation is, therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_{11} - c_1 & c_{12} - c_1 \\ c_{21} - c_2 & c_{22} - c_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

The n-tuple $\langle a_1, a_2, \ldots, a_n\rangle$ is often identified with the matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Therefore, in the foregoing case, the constant matrix of g is $g(0, 0)$ and the columns of the linear matrix of g are $g(1, 0) - g(0, 0)$, $g(0, 1) - g(0, 0)$. The extension to other dimensions is evident (see Proofs, exercise 1).

Example 1.2 The affine function g from $\mathbf{R}^3$ to $\mathbf{R}^2$ such that $g(\mathbf{0}) = \langle 2, 1\rangle, g(\mathbf{i}) = \langle 1, 3\rangle, g(\mathbf{j}) = \langle 6, 7\rangle, g(\mathbf{k}) = \langle -3, 0\rangle$ is described by the matrix representation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 - 2 & 6 - 2 & -3 - 2 \\ 3 - 1 & 7 - 1 & 0 - 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 4 & -5 \\ 2 & 6 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}.$$

Questions

1. The function $y = ax$ is a(n) ________ function; $y = b + ax$ is a(n) ________ function.
2. Associated with each affine function is a ________ matrix and a ________ matrix.
3. The ________ matrix of a linear function consists of zeros.
4. If g is an affine function from $\mathbf{R}^2$ to $\mathbf{R}^2$, then the column vector of the constant matrix of g is ________.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. Prove that if g is an affine function from $\mathbf{R}^3$ to $\mathbf{R}^3$, then the column vectors of the linear matrix of g are

$$g(\mathbf{i}) - g(\mathbf{0}), \quad g(\mathbf{j}) - g(\mathbf{0}), \quad \text{and} \quad g(\mathbf{k}) - g(\mathbf{0}).$$

2. *Affine and Linear Functions (Abstract Vector Spaces)*

In this section the affine and linear functions from a vector space $\mathbf{V}$ to a vector space $\mathbf{V}'$ will be characterized. The definition of linear function requires that the addition and scalar-multiplication operations be preserved. We therefore state the following definition.

Definition of Linear Function

A function f from a vector space $\mathbf{V}$ to a vector space $\mathbf{V}'$ is linear *provided that for each $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and each real number c:*

(a) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$, and
(b) $f(c\mathbf{u}) = cf(\mathbf{u})$.

Letting $\mathbf{0}'$ denote the zero vector of $\mathbf{V}'$, from the chain of equalities

$$f(\mathbf{0}) = f(0\mathbf{0}) = 0f(\mathbf{0}) = \mathbf{0}',$$

we see that a linear function from $\mathbf{V}$ to $\mathbf{V}'$ carries the zero vector of $\mathbf{V}$ into the zero vector of $\mathbf{V}'$. If f is a linear function from $\mathbf{R}^2$ to $\mathbf{R}^2$ and if

$$f(\langle 1, 0\rangle) = \langle a_{11}, a_{21}\rangle, \qquad f(\langle 0, 1\rangle) = \langle a_{12}, a_{22}\rangle,$$

then

$$\begin{aligned} f(\langle r, s\rangle) &= f(r\langle 1, 0\rangle + s\langle 0, 1\rangle) \\ &= rf(\langle 1, 0\rangle) + sf(\langle 0, 1\rangle) \\ &= r\langle a_{11}, a_{21}\rangle + s\langle a_{12}, a_{22}\rangle \\ &= \langle a_{11}r + a_{12}s, a_{21}r + a_{22}s\rangle. \end{aligned}$$

This indicates that the linear functions, as defined here, agree with those of the previous section.

It follows, by induction from the definition of linear function, that if f is linear and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are arbitrary vectors in $\mathbf{V}$, then

$$f(c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k) = c_1 f(\mathbf{u}_1) + \cdots + c_k f(\mathbf{u}_k)$$

for all numbers $c_1, \ldots, c_k$.

Example 2.1 The derivative and indefinite integral are examples of linear functions. For simplicity let $\mathbf{V} = \mathbf{V}'$ be the vector space of polynomials on [0, 1]. Then the functions D and I from $\mathbf{V}$ to $\mathbf{V}'$ defined by

$$Df = \frac{df}{dx} \quad \text{and} \quad If = \int_0^x f\,dx$$

are linear. The integral is linear, since the properties

$$\int_0^x (f+g)\,dx = \int_0^x f\,dx + \int_0^x g\,dx$$

and

$$\int_0^x cf\,dx = c\int_0^x f\,dx$$

are valid. Similiar equations show that D is linear.

We shall next show that for finite-dimensional spaces a linear function is uniquely determined by its values on a basis.

Interpolation Theorem

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbf{V}$ and $\mathbf{u}_1', \ldots, \mathbf{u}_n'$ are (not necessarily distinct) vectors in $\mathbf{V}'$, then there is a unique linear function f from $\mathbf{V}$ to $\mathbf{V}'$ such that $f(\mathbf{u}_1) = \mathbf{u}_1', \ldots, f(\mathbf{u}_n) = \mathbf{u}_n'$.

A proof will be indicated for the case where $n = 3$. An arbitrary vector in $\mathbf{V}$ has the form

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3;$$

and if f is linear, then

$$\begin{aligned} f(\mathbf{u}) &= c_1 f(\mathbf{u}_1) + c_2 f(\mathbf{u}_2) + c_3 f(\mathbf{u}_3) \\ &= c_1\mathbf{u}_1' + c_2\mathbf{u}_2' + c_3\mathbf{u}_3'. \end{aligned}$$

This shows the uniqueness of f, and it is not difficult to verify that the function f defined by the equation

$$f(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = c_1\mathbf{u}_1' + c_2\mathbf{u}_2' + c_3\mathbf{u}_3'$$

satisfies the defining properties of linearity.

Example 2.2 The derivative D is linear on the vector space $\mathbf{V}$ of polynomials of degree less than or equal to 2 (see Example 2.1). Also, $\{1, x, x^2\}$ is a basis of $\mathbf{V}$ as we learned in Chapter II. Hence, from

$$D(1) = 0, \qquad D(x) = 1, \qquad D(x^2) = 2x,$$

all derivatives of polynomials of degree less than or equal to 2 can be found using the linearity properties of D.

In Chapter I we proved that the set $\mathbf{F}[\mathbf{V}, \mathbf{V}']$ of all functions from $\mathbf{V}$ to $\mathbf{V}'$ forms a vector space. It will now be seen that the linear functions from $\mathbf{V}$ to $\mathbf{V}'$ are a subspace of $\mathbf{F}[\mathbf{V}, \mathbf{V}']$, and hence themselves constitute a vector space. It is sufficient to prove closure with respect to addition and scalar multiplication in order to show that the linear functions are a subspace.

Proposition 2.1 If f and g are linear from $\mathbf{V}$ to $\mathbf{V}'$, then so are (a) $f+g$, (b) cf.

A proof of (a) in Proposition 2.1 follows from the argument below together with a similar proof for $(f+g)(a\mathbf{u}) = a((f+g)(\mathbf{u}))$ (see Proofs, exercise 1).

$$\begin{aligned}
(f+g)(\mathbf{u}+\mathbf{v}) &= f(\mathbf{u}+\mathbf{v}) + g(\mathbf{u}+\mathbf{v}), && \text{add fcn}\\
&= (f(\mathbf{u}) + f(\mathbf{v})) + (g(\mathbf{u}) + g(\mathbf{v})), && \text{def linear}\\
&= (f(\mathbf{u}) + g(\mathbf{u})) + (f(\mathbf{v}) + g(\mathbf{v})), && \text{comm add, as add}\\
&= (f+g)(\mathbf{u}) + (f+g)(\mathbf{v}). && \text{add fcn}
\end{aligned}$$

For (b) see Proofs, exercise 2.

Affine functions bear the same relation to linear functions as n-planes do to subspaces. In fact, n-planes are sometimes called *affine spaces*. The *constant function* $\mathbf{u}_o'$ from $\mathbf{V}$ to $\mathbf{V}'$ assigns the same vector $\mathbf{u}_o'$ to every vector in $\mathbf{V}$. An affine function is the sum of a constant function and a linear function.

Definition of Affine Function

An affine function from $\mathbf{V}$ *to* $\mathbf{V}'$ *is a function having the form* $\mathbf{u}_o' + f$ *where* f *is linear.*

Since the linearity of f implies $f(\mathbf{0}) = \mathbf{0}'$, the equality chain

$$(u_o' + f)(\mathbf{0}) = u_o'(\mathbf{0}) + f(\mathbf{0}) = \mathbf{u}_o' + \mathbf{0}' = \mathbf{u}_o'$$

shows that $u_o' + f$ assigns $\mathbf{u}_o'$ to $\mathbf{0}$. If $\mathbf{V}$ is finite-dimensional, then an affine function from $\mathbf{V}$ to $\mathbf{V}'$ is determined by its values at $\mathbf{0}$ and on a basis. For instance, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of $\mathbf{V}$ and

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3,$$

then letting $g = u_o' + f$ gives

$$\begin{aligned} g(\mathbf{u}) &= \mathbf{u}_o' + c_1 f(\mathbf{u}_1) + c_2 f(\mathbf{u}_2) + c_3 f(\mathbf{u}_3) \\ &= \mathbf{u}_o' + c_1[g(\mathbf{u}_1) - \mathbf{u}_o'] + c_2[g(\mathbf{u}_2) - \mathbf{u}_o'] + c_3[g(\mathbf{u}_3) - \mathbf{u}_o'] \\ &= (1 - c_1 - c_2 - c_3)g(\mathbf{0}) + c_1 g(\mathbf{u}_1) + c_2 g(\mathbf{u}_2) + c_3 g(\mathbf{u}_3). \end{aligned}$$

The extension to an arbitrary finite-dimensional space is evident.

Example 2.3 Let g be an affine function from $\mathbf{R}^2$ to $\mathbf{R}^2$ with $g(\mathbf{0}) = \langle 1, 3\rangle$, $g(\mathbf{i}) = \langle 2, 5\rangle$, and $g(\mathbf{j}) = \langle 4, -1\rangle$. Then for all c_1, c_2

$$\begin{aligned} g(c_1, c_2) &= (1 - c_1 - c_2)\langle 1, 3\rangle + c_1\langle 2, 5\rangle + c_2\langle 4, -1\rangle \\ &= \langle 1 + c_1 + 3c_2, 3 + 2c_1 - 4c_2\rangle. \end{aligned}$$

The affine functions from $\mathbf{V}$ to $\mathbf{V}'$ also form a vector space. The proof is a straightforward application of definitions (see Proofs, exercise 3).

Questions

1. If f is linear, then for each number c and domain vectors $\mathbf{u}$, $\mathbf{v}$ the equalities ________ and ________ hold.
2. A linear function is determined by its values on a ________.
3. The image of a ________ function consists of a single vector.
4. An affine function is the sum of a ________ function and a ________ function.
5. An affine function is determined by its values at ________ and on a ________.
6. The linearity of the derivative follows from the derivative properties ________ and ________.

Exercises

1. Determine from the definition of linear whether or not the following rules describe a linear function.

   (a) $f(\langle a_1, a_2\rangle) = \langle a_1, a_2, 0\rangle$, (b) $f(\langle a_1, a_2\rangle) = \langle a_1, a_2, 1\rangle$.

2. Given that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis of $\mathbf{R}^2$ and that f is a linear function from $\mathbf{R}^2$ to $\mathbf{R}^2$ with $f(\mathbf{u}_1) = \langle 2, 4\rangle$ and $f(\mathbf{u}_2) = \langle 1, 3\rangle$, find the vector $f(3\mathbf{u}_1 - \mathbf{u}_2)$.
3. Find $g(\langle 1, 2\rangle)$ if g if an affine function with $g(\langle 1, 0\rangle) = \langle 3, 1\rangle$, $g(\langle 1, 1\rangle) = \langle 2, 1\rangle$, and $g(\langle 0, 1\rangle) = \langle 0, 0\rangle$.

Proofs

1. Justify the steps of the following proof when f and g are linear:
$$(f+g)(a\mathbf{u}) = f(a\mathbf{u}) + g(a\mathbf{u}) = a(f(\mathbf{u})) + a(g(\mathbf{u})) = a(f(\mathbf{u}) + g(\mathbf{u})) = a((f+g)(\mathbf{u})).$$
2. Prove that if f is linear, then so is cf.
3. Prove that if g_1 and g_2 are affine, then so is (a) $g_1 + g_2$ and (b) cg_1.

3. *Matrices*

In this section we shall define matrices and relate them to linear functions. Addition and scalar multiplication operations for matrices will also be introduced.

Definition of an $m \times n$ Matrix

An $m \times n$ matrix *is an ordered set of n m-tuples*

An $m \times n$ matrix may be symbolized by

$$A = \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle,$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are m-tuples, called the *column vectors* of A. It is often more convenient to write each column vector of A as a vertical array. Letting $\mathbf{u}_j = \langle a_{1j}, \ldots, a_{mj}\rangle$ we obtain the *rectangular array* form of A shown below.

$$\begin{bmatrix} a_{11} \ldots a_{1j} \ldots a_{1n} \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{i1} \ldots a_{ij} \ldots a_{in} \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1} \ldots a_{mj} \ldots a_{mn} \end{bmatrix}$$

The ith coordinate of $\mathbf{u}_j$ is called the *ij-entry* of A. The symbol $[a_{ij}]$ will sometimes be used to denote a matrix whose ij-entry is a_{ij}. The horizontal arrays in the

rectangular form of A are called *row vectors*. The entry a_{ij} appears in the ith row and jth column of A.

Example 3.1 The column vectors of $A = \langle\langle 1, 3\rangle, \langle 2, 5\rangle, \langle 4, 7\rangle\rangle$ are $\langle 1, 3\rangle$, $\langle 2, 5\rangle$, and $\langle 4, 7\rangle$. In rectangular array form,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}.$$

The 11-entry of A is 1, the 23-entry of A is 7, and so forth. The row vectors are $\langle 1, 2, 4\rangle$ and $\langle 3, 5, 7\rangle$.

The operations of addition and scalar multiplication of matrices are performed in a natural way, as seen in our next definition.

Definition of Addition of $m \times n$ Matrices

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_n\rangle + \langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle = \langle \mathbf{u}_1 + \mathbf{v}_1, \ldots, \mathbf{u}_n + \mathbf{v}_n\rangle$$

Definition of Scalar Multiplication of Matrices

$$c\langle \mathbf{u}_1, \ldots, \mathbf{u}_n\rangle = \langle c\mathbf{u}_1, \ldots, c\mathbf{u}_n\rangle$$

Addition is defined only for matrices having the same size in rectangular array form, and it is achieved by adding corresponding entries. Scalar multiplication by c causes each entry to be multiplied by c. Using these operations we can show the following proposition.

Proposition 3.1 The set of all $m \times n$ matrices is a vector space.

The proof of Proposition 3.1 is similar to the proof of the fact that $\mathbf{R}^n$ is a vector space. The zero $m \times n$ matrix is $\langle \mathbf{0}, \ldots, \mathbf{0}\rangle$, where $\mathbf{0}$ denotes the zero vector of $\mathbf{R}^m$. As a rectangular array it has all entries 0.

There is an intimate relationship between the linear functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ and the $m \times n$ matrices. It is established by the correspondence

$$\text{Mat}\, f = \langle f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\rangle,$$

where Mat associates with each linear f from $\mathbf{R}^n$ to $\mathbf{R}^m$ the $m \times n$ matrix whose jth column vector is the f-image of the standard basis vector $\mathbf{e}_j$. This correspondence is linear and one-to-one. It is also *onto* in the sense that each $m \times n$ matrix corresponds to some f. We now state this formally.

Isomorphism Theorem for Matrices and Linear Functions

(a) $\text{Mat}\,(f+g)=\text{Mat}\,f+\text{Mat}\,g$,
(b) $\text{Mat}\,cf=c\,\text{Mat}\,f$,
(c) *If* $f\neq g$ then $\text{Mat}\,f\neq\text{Mat}\,g$,
(d) *For each* $m\times n$ *matrix* A *there exists a linear function* f *from* $\mathbf{R}^n$ *to* $\mathbf{R}^m$ *such that* $\text{Mat}\,f=A$.

For (a) see Proofs, exercise 3.

A proof of (b) will be indicated for the case $n=3$. If $\text{Mat}\,f=\langle\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\rangle$, then

$$f(\mathbf{e}_1)=\mathbf{u}_1,f(\mathbf{e}_2)=\mathbf{u}_2,f(\mathbf{e}_3)=\mathbf{u}_3.$$

Therefore, by the definition of scalar multiplication of functions,

$$(cf)(\mathbf{e}_1)=c\mathbf{u}_1,\qquad (cf)(\mathbf{e}_2)=c\mathbf{u}_2,\qquad (cf)(\mathbf{e}_3)=c\mathbf{u}_3.$$

Hence,

$$\text{Mat}\,cf=\langle c\mathbf{u}_1,c\mathbf{u}_2,c\mathbf{u}_3\rangle=c\langle\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\rangle=c\,\text{Mat}\,f.$$

The proofs of (c) and (d) both follow directly from the interpolation theorem. If $f\neq g$ then f and g must differ on some standard basis vector; otherwise they would be equal by the interpolation theorem. Hence if $f\neq g$, then $\text{Mat}\,f\neq\text{Mat}\,g$, and this proves (c). For the proof of (d), if $A=\langle\mathbf{u}_1,\ldots,\mathbf{u}_n\rangle$ then by the interpolation theorem there exists a linear function f from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that $f(\mathbf{e}_1)=\mathbf{u}_1,\ldots,f(\mathbf{e}_n)=\mathbf{u}_n$, and hence $\text{Mat}\,f=A$.

Our isomorphism theorem implies that the spaces

(1) $m\times n$ matrices
(2) linear functions from $\mathbf{R}^n$ to $\mathbf{R}^m$

are, considering only vector space properties, indistinguishable. Also, for each matrix property there is a corresponding property for linear functions. This will be exploited throughout our study of linear functions and matrices.

If f and g are linear, then

$$(u_o'+f)+(v_o'+g)=(u_o'+v_o')+(f+g).$$

It follows that the constant matrix of the sum of two affine functions is the sum of their constant matrices. The linear matrix of the sum, for the same reason, is the sum of the linear matrices. Similar conclusions can be drawn from the equation

$$c(u_o'+f)=cu_o'+cf.$$

Thus, matrix operations are applicable to affine functions.

Example 3.2 The constant, linear matrices of the sum of the affine functions $u_o + f$, $v_o' + g$ where

$$u_o' = \langle 1, 3\rangle, \qquad \text{Mat } f = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},$$

and

$$v_o' = \langle 2, -1\rangle, \qquad \text{Mat } g = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$$

are, respectively,

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

Questions

1. A 2×3 matrix has ______ row and ______ column vectors.
2. The 12-entry of $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ is ______.
3. If f is linear from $\mathbf{R}^2$ to $\mathbf{R}^2$, then the column vectors of Mat f are ______ and ______.
4. If Mat $f = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, then $f(\langle 0, 1\rangle) =$ ______.
5. The domain of Mat is a vector space of ______.
   (a) matrices,
   (b) functions,
   (c) tuples.

Problems

1. Do Problem Set B at the end of the chapter.

Proofs

1. Prove that the ij-entry of Mat f is $\mathbf{e}_i \cdot f(\mathbf{e}_j)$.
2. Justify for $n = 2$ the steps of the following proof that matrix addition is commutative:

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle + \langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2\rangle = \langle \mathbf{v}_1 + \mathbf{u}_1, \mathbf{v}_2 + \mathbf{u}_2\rangle = \langle \mathbf{v}_1, \mathbf{v}_2\rangle + \langle \mathbf{u}_1, \mathbf{u}_2\rangle.$$

3. Prove for $n = 2$ that Mat $(f + g) =$ Mat $f +$ Mat g.

4. Composition and Product Operations

In this section we shall introduce the *composition* operation on functions and its corresponding *product* operation on matrices. The composition operation comes into play whenever a second function is applied to the image of a first function.

Definition of Composition

Given f in $F(X, Y)$ and g in $F(Y', Z)$, where Y is a subset of Y', the composition, $g \circ f$, of g and f is the function from X to Z having the following rule for each x in X.

$$g \circ f(x) = g(f(x)).$$

The composition $g \circ f$ is accomplished by successively applying the rules of f and g. This may be symbolized

$$x \to f(x) \to g(f(x)).$$

Given rules or formulas for f and g, it is achieved by substituting the rule of f into the rule for g. It may then be symbolized,

$$g \circ f(x) = g(y)\Big|^{y=f(x)}.$$

Example 4.1

(a) Let $f(x) = \sqrt{x}$, $g(y) = \sin y$, and $X = Y = Y' = Z =$ the set of real numbers. Then

$$g \circ f(x) = \sin y\Big|^{y=\sqrt{x}} = \sin \sqrt{x}.$$

(b) Let

$$f(x_1, x_2) = \langle x_1 - x_2, 2x_1 + x_2\rangle, \text{ and } g(y_1, y_2) = y_1 + 3y_2.$$

Then

$$\begin{aligned} g \circ f(x_1, x_2) &= y_1 + 3y_2\Big|^{\langle y_1, y_2\rangle = \langle x_1 - x_2,\, 2x_1 + x_2\rangle} \\ &= x_1 - x_2 + 3(2x_1 + x_2), \\ &= 7x_1 + 2x_2. \end{aligned}$$

Because of the domain and range restrictions on the center set of the composition, the existence of $g \circ f$ need not imply that $f \circ g$ be defined. Even when both $g \circ f$ and $f \circ g$ are defined, they may not be equal.

Example 4.2 If $f(x) = x - 1$ and $g(y) = 2y$, then

$$g \circ f(x) = 2y \Big|^{y=x-1} = 2x - 2$$

and

$$f \circ g(y) = x - 1 \Big|^{x=2y} = 2y - 1.$$

Therefore $f \circ g$ and $g \circ f$ do not have the same rule.

The composition operation is associative, however, and a proof is shown below.

$$\begin{aligned}(f \circ g) \circ h(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x)) \\ &= f((g \circ h)(x)) \\ &= (f \circ (g \circ h))(x).\end{aligned}$$

The composition of two linear functions is again linear as stated formally in the next proposition.

Proposition 4.1 If f is a linear function from $\mathbf{V}$ to $\mathbf{V}'$, and g a linear function from $\mathbf{V}'$ to $\mathbf{V}''$, then $g \circ f$ is a linear function from $\mathbf{V}$ to $\mathbf{V}''$.

We shall prove the additive property part of the linearity of $g \circ f$ in Proposition 4.1. If $\mathbf{u}$, $\mathbf{v}$ are in $\mathbf{V}$, then

$$\begin{aligned}g \circ f(\mathbf{u} + \mathbf{v}) &= g(f(\mathbf{u} + \mathbf{v})) && \text{comp fcn} \\ &= g(f(\mathbf{u}) + f(\mathbf{v})) && \text{def linear} \\ &= g(f(\mathbf{u})) + g(f(\mathbf{v})) && \text{def linear} \\ &= g \circ f(\mathbf{u}) + g \circ f(\mathbf{v}) && \text{comp fcn}\end{aligned}$$

We next seek a way to determine $\text{Mat}(g \circ f)$ from Mat g and Mat f. The result will provide a systematic way of substituting the rule of one linear function into the rule of another. For the proof we shall consider the case of the product of a 2×3 and a 3×2 matrix. Let

$$\text{Mat}\, f = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}; \qquad \text{Mat}\, g = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix};$$

and

$$\text{Mat } g \circ f = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} g \circ f(\mathbf{e}_1) &= g(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3) \\ &= a_{11}g(\mathbf{e}_1) + a_{21}g(\mathbf{e}_2) + a_{31}g(\mathbf{e}_3) \\ &= a_{11}(b_{11}\mathbf{e}_1 + b_{21}\mathbf{e}_2) + a_{21}(b_{12}\mathbf{e}_1 + b_{22}\mathbf{e}_2) \\ &\quad + a_{31}(b_{13}\mathbf{e}_1 + b_{23}\mathbf{e}_2) \\ &= (a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13})\mathbf{e}_1 + (a_{11}b_{21} + \\ &\quad a_{21}b_{22} + a_{31}b_{23})\mathbf{e}_2 ; \end{aligned}$$

and, hence,

$$c_{11} = a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13}$$

and

$$c_{21} = a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23}.$$

It may be noted that c_{11} is the dot product of the first row vector of Mat g and first column vector of Mat f. Also, c_{21} is the dot product of the second row of Mat g and the first column of Mat f. In order to state our conclusion generally, we shall introduce the following definition.

Definition of Matrix Product

If A is an $m \times n$ matrix and B is an $l \times m$ matrix, then the product *BA of A and B is the $l \times n$ matrix whose ij-entry is the dot product of the ith row of A and jth column of B.*

The matrix product is written in the nonstandard form BA, instead of AB, in order to conform to the composite $g \circ f$ of f and g. The requirement in the above definition that the number, m, of rows of A and columns of B be equal corresponds to the condition for $g \circ f$ that the dimensions of the range of f and the domain of g are equal. We now state the fundamental result which connects the matrix product and function composition.

Proposition 4.2 $\quad \text{Mat}(g \circ f) = (\text{Mat } g)(\text{Mat } f).$

Example 4.3 Given the two matrices

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix},$$

the 11-entry of BA is $\langle 1, 3, 4\rangle \cdot \langle 1, 0, 2\rangle = 9$; the 32-entry is $\langle 2, 1, 3\rangle \cdot \langle 3, 1, 4\rangle = 19$. Continuing this process gives the matrix

$$BA = \begin{bmatrix} 9 & 22 \\ 4 & 9 \\ 8 & 19 \end{bmatrix}.$$

It is often convenient to identify the n-tuple $\mathbf{u}$ with the $n \times 1$ matrix $\langle \mathbf{u}\rangle$. The matrix product can then be used to compute the f-image of a vector according to the following formula.

Proposition 4.3 $f(\mathbf{u}) = (\operatorname{Mat} f)\mathbf{u}$.

An example will precede the proof.

Example 4.4 Let f be the linear function from $\mathbf{R}^3$ to $\mathbf{R}^2$ such that $f(\mathbf{i}) = \langle 3, 2\rangle$, $f(\mathbf{j}) = \langle 1, 4\rangle$, and $f(\mathbf{k}) = \langle 2, -5\rangle$. Then $f(\langle 2, 0, 5\rangle)$ can be evaluated by the product

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ -21 \end{bmatrix}.$$

Hence, $f(\langle 2, 0, 5\rangle) = \langle 16, -21\rangle$.

A proof of $f(\mathbf{u}) = (\operatorname{Mat} f)\mathbf{u}$ will be given for the case where

$$\operatorname{Mat} f = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Then

$$\begin{aligned} f(\mathbf{u}) &= f(\langle c_1, c_2\rangle) \\ &= f(c_1\mathbf{e}_1 + c_2\mathbf{e}_2) \\ &= c_1 f(\mathbf{e}_1) + c_2 f(\mathbf{e}_2) \\ &= c_1(\langle a_{11}, a_{21}\rangle) + c_2(\langle a_{12}, a_{22}\rangle) \\ &= \langle c_1 a_{11} + c_2 a_{12}, c_1 a_{21} + c_2 a_{22}\rangle \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= (\operatorname{Mat} f)\mathbf{u}. \end{aligned}$$

The various properties of the matrix product are most easily proved using the correspondence with the composition of functions. We shall prove the

associative property, $C(BA) = (CB)A$ using the corresponding property $h \circ (g \circ f) = (h \circ g) \circ f$. In the proof, we let Mat $f = A$, Mat $g = B$, Mat $h = C$:

$$\begin{aligned} C(BA) &= (\text{Mat } h)(\text{Mat } g \text{ Mat } f) \\ &= (\text{Mat } h)(\text{Mat}(g \circ f)) \\ &= \text{Mat } (h \circ (g \circ f)) \\ &= \text{Mat } ((h \circ g) \circ f) \\ &= \text{Mat } (h \circ g) \text{ Mat } f \\ &= (\text{Mat } h \text{ Mat } g) \text{ Mat } f \\ &= (CB)A. \end{aligned}$$

Other properties which may be proved using similar correspondences are included in the next result.

Proposition 4.4 (a) $(C + B)A = CA + BA$,
(b) $C(B + A) = CB + CA$,
(c) $c(BA) = (cB)A = B(cA)$.

Part (a) of Proposition 4.4 corresponds to the property $(h + g) \circ f = h \circ f + g \circ f$, which is proved for linear functions f, g, and h as follows.

$$\begin{aligned} ((h + g) \circ f)(\mathbf{u}) &= (h + g)((f(\mathbf{u})) && \text{comp fcn} \\ &= h(f(\mathbf{u})) + g(f(\mathbf{u})) && \text{add fcn} \\ &= h \circ f(\mathbf{u}) + g \circ f(\mathbf{u}) && \text{comp fcn} \\ &= (h \circ f + g \circ f)(\mathbf{u}). && \text{add fcn} \end{aligned}$$

With small modifications, many of the results on composition of linear functions apply to composition of affine functions. The composition of two affine functions is again affine, as evidenced by the next result (see Figure 5.2), where f and g are assumed linear.

Proposition 4.5 $(u_o'' + g) \circ (u_o' + f) = [u_o'' + g(u_o')] + g \circ f$.

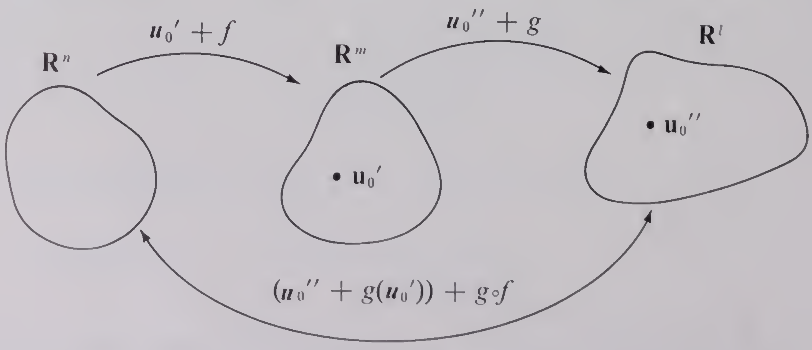


Figure 5.2

Matrices may be used for affine-function computations. The next example illustrates the technique to be used.

Example 4.5 Given the affine function $u_o' + f$, where

$$\mathbf{u}_o' = \langle 1, 4 \rangle, \qquad \text{Mat } f = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 5 & -2 \end{bmatrix};$$

the image of an arbitrary $\langle x, y, z \rangle$ is given by the expression

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

If $u_o'' + g$ is a second affine function, where

$$u_o'' = \langle 3, 0, 5 \rangle, \qquad \text{Mat } g = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 0 & 1 \end{bmatrix};$$

then, by Proposition 4.5, the composition image of an arbitrary $\langle x, y, z \rangle$ is given by

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 21 \\ 9 \end{bmatrix} + \begin{bmatrix} 13 & 13 & 14 \\ 13 & 26 & -4 \\ 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Questions

1. The composition operation is ______.
   (a) commutative,
   (b) associative.
2. The composition $g \circ f$ can always be defined when ______.
   (a) the domain of g equals the range of f,
   (b) the domain of f equals the range of g,
   (c) the ranges of f, g are equal.
3. If $f(x) = x^2$ and $g(y) = e^y$, then $g \circ f(x) =$ ______ and $f \circ g(y) =$ ______.
4. The matrix product BA is defined when the number of ______ of A and ______ of B are equal.
5. The matrix property $c(BA) = (cB)A$ corresponds to the function property ______.
6. The tuple $\langle 1, 3 \rangle$ is identified with the matrix ______.

Problems

1. Do Problem set C at the end of the chapter.

Exercises

1. By (a) substitution and (b) matrix operations find $g \circ f(\langle x_1, x_2\rangle)$ if

$$f(\langle x_1, x_2\rangle) = \langle x_1 - 2x_2, 1 + x_1 + 3x_2\rangle$$

and

$$g(\langle y_1, y_2\rangle) = \langle y_1 - y_2, y_1 + 2y_2\rangle.$$

Proofs

1. Prove that $\text{Mat}(g \circ f) = \text{Mat}\, g\, \text{Mat}\, f$ if f is from $\mathbf{R}^2$ to $\mathbf{R}^2$ and g from $\mathbf{R}^2$ to $\mathbf{Re}$.
2. Prove that $C(B + A) = CB + CA$ using the corresponding property for linear functions.

5. *Identity and Inverse Matrices*

In general, an identity with respect to an operation combines with each element x to give that same x. In any vector space the identity with respect to addition is the zero vector. Our interest here is the identity function with respect to the composition operation. We begin by defining the *identity function* from a set X to itself by the rule, valid for all x in X,

$$\text{id}_X(x) = x.$$

The justification for the use of the term "identity" follows from the equalities shown here.

(a) $f \circ \text{id}_X = f$ (X = domain of f)
(b) $\text{id}_X \circ f = f$ (X = range of f)

The equality chain

$$f \circ \text{id}_X(x) = f(\text{id}_X(x)) = f(x)$$

proves (a); for (b) see Proofs, exercise 1. If $\mathbf{V}$ is a vector space, then $\text{id}_\mathbf{V}$ is linear, as seen from the equalities

$$\text{id}_\mathbf{V}(\mathbf{u} + \mathbf{v}) = \mathbf{u} + \mathbf{v} = \text{id}_\mathbf{V}(\mathbf{u}) + \text{id}_\mathbf{V}(\mathbf{v}),$$
$$\text{id}_\mathbf{V}(c\mathbf{u}) = c\mathbf{u} = c\,\text{id}_\mathbf{V}(\mathbf{u}).$$

The matrix of $\mathrm{id}_{\mathbf{R}^n}$ is called the *identity* $n \times n$ matrix and is given by

$$I^n = \langle \mathrm{id}_{\mathbf{R}^n}(\mathbf{e}_1), \ldots, \mathrm{id}_{\mathbf{R}^n}(\mathbf{e}_n) \rangle = \langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle.$$

This matrix has 1 in each ij-entry where $i = j$, and 0 elsewhere. By correspondence with $\mathrm{id}_{\mathbf{R}^n}$, or directly, it may be verified that the following equations are valid.

$$\begin{aligned} I^n A &= A \qquad (A \text{ has } n \text{ rows}), \\ AI^n &= A \qquad (A \text{ has } n \text{ columns}). \end{aligned}$$

The existence of an identity element, such as our identity function or matrix, usually brings forth an investigation of inverses. The inverse, in general, combines with a given element to produce the identity. Not all elements can generally be expected to have an inverse. Our interest here concerns the inverse of a linear function with respect to the composition operation. The conditions for which a given linear function has an inverse will be studied in the next chapter.

Given a function f from X to Y, we mean by its *inverse* a function f^{-1} from Y to X such that

(a) $f^{-1} \circ f = \mathrm{id}_X$,
(b) $f \circ f^{-1} = \mathrm{id}_Y$.

A function which has an inverse is called *invertible*. If $f(x) = y$ and f^{-1} exists, then

$$f^{-1}(y) = f^{-1}(f(x)) = \mathrm{id}_X(x) = x.$$

This suggests the next result.

Proposition 5.1 The following statements are equivalent.
(a) f^{-1} is the inverse of f,
(b) $f(x) = y$ if and only if $f^{-1}(y) = x$ (for all x, y).

It may be noted that Proposition 5.1 implies the uniqueness of an inverse function. The next property follows from symmetry in the definition of f^{-1}.

Proposition 5.2 $(f^{-1})^{-1} = f$.

If f and g are invertible and $g \circ f$ is defined, then $g \circ f$ is also invertible with its inverse given by the equation set forth below.

Proposition 5.3 $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

A proof of Proposition 5.3 proceeds as follows.

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathrm{id} \circ g^{-1} = g \circ g^{-1} = \mathrm{id},$$

where id denotes the identity with respect to a suitable domain. The reverse equality

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{id}$$

is shown similarly (see Proofs, exercise 2). The fundamental result for linear functions says that the inverse of a linear function is again linear.

Proposition 5.4 If f is an invertible, linear function from $\mathbf{V}$ to $\mathbf{V}'$, then f^{-1} is linear.

The additive portion of the linearity property will be proved for f^{-1}. For $\mathbf{u}'$ and $\mathbf{v}'$ in $\mathbf{V}'$ let $f^{-1}(\mathbf{u}') = \mathbf{u}$, $f^{-1}(\mathbf{v}') = \mathbf{v}$. Then

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) = \mathbf{u}' + \mathbf{v}',$$

and hence

$$f^{-1}(\mathbf{u}' + \mathbf{v}') = \mathbf{u} + \mathbf{v} = f^{-1}(\mathbf{u}') + f^{-1}(\mathbf{v}'),$$

as desired. For the proof of $f^{-1}(c\mathbf{u}') = cf^{-1}(\mathbf{u}')$ see Proofs, exercise 3.

Corresponding to the inverse of functions is the inverse of matrices with respect to the product operation. The following matrix equalities, which are meaningful only for *square matrices* (having an equal number of rows and columns), characterize the inverse A^{-1} of an $n \times n$ matrix A.

$$A^{-1}A = AA^{-1} = I^n.$$

Square matrices which have an inverse are called *nonsingular*; those which do not are called *singular*. The correspondence between matrices and linear functions implies that if Mat $f = A$ then A is nonsingular if and only if f is invertible. Furthermore, the equality,

$$(\text{Mat } f)^{-1} = \text{Mat } f^{-1},$$

is valid when meaningful. Properties of the matrix inverse corresponding to those of linear functions are given by the proposition below.

Proposition 5.5 (a) $(A^{-1})^{-1} = A$,
(b) $(BA)^{-1} = A^{-1}B^{-1}$.

There are various methods of computing the inverse of a nonsingular matrix. An elementary technique requires the solving of a system of equations. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

then $AA^{-1} = I^2$ yields the equations

$$a_{11}x_{11} + a_{12}x_{21} = 1, \qquad a_{11}x_{12} + a_{12}x_{22} = 0,$$
$$a_{21}x_{11} + a_{22}x_{21} = 0, \qquad a_{21}x_{12} + a_{22}x_{22} = 1.$$

This system may be solved for $x_{11}, x_{21}, x_{12}, x_{22}$ to give

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

provided $a_{11}a_{22} - a_{21}a_{12} \neq 0$. This technique can be made reasonably effective for large size matrices by using efficient methods involving matrix techniques in solving the systems of equations.

Example 5.1 The inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is

$$\frac{1}{(1)(4) - (2)(3)} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

An affine function is invertible provided its linear portion is invertible. Furthermore, if f is invertible, then we have the following result (see Proofs, exercise 4).

Proposition 5.6 $(u_o' + f)^{-1} = -f^{-1}(u_o') + f^{-1}$.

Example 5.2 Given the affine function $u_o' + f$, where

$$\mathbf{u}_o' = \langle 1, 3 \rangle, \ \mathrm{Mat}\, f = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix},$$

then $\mathrm{Mat}\, f^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ and

$$(u_o' + f)^{-1}(\langle x, y \rangle) = -\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \langle 2 + 7x - 3y, -1 - 2x + y \rangle.$$

Questions

1. The function f^{-1} is the inverse of f with respect to the ________ operation.
2. A(n) ________ function has an inverse.
3. A(n) ________ matrix has an inverse.
4. The composition of a function and its inverse is the ________ function.

Problems

1. Do Problem Set D at the end of the chapter.

Exercises

1. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ by solving systems of equations.

Proofs

1. Prove that $\mathrm{id}_X \circ f = f$.
2. Prove that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}$.
3. Prove that if f is linear and invertible, then $f^{-1}(c\mathbf{u}') = cf^{-1}(\mathbf{u}')$.
4. Prove that if f is linear and invertible with $f(\mathbf{u}_o) = \mathbf{u}_o'$, then $(u_o' + f) \circ (-u_o + f^{-1}) = \mathrm{id}$.

6. Adjoint and Transpose

In this section f will denote a linear function from $\mathbf{V}$ to $\mathbf{V}'$, where $\mathbf{V}$ and $\mathbf{V}'$ are Euclidean vector spaces. In this case, there is an associated linear function from $\mathbf{V}'$ to $\mathbf{V}$ called its *adjoint* and denoted f^*. The adjoint plays a useful role in the study of rotations and quadratic forms which will come later. Its definition appears unnatural, although it has a simple matrix correspondence. The existence of the adjoint is justified by the following result (see Figure 5.3).

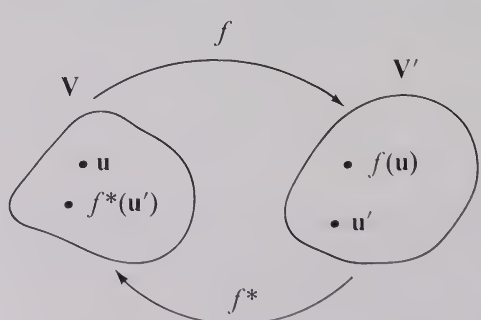


Figure 5.3

Proposition 6.1 There is a unique linear function, f^*, from $\mathbf{V}'$ to $\mathbf{V}$ such that for each $\mathbf{u}$ in $\mathbf{V}$ and $\mathbf{u}'$ in $\mathbf{V}'$

$$f(\mathbf{u}) \cdot \mathbf{u}' = \mathbf{u} \cdot f^*(\mathbf{u}').$$

We consider the proof of Proposition 6.1 for the case where $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis of $\mathbf{V}$. Then f^* may be defined by the expression

$$f^*(\mathbf{u}') = (f(\mathbf{u}_1) \cdot \mathbf{u}')\mathbf{u}_1 + (f(\mathbf{u}_2) \cdot \mathbf{u}')\mathbf{u}_2$$

for each $\mathbf{u}'$ in $\mathbf{V}$. If $\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ is an arbitrary vector in $\mathbf{V}$, then

$$\begin{aligned}\mathbf{u} \cdot f^*(\mathbf{u}') &= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2) \cdot [(f(\mathbf{u}_1) \cdot \mathbf{u}')\mathbf{u}_1 + (f(\mathbf{u}_2) \cdot \mathbf{u}')\mathbf{u}_2] \\ &= a_1(f(\mathbf{u}_1) \cdot \mathbf{u}') + a_2(f(\mathbf{u}_2) \cdot \mathbf{u}') \\ &= (a_1 f(\mathbf{u}_1) + a_2 f(\mathbf{u}_2)) \cdot \mathbf{u}' \\ &= f(\mathbf{u}) \cdot \mathbf{u}'.\end{aligned}$$

For the linearity and uniqueness of f^* see Proofs, exercises 2 and 3. The function f^* is called the *adjoint* of f, and is characterized by the following restatement of Proposition 6.1.

Proposition 6.2 If g is a function from $\mathbf{V}'$ to $\mathbf{V}$, then $g = f^*$ if and only if $f(\mathbf{u}) \cdot \mathbf{u}' = \mathbf{u} \cdot g(\mathbf{u}')$ for all $\mathbf{u}$ in $\mathbf{V}$ and $\mathbf{u}'$ in $\mathbf{V}'$.

We may apply Proposition 6.2 to obtain proofs of the following properties.

Proposition 6.3

(a) $(cf)^* = cf^*$ (b) $(f+g)^* = f^* + g^*$,
(c) $(f^*)^* = f$, (d) $\mathrm{id}_V{}^* = \mathrm{id}_V$,
(e) $(g \circ f)^* = f^* \circ g^*$, (f) $(f^*)^{-1} = (f^{-1})^*$.

A proof of (b), using Proposition 6.2 is

$$\begin{aligned}(f+g)(\mathbf{u}) \cdot \mathbf{u}' &= f(\mathbf{u}) \cdot \mathbf{u}' + g(\mathbf{u}) \cdot \mathbf{u}' \\ &= \mathbf{u} \cdot f^*(\mathbf{u}') + \mathbf{u} \cdot g^*(\mathbf{u}') \\ &= \mathbf{u} \cdot (f^* + g^*)(\mathbf{u}').\end{aligned}$$

A proof of (e) is

$$\begin{aligned}g \circ f(\mathbf{u}) \cdot \mathbf{u}'' &= g(f(\mathbf{u})) \cdot \mathbf{u}'' \\ &= f(\mathbf{u}) \cdot g^*(\mathbf{u}'') \\ &= \mathbf{u} \cdot f^*(g^*(\mathbf{u}'')) \\ & \qquad f^* \circ g^*)(\mathbf{u}'').\end{aligned}$$

If f is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$, then f^* is a linear function from $\mathbf{R}^m$ to $\mathbf{R}^n$. Thus, Mat f is an $m \times n$ matrix and Mat f^* an $n \times m$ matrix. We let

$$[a_{ij}] = \text{Mat } f \quad \text{and} \quad [b_{ij}] = \text{Mat } f^*,$$

and seek a relationship between the entries of Mat f and Mat f^*. From the equation,

$$a_{ij} = \mathbf{e}_i \cdot f(\mathbf{e}_j) = f^*(\mathbf{e}_i) \cdot \mathbf{e}_j = b_{ji},$$

it follows that the ij-entry of Mat f equals the ji-entry of Mat f^*. Therefore, the column vectors of Mat f are the row vectors of Mat f^*. The converse is also true. We define the *transpose* A^* of an $m \times n$ matrix A to be the $n \times m$ matrix whose ith column vector is the ith row vector of A. This gives the result shown below.

Proposition 6.4 Mat $f^* = (\text{Mat } f)^*$.

Example 6.1 If $f(\mathbf{i}) = \langle 3, 0, 2 \rangle$, $f(\mathbf{j}) = \langle 1, 1, 5 \rangle$, then

$$\text{Mat } f = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad \text{Mat } f^* = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 5 \end{bmatrix}.$$

Using the correspondence between the adjoint and the transpose, we have, from Proposition 6.3, the following properties for the transpose.

Proposition 6.5

(a) $(cA)^* = cA^*$, (b) $(A + B)^* = A^* + B^*$,
(c) $(A^*)^* = A$, (d) $(I^n)^* = I^n$,
(e) $(BA)^* = A^*B^*$, (f) $(A^*)^{-1} = (A^{-1})^*$.

Example 6.2 Given $A = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$, then

$$(A^*)^{-1} = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}^{-1} = -\tfrac{1}{2}\begin{bmatrix} 8 & -5 \\ -2 & 1 \end{bmatrix},$$

$$(A^{-1})^* = \left(-\tfrac{1}{2}\begin{bmatrix} 8 & -2 \\ -5 & 1 \end{bmatrix}\right)^* = -\tfrac{1}{2}\begin{bmatrix} 8 & -5 \\ -2 & 1 \end{bmatrix}.$$

This verifies Proposition 6.5(f).

Questions

1. The domain of f is the ______ of f^*.
2. The matrix term corresponding to adjoint is ______.
3. If g is the adjoint of f then $f(\mathbf{u}) \cdot \mathbf{u}' =$ ______.
4. If Mat f is a 2×3 matrix, then Mat f^* is a ______ matrix.
5. The 42-entry of Mat f equals the ______ -entry of Mat f^*.

Exercises

1. Verify that $(A^*)^{-1} = (A^{-1})^*$ if $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.
2. Given Mat $f = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find
   (a) Mat f^*, (b) Mat $(f^*)^*$, (c) Mat $(3f)^*$.
3. Expand, and simplify when possible, the following expressions.
   (a) $(2f + 3g)^*$, (b) $(h^* \circ g \circ f)^*$,
   (c) $(f^{-1} \circ (f^{-1})^*)^{-1}$.
4. Given $f(\langle 1, 0, 0\rangle) = \langle 2, 1\rangle$, $f(\langle 0, 1, 0\rangle) = \langle -1, 2\rangle$, and $f(\langle 0, 0, 1\rangle) = \langle 2, 0\rangle$, find $f^*(\langle 1, 3\rangle)$. (Use matrices.)

Proofs

1. Prove that $(cf)^* = cf^*$. (*Hint*: Show $(cf)(\mathbf{u}) \cdot \mathbf{u}' = \mathbf{u} \cdot (cf^*)(\mathbf{u}')$ for all $\mathbf{u}, \mathbf{u}'$.)
2. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis of $\mathbf{V}$ and f is a linear function from $\mathbf{V}$ to $\mathbf{V}'$, then the function f^* from $\mathbf{V}'$ to $\mathbf{V}$, defined by $f^*(\mathbf{u}') = (f(\mathbf{u}_1) \cdot \mathbf{u}')\mathbf{u}_1 + (f(\mathbf{u}_2) \cdot \mathbf{u}')\mathbf{u}_2$, is linear.
3. Prove that if $\mathbf{u} \cdot g(\mathbf{u}') = \mathbf{u} \cdot f^*(\mathbf{u}')$ for all $\mathbf{u}$ in $\mathbf{V}$ and $\mathbf{u}'$ in $\mathbf{V}'$, then $g = f^*$. (*Hint*: Show $g(\mathbf{u}') - f^*(\mathbf{u}') = \mathbf{v}$ implies $\mathbf{v} \cdot \mathbf{v} = 0$.)

7. Orthogonal Functions; Rotations

In this section we shall study a property of linear functions which can be closely related to the orthogonal property of vectors. This property of functions is used in the study of rotations of the Euclidean plane or 3-dimensional space. It will be assumed here that $\mathbf{V}$ and $\mathbf{V}'$ are Euclidean vector spaces having the same dimension.

Definition of Orthogonal Function

A linear function f from $\mathbf{V}$ *to* $\mathbf{V}'$ *is orthogonal if and only if f is invertible and* $f^{-1} = f^*$

Orthogonal functions preserve an inner product in the sense indicated by the next result.

Proposition 7.1 If f is orthogonal from $\mathbf{V}$ to $\mathbf{V}'$, then for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$

$$f(\mathbf{u}) \cdot f(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

A proof of Proposition 7.1 is

$$f(\mathbf{u}) \cdot f(\mathbf{v}) = \mathbf{u} \cdot f^*(f(\mathbf{v})) = \mathbf{u} \cdot f^{-1}(f(\mathbf{v})) = \mathbf{u} \cdot \mathbf{v}.$$

We can now easily prove the following corollary of Proposition 7.1. It says that orthogonal functions also preserve norms and orthogonality of vectors.

Proposition 7.2 If f is orthogonal, then

(a) $|f(\mathbf{u})| = |\mathbf{u}|$,
(b) $\mathbf{u} \cdot \mathbf{v} = 0$ implies $f(\mathbf{u}) \cdot f(\mathbf{v}) = 0$.

A proof of (a) is

$$|f(\mathbf{u})| = [f(\mathbf{u}) \cdot f(\mathbf{u})]^{1/2} = (\mathbf{u} \cdot \mathbf{u})^{1/2} = |\mathbf{u}|.$$

From properties of the adjoint we also have other properties of orthogonal functions.

Proposition 7.3 If f is orthogonal, then so are

$$\text{(a) } f^{-1} \quad \text{and} \quad \text{(b) } f^*.$$

The proof of (a) is $(f^{-1})^{-1} = (f^*)^{-1} = (f^{-1})^*$.
For (b) see Proofs, exercise 1.

We next investigate the matrix correspondences for orthogonal functions. A nonsingular matrix A is defined to be *orthogonal* provided $A^{-1} = A^*$. Using previous correspondences it can be shown that if $\operatorname{Mat} f = A$, then f is orthogonal if and only if A is orthogonal. Since orthogonal functions preserve norm and orthogonality, an orthogonal function f from $\mathbf{R}^n$ to $\mathbf{R}^n$ yields an orthonormal set $\{f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\}$. This set comprises the column vectors of A, and this proves the following proposition.

Proposition 7.4 If A is an orthogonal matrix, then the column vectors of A are an orthonormal set.

Corresponding to Proposition 7.3 we find that the next proposition is true.

Proposition 7.5 If A is orthogonal then so are (a) A^{-1}, (b) A^*.

It is a corollary of Propositions 7.4 and 7.5(b) that if A is orthogonal then the row vectors of A (which are the column vectors of A^*) form an orthonormal set.

Example 7.1 We seek values of a and b so that $\begin{bmatrix} \frac{3}{5} & a \\ \frac{4}{5} & b \end{bmatrix}$ is an orthogonal matrix. The condition that the column vectors form an orthonormal set gives

$$1 = |\langle a, b\rangle| = \sqrt{a^2 + b^2},$$
$$0 = \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle \cdot \langle a, b\rangle = \tfrac{1}{5}(3a + 4b).$$

These equations yield

$$a = \tfrac{4}{5}, b = -\tfrac{3}{5}; \qquad a = -\tfrac{4}{5}, b = \tfrac{3}{5}.$$

Each of these may be verified to give an orthogonal matrix.

Example 7.2 We seek values of a, b, c, d to make the matrix

$$\begin{bmatrix} 2/3 & 0 & b \\ 2/3 & a & c \\ 1/3 & 2/\sqrt{5} & d \end{bmatrix}$$

orthogonal. Since the dot product of the first two columns must be 0,

$$\tfrac{2}{3}a + \frac{2}{3\sqrt{5}} = 0,$$

from which $a = -\sqrt{1/5}$. Since the third column is orthogonal to each of the first two columns,

$$\langle b, c, d\rangle = \pm \langle \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\rangle \times \langle 0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle,$$

from which $\langle b, c, d\rangle$ has the values $\pm 1/3\sqrt{5}\ \langle 5, -4, -2\rangle$. Each of these gives an orthogonal matrix.

We next study functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ which rotate each vector through a fixed angle θ. Application of this requirement to $\mathbf{i}$ and $\mathbf{j}$ gives the f-image vectors (see Figure 5.4).

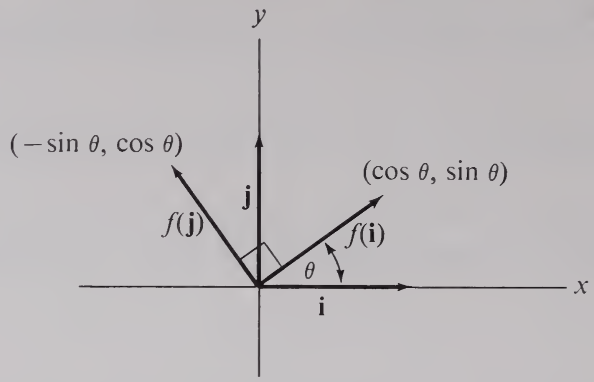


Figure 5.4

$$f(\mathbf{i}) = \langle \cos\theta, \sin\theta \rangle, f(\mathbf{j}) = \langle -\sin\theta, \cos\theta \rangle.$$

Hence,

$$\operatorname{Mat} f = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and f is called a *rotation* of angle θ. It is evident that Mat f is orthogonal. There is another type of orthogonal function on $\mathbf{R}^2$. If $f(\mathbf{i})$ and $f(\mathbf{j})$ have the positions shown in Figure 5.5 then they again form an orthonormal set and

$$\operatorname{Mat} f = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

In this case f is called a *reflection*. If $\theta = 0$, then f describes a reflection about the x axis; if $\theta = \pi$, then f describes a reflection about the y axis.

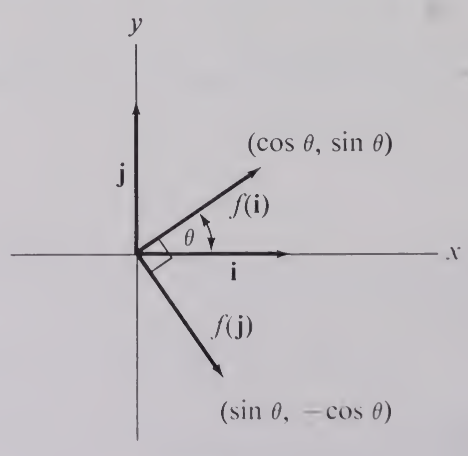


Figure 5.5

We next consider rotations and reflection of $\mathbf{R}^3$; these are again described by orthogonal functions since a rotation or reflection of the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ vector system produces an orthonormal set. Let

$$f(\mathbf{i}) = \mathbf{u}_1 = \mathbf{OP},\ f(\mathbf{j}) = \mathbf{u}_2 = \mathbf{OQ}, \quad \text{and} \quad f(\mathbf{k}) = \mathbf{u}_3 = \mathbf{OR}$$

where f is orthogonal. If $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ form a right hand system, then f is a rotation; if $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ form a left hand system, then f is a reflection. From the previous study of the cross product we can state the following rules.

(a) f is a rotation if $\mathbf{OP} \times \mathbf{OQ} = \mathbf{OR}$.
(b) f is a reflection if $\mathbf{OP} \times \mathbf{OQ} = -\mathbf{OR}$.

A general criterion for distinguishing rotations from reflections among orthogonal matrices uses the determinant, to be studied in Chapter VII. An orthogonal function f is a rotation if the determinant of Mat f is positive, and a reflection if it is negative.

Questions

1. The column vectors of an orthogonal matrix form a(n) _______ set.
2. An orthogonal function from $\mathbf{R}^2$ to $\mathbf{R}^2$ describes either a _______ or _______.
3. If the 3×3 matrix $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$, describes a rotation, then $\langle \mathbf{u}_1, \mathbf{u}_2, -\mathbf{u}_3 \rangle$ describes _______.
   (a) a rotation,
   (b) a reflection,
   (c) neither a rotation nor reflection.

Exercises

1. Complete the lettered entries to give the possible orthogonal matrices, and determine in each case if it is a rotation or a reflection.

   (a) $\begin{bmatrix} 12/13 & a \\ 5/13 & b \end{bmatrix}$, (b) $\begin{bmatrix} -2/3 & 1/\sqrt{5} & b \\ 1/3 & a & c \\ 2/3 & 0 & d \end{bmatrix}$.

2. Give Mat f if f describes from $\mathbf{R}^2$ to $\mathbf{R}^2$,
   (a) a rotation of angle $3\pi/4$,
   (b) a reflection about the x axis,
   (c) a reflection about the y axis followed by a rotation of angle $\pi/4$.

Proofs

1. Prove that if f is orthogonal then so is f^*.
2. Prove that every reflection of $\mathbf{R}^2$ can be obtained as a reflection about the x axis followed by a rotation.
3. Prove that

$$\operatorname{Mat} f = \frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

if f is the reflection of $\mathbf{R}^2$ about the line $y = mx$. (The line is the perpendicular bisector of the segment from $\mathbf{u}$ to $f(\mathbf{u})$ for each $\mathbf{u}$.)

8. *Matrices Relative to Different Bases*

Our definition of Mat f uses the f values at the standard basis vectors. Sometimes, simpler matrices may be obtained using the f values at other basis vectors. An application of this occurs later in the study of conics whose axes are not parallel to the rectangular coordinate axes.

We let f be a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and

$$\mathbf{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\},\ \mathbf{B}' = \{\mathbf{u}_1', \ldots, \mathbf{u}_m'\}$$

respectively ordered bases of $\mathbf{R}^n$, $\mathbf{R}^m$. Then the *matrix* of f *relative* to $\mathbf{B}$ and $\mathbf{B}'$, to be denoted $\operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f$, is the $m \times n$ matrix whose *ij*-entry is a_{ij}, where

$$f(\mathbf{u}_j) = a_{1j}\mathbf{u}_1' + \cdots + a_{mj}\mathbf{u}_m'.$$

Thus a_{ij} is the ith coordinate of $f(\mathbf{u}_j)$ relative to $\mathbf{B}'$. If $\mathbf{B}$ and $\mathbf{B}'$ are the standard bases, then $\operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f = \operatorname{Mat} f$.

Example 8.1 Given ordered bases $\mathbf{B}$, $\mathbf{B}'$ and linear f where

$$\mathbf{B} = \{\langle 1, 1\rangle, \langle 0, 2\rangle\},\quad \mathbf{B}' = \{\langle 1, 2\rangle, \langle 0, 1\rangle\},\quad \operatorname{Mat} f = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},$$

then the matrix equalities,

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad ; \quad \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

give

$$f(\langle 1, 1\rangle) = \langle 2, 4\rangle = 2\langle 1, 2\rangle + 0\langle 0, 1\rangle,$$
$$f(\langle 0, 2\rangle) = \langle 0, 6\rangle = 0\langle 1, 2\rangle + 6\langle 0, 1\rangle.$$

Hence $\operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

The following properties are derived in the same manner as the corresponding properties of Mat f:

Proposition 8.1

(a) $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, cf = c\, \mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f$,
(b) $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\,(f+g) = \mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f + \mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, g$,
(c) $\mathrm{Mat}_{\mathbf{B}''\mathbf{B}}(g \circ f) = (\mathrm{Mat}_{\mathbf{B}''\mathbf{B}'}g)(\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f)$,
(d) $\mathrm{Mat}_{\mathbf{B}\mathbf{B}'}\, f^{-1} = (\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f)^{-1}$.

Some results will now be established in connection with changes of bases. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are bases of $\mathbf{R}^n$, then the *change of basis matrix* from $\mathbf{B}_1$ to $\mathbf{B}_2$ is defined by

$$C_{\mathbf{B}_2\mathbf{B}_1} = \mathrm{Mat}_{\mathbf{B}_2\mathbf{B}_1}\mathrm{id}_{\mathbf{R}^n}.$$

Also if $\mathbf{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbf{R}^n$, then the *matrix of* $\boldsymbol{B}$ is

$$C_{\mathbf{B}} = \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle.$$

Letting $\mathbf{S}$ denote the standard basis of $\mathbf{R}^n$ we have the next proposition.

Proposition 8.2 $\quad C_{\mathbf{B}} = C_{\mathbf{SB}}$.

For $\mathbf{B} = \{\langle a_{11}, a_{21}\rangle, \langle a_{12}, a_{22}\rangle\}$ a proof follows from

$$\mathrm{id}_{\mathbf{R}^2}(\langle a_{11}, a_{21}\rangle) = a_{11}\langle 1, 0\rangle + a_{21}\langle 0, 1\rangle,$$
$$\mathrm{id}_{\mathbf{R}^2}(\langle a_{12}, a_{22}\rangle) = a_{12}\langle 1, 0\rangle + a_{22}\langle 0, 1\rangle.$$

Therefore, the change of basis matrix is easily found when the basis of the range space $\mathbf{R}^m$ is the standard basis. The following formula for finding $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f$ from Mat f will now be established.

Proposition 8.3 $\quad \mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f = C_{\mathbf{B}'}^{-1}(\mathrm{Mat}\, f)C_{\mathbf{B}}$.

For the proof we let $\mathbf{S}$ and $\mathbf{S}'$ be respective standard bases of $\mathbf{R}^n$, $\mathbf{R}^m$. Application of Proposition 8.1(c) and (d) to $f = \mathrm{id}_{\mathbf{R}^m} \circ f \circ \mathrm{id}_{\mathbf{R}^n}$ gives

$$\begin{aligned}\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f &= (\mathrm{Mat}_{\mathbf{B}'\mathbf{S}'}\mathrm{id}_{\mathbf{R}^m})(\mathrm{Mat}_{\mathbf{S}'\mathbf{S}} f)(\mathrm{Mat}_{\mathbf{S}\mathbf{B}}\mathrm{id}_{\mathbf{R}^n})\\ &= (\mathrm{Mat}_{\mathbf{S}'\mathbf{B}'}\mathrm{id}_{\mathbf{R}^m})^{-1}(\mathrm{Mat}\, f)(\mathrm{Mat}_{\mathbf{S}\mathbf{B}}\mathrm{id}_{\mathbf{R}^n})\\ &= C_{\mathbf{B}'}^{-1}(\mathrm{Mat}\, f)C_{\mathbf{B}}.\end{aligned}$$

We set $f = \mathrm{id}_{\mathbf{R}^n}$ in Proposition 8.3 and obtain a useful special case.

Proposition 8.4 $C_{\mathbf{B}'\mathbf{B}} = C_{\mathbf{B}'}^{-1} C_{\mathbf{B}}$.

Example 8.2 (a) If $\mathbf{B} = \{\langle 1, 3\rangle, \langle 0, -1\rangle\}$ and $\mathbf{B}' = \{\langle 0, 2\rangle, \langle 1, 4\rangle\}$, then the change of basis matrix from $\mathbf{B}$ to $\mathbf{B}'$ is

$$\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

(b) In (a) if also $f(\mathbf{i}) = \langle 1, 3\rangle$, $f(\mathbf{j}) = \langle 2, -4\rangle$, then

$$\begin{aligned} \operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f &= \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -37/2 & 6 \\ 7 & -2 \end{bmatrix}. \end{aligned}$$

We next observe the following more general conversion formula for change of basis.

Proposition 8.5 If $\mathbf{B}_1$ and $\mathbf{B}_2$ are bases of $\mathbf{R}^n$ and $\mathbf{B}_1'$, $\mathbf{B}_2'$ are bases of $\mathbf{R}^m$, then

$$\operatorname{Mat}_{\mathbf{B}_2'\mathbf{B}_2} f = C_{\mathbf{B}_2'\mathbf{B}_1'}(\operatorname{Mat}_{\mathbf{B}_1'\mathbf{B}_1} f) C_{\mathbf{B}_1\mathbf{B}_2}.$$

A proof of Proposition 8.5 can be made by solving the equality in Proposition 8.3 to get the expressions,

$$\operatorname{Mat} f = C_{\mathbf{B}_1'}(\operatorname{Mat}_{\mathbf{B}_1'\mathbf{B}_1} f) C_{\mathbf{B}_1}^{-1}$$

$$\operatorname{Mat} f = C_{\mathbf{B}_2'}(\operatorname{Mat}_{\mathbf{B}_2'\mathbf{B}_2} f) C_{\mathbf{B}_2}^{-1}.$$

Equating the expressions for Mat f and application of the property $C_{\mathbf{B}_2\mathbf{B}_1} = C_{\mathbf{B}_2}^{-1} C_{\mathbf{B}_1}$ completes the proof.

It may be desirable to choose $\mathbf{B}'$ or $\mathbf{B}$ so that $\operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f$ has a special form such as the identity matrix. This can be done if Mat f is nonsingular by solving the formula in Proposition 8.3 for $C_{\mathbf{B}'}$ or $C_{\mathbf{B}}$.

Example 8.3 Given $\mathbf{B} = \{\langle 1, 2\rangle, \langle 0, 1\rangle\}$, Mat $f = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, we seek $\mathbf{B}'$ so that $\operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f$ is the identity. From Proposition 8.3,

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \operatorname{Mat}_{\mathbf{B}'\mathbf{B}} f = C_{\mathbf{B}'}^{-1}(\operatorname{Mat} f) C_{\mathbf{B}} \\ &= C_{\mathbf{B}'}^{-1} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

gives

$$C_{\mathbf{B}'} = \begin{bmatrix} 6 & 1 \\ 5 & 1 \end{bmatrix}$$

and hence, $\mathbf{B}' = \{\langle 6, 5\rangle, \langle 1, 1\rangle\}$.

An important special case occurs when $m = n$ and $\mathbf{B} = \mathbf{B}'$. The *matrix of a linear function* f from $\mathbf{R}^n$ to $\mathbf{R}^n$ *relative to a basis* $\mathbf{B}$ of $\mathbf{R}^n$ is defined by

$$\mathrm{Mat}_{\mathbf{B}} f = \mathrm{Mat}_{\mathbf{BB}} f.$$

Setting $\mathbf{B}_1' = \mathbf{B}_1$ and $\mathbf{B}_2' = \mathbf{B}_2$ in Proposition 8.5 gives the next result.

Proposition 8.6 $\quad \mathrm{Mat}_{\mathbf{B}_2} f = (C_{\mathbf{B}_1\mathbf{B}_2})^{-1}(\mathrm{Mat}_{\mathbf{B}_1} f) C_{\mathbf{B}_1\mathbf{B}_2}$.

Example 8.4 Given

$$\mathbf{B}_1 = \{\langle 2, 1\rangle, \langle 1, 0\rangle\},$$

$$\mathbf{B}_2 = \{\langle -1, 2\rangle, \langle 3, 1\rangle\},$$

and

$$\mathrm{Mat}_{\mathbf{B}_1} f = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},$$

then from

$$C_{\mathbf{B}_1\mathbf{B}_2} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix}$$

it follows that

$$\mathrm{Mat}_{\mathbf{B}_2} f = \begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix}$$

$$= \tfrac{1}{7} \begin{bmatrix} 17 & -2 \\ -6 & 18 \end{bmatrix}.$$

The change-of-basis matrix $C_{\mathbf{B}_2\mathbf{B}_1}$ may also be used to obtain a change of coordinates of a vector. A coordinate change occurs, for example, when the coordinate axes in the Cartesian plane are rotated and each point has new coordinates relative to the rotated axes. If $\mathbf{B}_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbf{R}^n$ and $\mathbf{u}$ is in $\mathbf{R}^n$, then the *coordinate vector of* $\mathbf{u}$ *relative to* $\mathbf{B}_1$ is

$$\langle \mathbf{u}\rangle_{\mathbf{B}_1} = \langle a_1, \ldots, a_n\rangle,$$

where $\mathbf{u} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n$. If $\mathbf{B}_2$ is another basis of $\mathbf{R}^n$ then we have:

Proposition 8.7 $\langle \mathbf{u} \rangle_{\mathbf{B}_2} = C_{\mathbf{B}_2\mathbf{B}_1} \langle \mathbf{u} \rangle_{\mathbf{B}_1}$.

Example 8.5 Given $\mathbf{B}_1 = \{\langle 2, 1\rangle, \langle 0, 3\rangle\}$, $\mathbf{B}_2 = \{\langle 1, 1\rangle, \langle 4, 1\rangle\}$. $\langle \mathbf{u} \rangle_{\mathbf{B}_1} = \langle 2, 3\rangle$, it follows from

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 40 \\ -7 \end{bmatrix}$$

that $\langle \mathbf{u} \rangle_{\mathbf{B}_2} = \langle 40/3, -7/3\rangle$.

Example 8.6 Let $\mathbf{u} = \langle x, y\rangle$; also let $\langle x', y'\rangle$ denote coordinates of $\mathbf{u}$ relative to the vectors obtained by rotation of $\mathbf{i}$, $\mathbf{j}$ through a counterclockwise angle θ (see Figure 5.6). Setting

$$\mathbf{B}_1 = \{\mathbf{i}, \mathbf{j}\},\ \mathbf{B}_2 = \{\langle \cos\theta, \sin\theta\rangle, \langle -\sin\theta, \cos\theta\rangle\},$$

we have $\langle x', y'\rangle = \langle \mathbf{u} \rangle_{\mathbf{B}_2}$ and $\langle x, y\rangle = \langle \mathbf{u} \rangle_{\mathbf{B}_1}$. Hence

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

from which equations relating x, y and x', y' can be obtained.

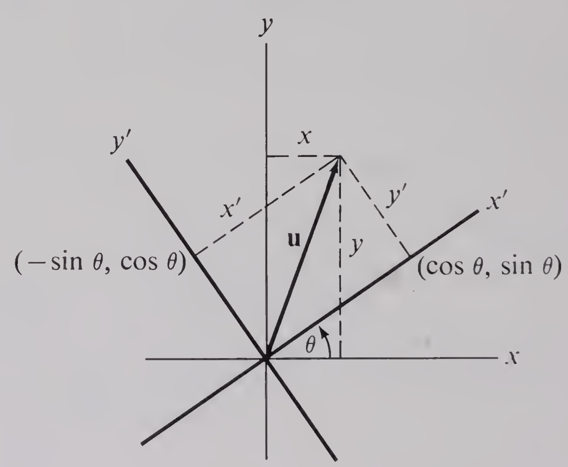


Figure 5.6

We consider now the proof of Proposition 8.6 for

$$\mathbf{B}_1 = \{\mathbf{u}_1, \mathbf{u}_2\},\ \mathbf{B}_2 = \{\mathbf{v}_1, \mathbf{v}_2\};\ \langle \mathbf{u} \rangle_{\mathbf{B}_1} = \langle a_1, a_2\rangle,$$

$$\langle \mathbf{u} \rangle_{\mathbf{B}_2} = \langle b_1, b_2\rangle,\ C_{\mathbf{B}_2\mathbf{B}_1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

From $C_{\mathbf{B}_2\mathbf{B}_1} = \mathrm{Mat}_{\mathbf{B}_2\mathbf{B}_1}\mathrm{id}_{\mathbf{R}^2}$ it may be seen that

$$\mathbf{u}_1 = \mathrm{id}_{\mathbf{R}^2}(\mathbf{u}_1) = a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2,$$
$$\mathbf{u}_2 = \mathrm{id}_{\mathbf{R}^2}(\mathbf{u}_2) = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2.$$

Therefore, $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$

$$= (a_{11}a_1 + a_{12}a_2)\mathbf{v}_1 + (a_{21}a_1 + a_{22}a_2)\mathbf{v}_2.$$

In matrix notation, this gives the desired equality,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Questions

1. $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f$ agrees with $\mathrm{Mat}\, f$ when $\mathbf{B}$, $\mathbf{B}'$ are ______.
2. If $\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, $\mathbf{B}' = \{\mathbf{u}_1', \mathbf{u}_2'\}$, and $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f = [a_{ij}]$, then $f(\mathbf{u}_1) =$ ______.
3. If $\mathbf{B} = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ and the coordinate vector of $\mathbf{u}$ relative to $\mathbf{B}$ is $\langle a_1, a_2 \rangle$, then $\mathbf{u} =$ ______.

Exercises

1. Find $C_{\mathbf{B}}$ if (a) $\mathbf{B} = \{\langle 2, 3\rangle, \langle 5, 6\rangle\}$; (b) $\mathbf{B} = \{\langle 0, 1, 2\rangle, \langle 7, 3, 4\rangle, \langle 1, 1, 3\rangle\}$.
2. Find $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f$ if
   (a) $\mathbf{B} = \{\langle 1, 0\rangle, \langle 1, -1\rangle\}$, $\mathbf{B}' = \{\langle 1, 2\rangle, \langle 1, 1\rangle\}$, $\mathrm{Mat} f = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$;
   (b) $\mathbf{B} = \{\mathbf{i}, \mathbf{j}, \mathbf{i} + \mathbf{k}\}$, $\mathbf{B}' = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and $\mathrm{Mat}\, f$ is the identity matrix.
3. Given $\mathrm{Mat}\ f = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B}' = \{\langle 1, -1\rangle, \langle 2, 0\rangle\}$, find $\mathbf{B}$ so that $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$.
4. Given $\mathrm{Mat}\ f = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \{\langle 1, 0\rangle, \langle 2, 3\rangle\}$, find $\mathbf{B}'$ so that $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}} f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
5. Find the coordinate vector of
   (a) $\langle 3, 2\rangle$ relative to $\{\langle 1, 1\rangle, \langle 2, 0\rangle\}$,
   (b) $\langle 2, 5\rangle$ relative to the axis system obtained by rotating the xy-system through a counterclockwise angle of $5\pi/6$.
6. Given $\mathrm{Mat}\, f = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, find the matrix of f relative to $\{\langle 1, 1\rangle, \langle 2, 0\rangle\}$.

Proofs

1. Prove that if f is a linear function from $\mathbf{R}^2$ to $\mathbf{R}^2$, then $\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, cf = c\mathrm{Mat}_{\mathbf{B}'\mathbf{B}}\, f$.

Problems

A. *Matrices of Affine Functions*

An *affine* function g from $\mathbf{R}^2$ to $\mathbf{R}^2$ has a rule of the form $g(r, s) = \langle x, y\rangle$ where

A.1 $$\begin{aligned} x &= b_1 + a_{11}r + a_{12}s, \\ y &= b_2 + a_{21}r + a_{22}s. \end{aligned}$$

In A.1 the symbols $b_1, b_2, a_{11}, a_{12}, a_{21}, a_{22}$ denote real numbers. This system may be written in matrix form.

A.2 $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}.$$

The matrices,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

are respectively called the *constant* and *linear matrices* of g.

1. Write each system in matrix form.

(a) $\begin{aligned} x &= 2 + 3r - s \\ y &= 4 - 2r + 3s, \end{aligned}$

(b) $\begin{aligned} x &= 4 + 5r + 3s \\ y &= r. \end{aligned}$

2. Describe as a set of equations each of the matrix expressions below.

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$,

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

Matrix descriptions for functions from $\mathbf{R}^n$ to $\mathbf{R}^m$, where m and n are positive integers, assume a similar form. We shall let r, s, t denote domain variables and x, y, z range variables.

3. Write in matrix form each system of equations.

(a) $\begin{aligned} x &= 1 + r - s \\ y &= 2 - r + 3s \\ z &= 4 + r + 2s, \end{aligned}$ (b) $\begin{aligned} x &= 3 - r \\ y &= 2 + 5r \\ z &= 7 - 3r, \end{aligned}$

(c) $\begin{aligned} x &= 1 + r + s + t \\ y &= 2 - r - s + 3t \\ z &= 5 + r - t, \end{aligned}$ (d) $\begin{aligned} x &= 1 - r \\ y &= 3, \end{aligned}$

(e) $\begin{aligned} x &= 1 + r - s + t \\ y &= 1 + 2r + s - t, \end{aligned}$ (f) $\begin{aligned} x &= 1 - r - s \\ y &= 2 + 3r + 2s. \end{aligned}$

Affine functions are completely determined by the values at the origin and standard basis elements. If g is an affine function from $\mathbf{R}^3$ to $\mathbf{R}^3$, then its constant matrix has $g(\mathbf{0})$ as its column vector, and its linear matrix has

$$g(\mathbf{i}) - g(\mathbf{0}), \quad g(\mathbf{j}) - g(\mathbf{0}), \quad \text{and} \quad g(\mathbf{k}) - g(\mathbf{0})$$

as column vectors. Analogous conclusions hold for other dimensions.

4. Give the matrix form of the affine function g from $\mathbf{R}^3$ to $\mathbf{R}^3$ which satisfies the conditions given.

(a) $g(\mathbf{0}) = \langle 1, 1, 0\rangle, g(\mathbf{i}) = \langle 0, 1, 2\rangle, g(\mathbf{j}) = \langle 1, 3, 4\rangle, g(\mathbf{k}) = \langle 2, 0, 1\rangle$;

(b) $g(\mathbf{0}) = \langle 3, 5, 4\rangle, g(\mathbf{i}) = \langle 3, 2, 6\rangle, g(\mathbf{j}) = \langle 1, -1, 2\rangle,$
$g(\mathbf{k}) = \langle 8, 2, 7\rangle$.

5. Give the matrix form of the affine function g with properties as described.

(a) $g(\mathbf{0}) = \langle 1, 2\rangle, g(\mathbf{i}) = \langle 3, 0\rangle, g(\mathbf{j}) = \langle 2, -1\rangle$;

(b) $g(\mathbf{0}) = \langle 1, 6, 2\rangle, g(\mathbf{i}) = \langle 2, -5, 4\rangle, g(\mathbf{j}) = \langle 3, 0, 2\rangle$;

(c) $g(\mathbf{0}) = \langle 1, 1\rangle, g(\mathbf{i}) = \langle 1, 0\rangle, g(\mathbf{j}) = \langle 3, -1\rangle, g(\mathbf{k}) = \langle 0, 0\rangle$.

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6. Write, in matrix form, the following systems of equations.

(a) $\begin{aligned} x &= 1 + r - s + t \\ y &= 2 + r + 3s, \end{aligned}$ (b) $\begin{aligned} x &= 1 - 2r \\ y &= 3 + 7r, \end{aligned}$

(c) $\begin{aligned} x &= 4 - r + 2s \\ y &= 3 - 7r + 5s. \end{aligned}$

7. Give the matrix form and equation form of the affine function g which satisfies the given properties.

   (a) $g(\mathbf{0}) = \langle 1, 3\rangle$, $g(\mathbf{i}) = \langle 2, -1\rangle$, $g(\mathbf{j}) = \langle 3, 6\rangle$, $g(\mathbf{k}) = \langle 8, 0\rangle$;
   (b) $g(\mathbf{0}) = \langle 1, 0, 7\rangle$, $g(\mathbf{i}) = \langle 2, 0, 6\rangle$, $g(\mathbf{j}) = \langle 3, -2, 9\rangle$.

B. *Addition and Scalar Multiplication of Matrices*

Two matrices A and B have the *same size* provided they have the same number of rows and the same number of columns. Matrices of the same size are added in the following way.

B.1 $A + B$ is the matrix obtained from A and B by adding corresponding entries.

1. Add the matrices below.

   (a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & -4 \end{bmatrix}$,

   (c) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ -4 & 3 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ -2 & 5 & 2 \\ 3 & 9 & -8 \end{bmatrix}$.

The scalar product of a matrix is given next.

B.2 cA is the matrix obtained from A by multiplying each entry by c.

2. Find the indicated product.

   (a) $6\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, (b) $2\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$, (c) $3\begin{bmatrix} 1 & 0 & 2 \\ 5 & 1 & 8 \end{bmatrix}$.

3. Find:

   (a) $7\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$, (b) $8\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \end{bmatrix}$,

   (c) $4\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 \\ 1 & 7 \\ 6 & 4 \end{bmatrix}$.

Matrix forms for the sum and scalar multiplication of affine functions are obtained as described in B.3 and B.4.

B.3 The constant matrix of $f + g$ is the sum of the constant matrices of f and g; the linear matrix of $f + g$ is the sum of the linear matrices of f and g.

B.4 The constant and linear matrices of cf are respectively the scalar product of c by the constant and linear matrices of f.

4. Find the matrix form of $f+g$ and $3f$ if f and g have the respective matrix forms shown.

(a) $\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}0&1\\5&3\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1&2\\-3&-5\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix};$

(b) $\begin{bmatrix}2\\0\\1\end{bmatrix}+\begin{bmatrix}1&0\\2&1\\5&3\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}+\begin{bmatrix}3&9\\6&1\\0&2\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}.$

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5. Find the matrix form of $f+g$ and $6f$ if f, g have respective matrix forms given below.

(a) $\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}2&1&-3\\4&0&2\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}+\begin{bmatrix}5&3&9\\-1&2&7\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix};$

(b) $\begin{bmatrix}1\\0\\2\end{bmatrix}+\begin{bmatrix}2&5\\3&7\\1&1\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}, \begin{bmatrix}3\\9\\5\end{bmatrix}+\begin{bmatrix}8&6\\0&1\\7&-3\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}.$

C. *Matrix Product*

The *product BA* of matrices B and A is defined whenever the number of rows of A and columns of B are equal. Then BA has as many rows as B and columns as A. Its entries are computed in the following manner.

C.1 The number in the ith row and jth column of BA is the dot product of the ith row of B and jth column of A.

1. Find the matrix product indicated.

(a) $\begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix},$ (b) $\begin{bmatrix}2&1\\3&1\end{bmatrix}\begin{bmatrix}1&2\\-1&4\end{bmatrix},$

(c) $\begin{bmatrix}3&1&1\\2&1&4\\1&0&3\end{bmatrix}\begin{bmatrix}1&1\\2&1\\1&4\end{bmatrix}.$

The *affine image* of a vector may be obtained from the matrix form by substituting the coordinates of the vector for r, s, t and applying the matrix operations. The column matrices

$$\begin{bmatrix}r\\s\end{bmatrix}, \quad \begin{bmatrix}r\\s\\t\end{bmatrix}$$

are respectively identified with the tuples $\langle r, s\rangle$ and $\langle r, s, t\rangle$.

2. Given that f is affine and has the matrix form

$$\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}2 & 6\\-1 & 5\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix},$$

find

(a) $f(\langle 1, 4\rangle)$, (b) $f(\langle 2, -3\rangle)$.

3. Find $f(\langle 1, 2, 0\rangle)$ if f is affine and has the matrix form

$$\begin{bmatrix}2\\0\\7\end{bmatrix} + \begin{bmatrix}3 & 2 & 6\\0 & 5 & 7\\-1 & 4 & 3\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}.$$

4. Find $f(\langle 1, 0, 3\rangle)$ if f is affine and $f(\mathbf{0}) = \langle 2, 1, 1\rangle$, $f(\mathbf{i}) = \langle 4, 0, 7\rangle$, $f(\mathbf{j}) = \langle 3, 2, -1\rangle$, and $f(\mathbf{k}) = \langle 0, 6, 2\rangle$.

The matrix form of the composition $g \circ f$ of two affine functions having $\mathbf{R}^2$ as both domain and range is given next.

C.2 If f has the matrix form $C_1 + A\begin{bmatrix}r\\s\end{bmatrix}$ and g has the matrix form $C_2 + B\begin{bmatrix}r\\s\end{bmatrix}$, then $g \circ f$ has the matrix form

$$(C_2 + BC_1) + BA\begin{bmatrix}r\\s\end{bmatrix} = C_2 + B\left(C_1 + A\begin{bmatrix}r\\s\end{bmatrix}\right).$$

5. Find the matrix form of $g \circ f$ if f and g are affine and have respective matrix forms as shown.

(a) $\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}3 & 0\\2 & 5\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, $\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}2 & -1\\5 & 6\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$;

(b) $\begin{bmatrix}6\\-2\end{bmatrix} + \begin{bmatrix}9 & 3\\4 & 1\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, $\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}3 & -2\\5 & 8\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$.

Similar conclusions to that in C.2 can be obtained for other dimensions.

6. Find the matrix form of $g \circ f$ if f and g are affine and have respective matrix forms given below.

(a) $\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-1 & 2 & 0\\3 & -9 & 1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$, $\begin{bmatrix}1\\0\\6\end{bmatrix} + \begin{bmatrix}1 & 0\\2 & 4\\-5 & 3\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$;

(b) $\begin{bmatrix}3\\1\\1\end{bmatrix} + \begin{bmatrix}4\\1\\2\end{bmatrix}[r]$, $\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}1 & 1 & 9\\2 & 0 & 6\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$.

7. Find $g \circ f(\langle 1, 3\rangle)$ if f and g are affine functions with the following properties.
(a) $f(\mathbf{0}) = \langle 1, 1\rangle$, $f(\mathbf{i}) = \langle 2, 0\rangle$, $f(\mathbf{j}) = \langle -1, 1\rangle$, $g(\mathbf{0}) = \langle 3, 0\rangle$, $g(\mathbf{i}) = \langle 1, 2\rangle$, and $g(\mathbf{j}) = \langle 3, 1\rangle$;
(b) $f(\mathbf{0}) = \langle 1, 0, 1\rangle$, $f(\mathbf{i}) = \langle 2, 0, -1\rangle$, $f(\mathbf{j}) = \langle 1, 1, -5\rangle$, $g(\mathbf{0}) = \langle 1, 0\rangle$, $g(\mathbf{i}) = \langle 2, -3\rangle$, $g(\mathbf{j}) = \langle 3, -4\rangle$, and $g(\mathbf{k}) = \langle 4, -1\rangle$.

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8. Find $g \circ f(\langle 2, -3\rangle)$ if f and g are affine functions described respectively by the expressions below.

(a) $\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3 & 4\\-1 & 2\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, $\begin{bmatrix}0\\3\end{bmatrix} + \begin{bmatrix}2 & 2\\1 & 5\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$,

(b) $\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, $\begin{bmatrix}1\\0\\2\end{bmatrix} + \begin{bmatrix}0 & 2 & 1\\3 & -1 & 0\\5 & -4 & 1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$,

(c) $f(\mathbf{0}) = \langle 1, 2\rangle$, $f(\mathbf{i}) = \langle 2, 3\rangle$, $f(\mathbf{j}) = \langle 3, 0\rangle$, $g(\mathbf{0}) = \langle 1, 1\rangle$, $g(\mathbf{i}) = \langle 3, -1\rangle$, and $g(\mathbf{j}) = \langle 2, -7\rangle$.

D. *Inverse of Matrices*

The *identity* 2×2 and 3×3 matrices are, respectively:

D.1 $$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

The inverse, A^{-1}, of a 2×2 matrix satisfies the property in D.2.

D.2 $$A^{-1}A = AA^{-1} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

It may be verified that the inverse of $A = \begin{bmatrix}a & c\\b & d\end{bmatrix}$ exists provided $ad - bc \neq 0$, in which case it is given by the following formula.

D.3 $$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -c\\-b & a\end{bmatrix}.$$

1. Find the inverse of each of the given matrices.

(a) $\begin{bmatrix}2 & 1\\-3 & 5\end{bmatrix}$, (b) $\begin{bmatrix}3 & 4\\6 & 1\end{bmatrix}$.

The formula in D.3 can be obtained by starting with the matrix equality,

$$\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\x_{21} & x_{22}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$

and solving for the x_{ij}'s. Proceeding in a similar manner with the 3×3 case, it would be possible to find a formula for the inverse of a 3×3 matrix. It is more practical,

however, to systematize the solving of the system of equations than to employ the formula. If this is done, the inverse of a 3×3 matrix, if it exists, can be obtained by the following process.

D.4 (1) Augment A by the identity matrix to obtain

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}.$$

(2) By elementary row operations convert to the form

$$\begin{bmatrix} c & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & c & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & c & b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

(3) The inverse A^{-1} of A is then

$$\frac{1}{c}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

2. Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$,

(a) augment A by the identity matrix,
(b) add (-1) times the first row to the third,
(c) add the second row to the third,
(d) add (-1) times the second row to the first,
(e) multiply the third row by $(1/2)$,
(f) from (e) give A^{-1},
(g) verify AA^{-1} is the identity.

3. Find the inverses of the following matrices.

(a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$.

If f is an affine function, then the *inverse* f^{-1} of f is again an affine function, assuming it exists. It is given in matrix form for functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ as follows:

D.5 If the matrix form of f is

$$C + A\begin{bmatrix} r \\ s \end{bmatrix},$$

then the matrix form of f^{-1} is

$$(-A^{-1}C) + A^{-1}\begin{bmatrix} x \\ y \end{bmatrix}.$$

A similar formula holds for affine functions from $\mathbf{R}^3$ to $\mathbf{R}^3$.

4. Find the matrix form of f^{-1} if f is an affine function with matrix form as given below.

(a) $\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}2 & 0\\-1 & 4\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, (b) $\begin{bmatrix}2\\-1\end{bmatrix} + \begin{bmatrix}6 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$,

(c) $\begin{bmatrix}1\\0\\2\end{bmatrix} + \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$.

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5. Find the inverse of each matrix.

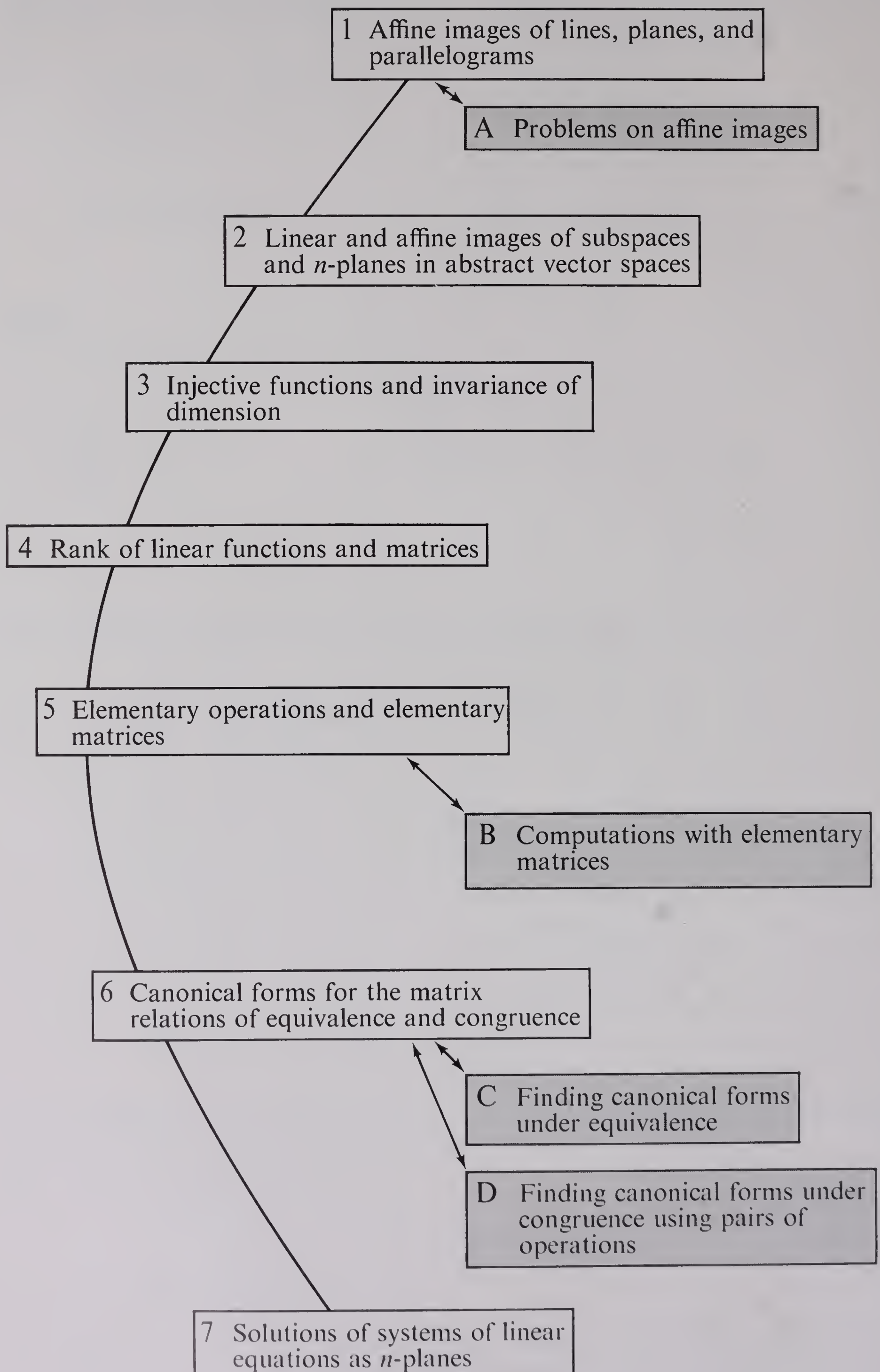
(a) $\begin{bmatrix}2 & 0\\1 & 4\end{bmatrix}$, (b) $\begin{bmatrix}3 & 6\\2 & 5\end{bmatrix}$, (c) $\begin{bmatrix}1 & 3 & 0\\0 & 1 & 4\\2 & 1 & 2\end{bmatrix}$,

(d) $\begin{bmatrix}1 & 2 & 4\\3 & 1 & 0\\2 & 1 & -1\end{bmatrix}$.

6. Find the matrix form of f^{-1} if f is an affine function having the given matrix form.

(a) $\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$, (b) $\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}$,

(c) $\begin{bmatrix}1\\1\\-1\end{bmatrix} + \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 6\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$.


1 Affine images of lines, planes, and parallelograms
A Problems on affine images
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3 Injective functions and invariance of dimension
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VI

Images and Rank

In this chapter we relate the concepts of dimension and function. If X_1 is a subset of the domain of a function f, there is an associated image set $f(X_1)$ consisting of all the images of elements in X_1. It is of interest to know relationships involving the dimension of $f(X_1)$, the dimension of X_1, and properties of f. As a crude example, suppose we define the "dimension" of a finite set to be the number of elements in that set. The definition of a function requires that each domain element in X_1 has only one f-image, and it is easy to see that if X_1 is finite, then $f(X_1)$ has no more elements than X_1. Therefore, $\dim f(X_1) \leq \dim X_1$ for this situation, and we say *f does not increase dimension.*

Intuitively it might seem that no function should increase dimension, since an element cannot have more than one image. However, it was shown by the German mathematician Georg Cantor (1845–1918) that there exists a function f which has the set of real numbers as its domain and the Cartesian plane as its image. Thus, f increases dimension from one to two. Even more surprising is a later example, produced in 1890 by the Italian mathematician G. Peano (1858–1932), which showed the existence of a *continuous* function which has [0, 1] as its domain and a rectangle as its image. This example promoted interest in the meaning of dimension and of function properties which did not permit increase of dimension. Of special interest to us are those functions

which "preserve dimension," that is, functions for which the dimensions of the domain and image are equal. Geometric cases will be considered first.

1. Images of Lines, Planes, Parallelograms

We shall assume the following notation:

$\boldsymbol{u}_o' + f$, an affine function from $\mathbf{R}^2$ to $\mathbf{R}^2$,
$\mathbf{u}_o + r\mathbf{u}_1$, a line in $\mathbf{R}^2$.

We use the additional notation

$$\mathbf{u}_o' = \langle b_1', b_2' \rangle; \qquad \text{Mat}\, f = \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix},$$

$$\mathbf{u}_o = \langle b_1, b_2 \rangle; \qquad \mathbf{u}_1 = \langle a_{11}, a_{21} \rangle,$$

from which it follows that an arbitrary vector in $\mathbf{u}_o + r\mathbf{u}_1$ has the form

$$\langle b_1 + a_{11}r, b_2 + a_{21}r \rangle,$$

and its $(\boldsymbol{u}_o' + f)$-image has the matrix form

$$\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} + \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \begin{bmatrix} b_1 + a_{11}r \\ b_2 + a_{21}r \end{bmatrix}.$$

By application of matrix properties, this image vector may be written

$$\left(\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} + \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + r \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}.$$

If we use the identification of a tuple and column vector, we see that this image vector is

$$(\mathbf{u}_o' + (\text{Mat}\, f)\mathbf{u}_o) + r(\text{Mat}\, f)\mathbf{u}_1 = (\mathbf{u}_o' + f(\mathbf{u}_o)) + rf(\mathbf{u}_1).$$

Hence, the set of all image vectors is again a line, provided $f(\mathbf{u}_1) \neq \mathbf{0}$. The position vector is $\mathbf{u}_o' + f(\mathbf{u}_o)$ and the direction vector is $f(\mathbf{u}_1)$.

Example 1.1 Given the line $\langle 1, 3 \rangle + r\langle 2, 5 \rangle$ and the affine function $\boldsymbol{u}_o' + f$, where $\mathbf{u}_o' = \langle 3, 4 \rangle$ and Mat $f = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, the affine image of the line is given by

$$\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) + r \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

A simple computation shows the image line is $\langle 14, 6 \rangle + r\langle 19, 5 \rangle$.

The foregoing analysis may be extended to general lines and planes. For instance, given

$u_o' + f$, an affine function from $\mathbf{R}^3$ to $\mathbf{R}^3$,

and

$\mathbf{u}_o + r\mathbf{u}_1 + s\mathbf{u}_2$, a plane in $\mathbf{R}^3$,

then the affine image of the given plane is

$$(\mathbf{u}_o' + f(\mathbf{u}_o)) + rf(\mathbf{u}_1) + sf(\mathbf{u}_2),$$

except in degenerate cases. In this case degeneracy occurs whenever $\{f(\mathbf{u}_1), f(\mathbf{u}_2)\}$ is dependent; the image is then a line or point.

Example 1.2 Given the plane $\langle 1, 2, 5\rangle + r\langle 3, 0, 7\rangle + s\langle 0, 1, 5\rangle$ and affine function $u_o' + f$, where

$$\mathbf{u}_o' = \langle 2, 0, -1\rangle \quad \text{and} \quad \operatorname{Mat} f = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix},$$

then the image plane is given by

$$\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}\right) + r \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$$
$$+ s \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

Computation gives $\langle 22, -4, 17\rangle + r\langle 24, -4, 16\rangle + s\langle 17, -5, 10\rangle$ as the image plane.

Similar conclusions may be obtained for affine images of segments, parallelograms, and parallelepipeds. Restriction of domain values of r, s, and t to the interval [0, 1] induces a similar restriction of the image values, and, except for degenerate cases, the domain and image set have the same character. For example, the $(u_o' + f)$-image is shown in the following table.

Set	Image
$\mathbf{u}_o + [r\mathbf{u}_1]$	$(\mathbf{u}_o' + f(\mathbf{u}_o)) + [rf(\mathbf{u}_1)]$,
$\mathbf{u}_o + [r\mathbf{u}_1 + s\mathbf{u}_2]$	$(\mathbf{u}_o' + f(\mathbf{u}_o)) + [rf(\mathbf{u}_1) + sf(\mathbf{u}_2)]$,
$\mathbf{u}_o + [r\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3]$	$(\mathbf{u}_o' + f(\mathbf{u}_o)) + [rf(\mathbf{u}_1) + sf(\mathbf{u}_2) + tf(\mathbf{u}_3)]$.

Matrix methods are again convenient for computing the images.

Example 1.3 Given the parallelogram $\langle 1, 3\rangle + [r\langle 2, 5\rangle + s\langle 1, -1\rangle]$ and the affine function $\boldsymbol{u}_0' + f$, where

$$\mathbf{u}_0' = \langle 1, -2\rangle \quad \text{and} \quad \text{Mat}\, f = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix},$$

then the image of the parallelogram is

$$\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) + r \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$
$$+ s \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where $0 \leq r, s \leq 1$. Thus the image is the parallelogram

$$\langle 8, -3\rangle + [r\langle 12, -2\rangle + s\langle -1, -1\rangle]$$

(see Figure 6.1).

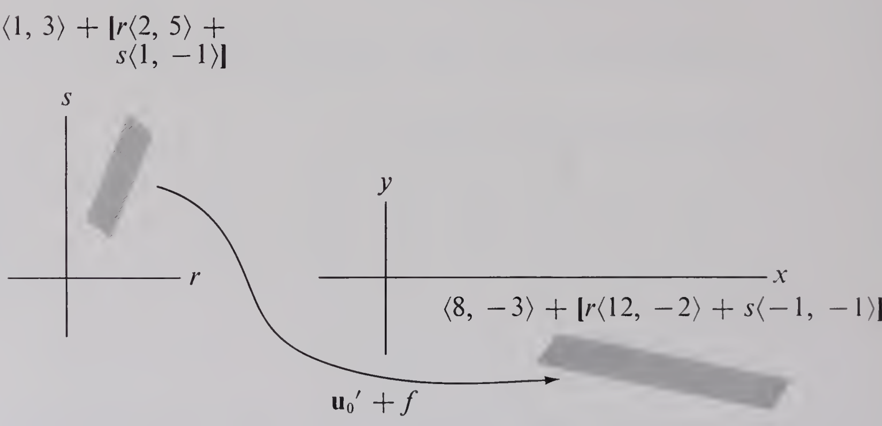

Figure 6.1

We now turn to the degenerate cases. The extreme cases occur when f is the zero function. In that case $\boldsymbol{u}_0' + f$ is the constant function $\boldsymbol{u}_0'$ and the image of each set consists of a single element. Even if f is not the zero function, however, the f-image of a direction vector may be the zero vector. In general, degeneracy takes place when the f-images of the direction vectors form a dependent set. The image sets may then be described by casting out the images of certain direction vectors. We consider various cases.

(a) If $f(\mathbf{u}_1) = \mathbf{0}$, then the $(\boldsymbol{u}_0' + f)$-image of the line $\mathbf{u}_0 + r\mathbf{u}_1$ consists of the single vector $\mathbf{u}_0' + f(\mathbf{u}_0)$.

(b) If $f(\mathbf{u}_1) = f(\mathbf{u}_2) = \mathbf{0}$, then the $(\boldsymbol{u}_0' + f)$-image of the plane $\mathbf{u}_0 + \mathbf{r}\mathbf{u}_1 + s\mathbf{u}_2$ consists of the single vector $\mathbf{u}_0' + f(\mathbf{u}_0)$.

(c) If $f(\mathbf{u}_2) = cf(\mathbf{u}_1)$, where $f(\mathbf{u}_1) \neq \mathbf{0}$, then the $(\boldsymbol{u}_o' + f)$-image of the plane $\mathbf{u}_o + r\mathbf{u}_1 + s\mathbf{u}_2$ is the line $(\mathbf{u}_o' + f(\mathbf{u}_o)) + rf(\mathbf{u}_1)$.

The reason for (c) comes from the equality

$$rf(\mathbf{u}_1) + sf(\mathbf{u}_2) = (r + sc)f(\mathbf{u}_1),$$

which shows that $rf(\mathbf{u}_1) + sf(\mathbf{u}_2)$ consists of all scalar multiples of $f(\mathbf{u}_1)$.

Example 1.4 (a) Given the line $\langle 1, 3\rangle + r\langle 4, 5\rangle$ and the affine function $\boldsymbol{u}_o' + f$, where

$$\mathbf{u}_o' = \langle 1, 6\rangle \quad ; \qquad \operatorname{Mat} f = \begin{bmatrix} -5 & 4 \\ 10 & -8 \end{bmatrix},$$

the affine image of the line is given by

$$\left(\begin{bmatrix} 1 \\ 6 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ 10 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) + r\begin{bmatrix} -5 & 4 \\ 10 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ -8 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus the image consists of the single vector $\langle 8, -8\rangle$.

(b) Given the plane $\langle 1, 0, 5\rangle + r\langle 2, 1, 0\rangle + s\langle 3, 6, 2\rangle$ and the affine function $\boldsymbol{u}_o' + f$, where

$$\mathbf{u}_o' = \langle 2, 1, 7\rangle \quad ; \qquad \operatorname{Mat} f = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 4 & 9 & 1 \end{bmatrix},$$

the affine image of the plane is given by

$$\left(\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 4 & 9 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}\right) + r\begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 4 & 9 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$
$$+ s\begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 4 & 9 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 16 \end{bmatrix} + r\begin{bmatrix} 9 \\ 1 \\ 17 \end{bmatrix} + s\begin{bmatrix} 36 \\ 4 \\ 68 \end{bmatrix}.$$

Since $\langle 36, 4, 68\rangle = 4\langle 9, 1, 17\rangle$, the image is the line $\langle 4, -4, 16\rangle + r\langle 9, 1, 17\rangle$.

Similar degeneracy may occur in the affine image of a segment, parallelogram, or parallelepiped. The affine image of a parallelogram will then be a segment or point. The affine image of a parallelepiped, however, may fail to be a parallelotope.

Example 1.5 Given the parallelogram $\langle 1, 3\rangle + [r\langle 2, 1\rangle + s\langle 3, 0\rangle]$ and affine function $\boldsymbol{u}_o' + f$, where

$$\mathbf{u}_o' = \langle 2, 3, 4\rangle; \quad \operatorname{Mat} f = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix},$$

the $(\boldsymbol{u}_o' + f)$-image of the parallelogram is given by

$$\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) + r \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

where $0 \le r, s \le 1$. This is the set of vectors,

$$\{\langle 0, -1, -2\rangle + r\langle 1, 2, 3\rangle + s\langle 3, 6, 9\rangle ; 0 \le r, s \le 1\}.$$

Since $\langle 3, 6, 9\rangle = 3\langle 1, 2, 3\rangle$, the image is a segment whose end points are found by successively setting $r = 0$, s $= 0$, and $r = 1$, $s = 1$. Thus, the image is $\langle 0, -1, -2\rangle + [r\langle 4, 8, 12\rangle]$.

Example 1.6 Let $\mathbf{T} = [r\mathbf{i} + s\mathbf{j} + t\mathbf{k}]$ and let f be the linear function with

$$\operatorname{Mat} f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the f-image of $\mathbf{T}$ is $\{r\mathbf{i} + s\mathbf{j} + t(\mathbf{i} + \mathbf{j}); 0 \le r, s, t \le 1\}$. Its graph lies in the xy plane, and it is not a parallelotope (see Figure 6.2).

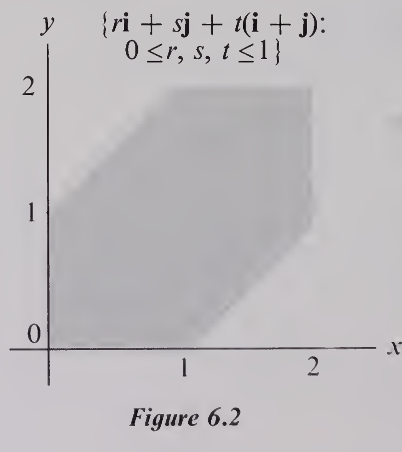

Figure 6.2

Questions

1. The affine image of a line is either a ______ or a ______.
2. The affine image of a plane is a ______, ______, or ______.

3. The affine image of a parallelogram is a ______, ______, or ______.
4. The $(\mathbf{u}_o' + f)$-image of $\mathbf{u}_o + r\mathbf{u}_1 + s\mathbf{u}_2$ is degenerate provided the set ______ is dependent.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. Find the $(\mathbf{u}_o' + f)$-image of $\langle 0, 1, 1\rangle + r\langle 4, 0, 6\rangle + s\langle 1, -1, 2\rangle$ if $\mathbf{u}_o' = \langle 1, 1, 3\rangle$ and Mat $f =$

$$\text{(a)} \begin{bmatrix} -1 & -1 & 1 \\ 2 & -2 & -1 \\ 3 & -1 & -2 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} 4 & -4 & -2 \\ -2 & 2 & 1 \\ 3 & -5 & -1 \end{bmatrix}.$$

2. Find an affine function for which
   (a) the image of $\langle 1, 0\rangle + r\langle 3, 2\rangle$ is $\langle 2, 1\rangle + s\langle 5, 0\rangle$,
   (b) the image of $\langle 1, 0\rangle + [r\langle 2, 1\rangle + s\langle 3, 0\rangle]$ is

$$\langle 1, 0, 1\rangle + [r\langle 1, 1, 0\rangle + s\langle 2, 1, -1\rangle].$$

   (*Hint*: The solution is not unique. Obtain a suitable system of equations and assign arbitrary, convenient values to certain unknowns.)

Proofs

1. Given the lines $\mathbf{L}_1 = r\mathbf{u}_1$ and $\mathbf{L}_2 = r\mathbf{u}_2$ in $\mathbf{R}^2$, prove that there is a linear function f such that the f-image of $\mathbf{L}_1$ is $\mathbf{L}_2$. (*Hint*: Express Mat f in terms of coordinates of $\mathbf{u}_1$, $\mathbf{u}_2$.)
2. Given the parallelograms $\mathbf{T}_1 = [r\mathbf{u}_1 + s\mathbf{u}_2]$ and $\mathbf{T}_2 = [r\mathbf{v}_1 + s\mathbf{v}_2]$, prove there is a linear function f such that the f-image of $\mathbf{T}_1$ is $\mathbf{T}_2$.

2. Images of Subspaces and n-Planes

If f is a function from X to Y and X_1 is a subset of X, then the *f-image* $f(X_1)$ of X_1 is the set of all elements in Y which are assigned to some element in X_1 (see Figure 6.3). Thus, y is in $f(X_1)$ provided there is an x in X_1 such that $f(x) = y$.

Example 2.1 (a) If $f(x) = x^2$, then the f-image of $[0, 2]$ is $f([0, 2]) = [0, 4]$.
(b) If X and Y are sets of real numbers then the f-image of X_1 can be geometrically described by projecting X_1 onto the graph of f and then projecting this portion of the graph onto the y axis (see Figure 6.4).

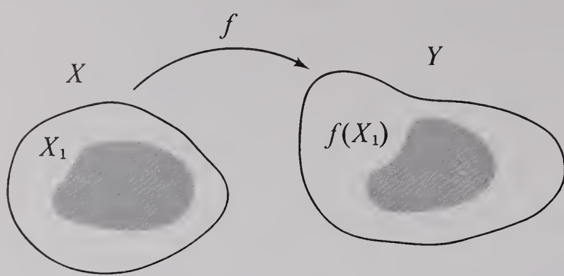

Figure 6.3

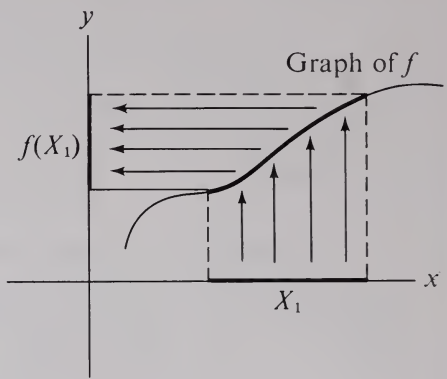

Figure 6.4

Our interest here concerns the f-image of subspaces and n-planes, where f is linear or affine. If we let $\mathbf{V}$ and $\mathbf{V}'$ denote Euclidean vector spaces, the fundamental result says that the image of a set has the same character as the set.

Invariance Theorem for Sets

(a) *If* $\mathbf{S}$ *is a subspace and* f *is linear, then* $f(\mathbf{S})$ *is a subspace and* dim $f(\mathbf{S}) \leq \dim \mathbf{S}$.
(b) *If* $\mathbf{P}$ *is an n-plane and* $\boldsymbol{u}_o' + f$ *is affine, then* $(\boldsymbol{u}_o' + f)(\mathbf{P})$ *is an* n'*-plane where* $n' \leq n$.

Before considering its proof, we shall investigate a consequence of this theorem.

Example 2.2 Let f be a linear function from $\mathbf{R}^3$ to $\mathbf{R}^3$. If $\mathbf{S}$ is a 2-dimensional subspace of $\mathbf{R}^3$, then its graph is a plane through the origin. By the invariance theorem for sets, the graph of $f(\mathbf{S})$ is either a plane through the origin, a line through the origin, or the origin.

The proof of (a) of the invariance theorem for sets will now be made. It must be shown that $f(\mathbf{S})$ is closed with respect to addition and scalar multiplication. If $\mathbf{u}'$ and $\mathbf{v}'$ are in $f(\mathbf{S})$, then by the definition of f-image there exist

$\mathbf{u}$ and $\mathbf{v}$ in S such that $f(\mathbf{u}) = \mathbf{u}'$ and $f(\mathbf{v}) = \mathbf{v}'$. Since S is a subspace, it is also true that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are in S for each number c. The equalities

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) = \mathbf{u}' + \mathbf{v}'$$

and

$$f(c\mathbf{u}) = cf(\mathbf{u}) = c\mathbf{u}'$$

show that $\mathbf{u}' + \mathbf{v}'$ and $c\mathbf{u}'$ are in $f(\mathrm{S})$ as desired. A proof of the statement $\dim f(\mathrm{S}) \leq \dim \mathrm{S}$ will be made for the case where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of S. Then an arbitrary element in S has the form $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, and therefore an arbitrary element in $f(\mathrm{S})$ has the form

$$f(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = c_1 f(\mathbf{u}_1) + c_2 f(\mathbf{u}_2) + c_3 f(\mathbf{u}_3).$$

This shows that $f(\mathrm{S})$ is the span set of $\{f(\mathbf{u}_1), f(\mathbf{u}_2), f(\mathbf{u}_3)\}$ and hence has dimension less than or equal to 3.

The proof of (b) of the invariance theorem follows from (a) by letting $\mathbf{P} = \mathbf{u}_o + \mathrm{S}$ and applying the set equality (see Proofs, exercises 1 and 2).

$$(\mathbf{u}_o' + f)(\mathbf{u}_o + \mathrm{S}) = (\mathbf{u}_o' + f(\mathbf{u}_o)) + f(\mathrm{S}).$$

Questions

1. If $f(x) = x^3$, then the f-image of $[0, 2]$ is ______.
2. If f is linear and S is a subspace, then the dimension of $f(\mathrm{S})$ is necessarily ______ the dimension of S.
   (a) less than,
   (b) less than or equal to,
   (c) more than.
3. If f is affine and S is a subspace, then the f-image of S is necessarily ______.
   (a) a subspace,
   (b) a k-plane for some k,
   (c) a k-parallelotope for some k.

Exercises

1. Find the f image of $\mathrm{Sp}\{\langle 1, 0, 0, 3\rangle, \langle 2, 1, 0, 7\rangle, \langle 0, 3, 0, 1\rangle\}$ given that f is linear and Mat $f =$

(a) $\begin{bmatrix} -3 & 0 & 0 & 1 \\ 1 & 5 & 3 & -1 \\ 2 & 1 & 1 & -3 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ 7 & 0 & 4 & 2 \\ 1 & -1 & 1 & 3 \end{bmatrix}$.

Proofs

1. Prove that if $\mathbf{T}$ is any set in $\mathbf{V}$ and $\boldsymbol{u}_o' + f$ is an affine function from $\mathbf{V}$ to $\mathbf{V}'$, then
$$(\boldsymbol{u}_o' + f)(\mathbf{T}) = \mathbf{u}_o' + f(\mathbf{T}).$$
2. Prove that if f is linear and $\mathbf{u}_o + \mathbf{S}$ is a k-plane, then
$$f(\mathbf{u}_o + \mathbf{S}) = f(\mathbf{u}_o) + f(\mathbf{S}).$$

3. *Injective Functions*

In the previous section we saw that affine functions cannot increase dimension, but may decrease dimension. In this section we study those affine functions which preserve dimension. Intuitively, a function is most likely to decrease dimension if many domain elements go into a single range element, and least likely to decrease dimension if different elements necessarily have different images. A function f from X to Y is *injective* provided $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Common synonyms for injective are *one-to-one* and *univalent*.

Example 3.1 (a) The function $f(x) = x^2$, from $[-1, 1]$ to **Re**, is not injective, since $f(a) = f(-a) = a^2$ for each $a \neq 0$. The function $f(x) = x^2$ from $[0, 1]$ to **Re** is injective, since it assigns different values to every pair x_1, x_2 in $[0, 1]$.
(b) A set of points in the Cartesian plane is the graph of a function if each vertical line intersects the set in at most one point. If it is also true that each horizontal line intersects the set in at most one point, then the function is injective.

An affine function $\boldsymbol{u}_o' + f$ is injective if and only if its linear part f is injective. This is implied by the equivalence of the following inequality statements:

$$(\boldsymbol{u}_o' + f)(\mathbf{u}) \neq (\boldsymbol{u}_o' + f)(\mathbf{v}),$$
$$\mathbf{u}_o' + f(\mathbf{u}) \neq \mathbf{u}_o' + f(\mathbf{v}),$$
$$f(\mathbf{u}) \neq f(\mathbf{v}).$$

The important result of this section is that affine functions preserve dimension if and only if they are injective. We first consider a preliminary result for linear functions.

Proposition 3.1 Let f be a linear function from $\mathbf{V}$ to $\mathbf{V}'$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ an independent set in $\mathbf{V}$. Then
(a) the f-image of Sp $(\{\mathbf{u}_1, \ldots, \mathbf{u}_n\})$ is Sp $(\{f(\mathbf{u}_1), \ldots, f(\mathbf{u}_n)\})$;
(b) if f is also injective, then $\{f(\mathbf{u}_1), \ldots, f(\mathbf{u}_n)\}$ is independent;
(c) if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbf{V}$ then f is injective if and only if $\{f(\mathbf{u}_1), \ldots, f(\mathbf{u}_n)\}$ is independent.

We consider the proof for $n = 3$. An arbitrary element of Sp $(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$ has the form

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3,$$

and its f-image is

$$c_1 f(\mathbf{u}_1) + c_2 f(\mathbf{u}_2) + c_3 f(\mathbf{u}_3)$$

by the linearity of f. This essentially proves part (a). For the proof of (b), suppose $\{f(\mathbf{u}_1), f(\mathbf{u}_2), f(\mathbf{u}_3)\}$ is dependent. Then there exist c_1, c_2, and c_3, not all 0, such that

$$c_1 f(\mathbf{u}_1) + c_2 f(\mathbf{u}_2) + c_3 f(\mathbf{u}_3) = \mathbf{0}'.$$

Hence, by the linearity of f,

$$f(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = \mathbf{0}' = f(\mathbf{0}).$$

Since $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \neq \mathbf{0}$ by the independence of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, it follows that f is not injective and we have a contradiction. For (c) see Proofs, exercise 3.

From Proposition 3.1(c) we can obtain the matrix property which corresponds to the injective property of functions. If we let $\mathbf{V} = \mathbf{R}^n$ and $\mathbf{V}' = \mathbf{R}^m$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be the standard basis of $\mathbf{R}^n$, then (c) yields the following result.

Proposition 3.2 A linear function f from $\mathbf{R}^n$ to $\mathbf{R}^m$ is injective if and only if the column vectors of Mat f are independent.

For $m = n$ there is the following easy corollary of Proposition 3.1.

Proposition 3.3 If f is linear from $\mathbf{R}^n$ to $\mathbf{R}^n$, then f is injective if and only if $f(\mathbf{R}^n) = \mathbf{R}^n$.

We are now ready for the important result of this section. It is valid for all finite dimensional spaces.

Invariance Theorem for Dimension

(a) *If* S *is a subspace and* f *is linear and injective, then* $\dim f(\mathbf{S}) = \dim \mathbf{S}$.

(b) *If* **P** *is an n-plane and* $\boldsymbol{u}_o' + f$ *is affine and injective, then* $(\boldsymbol{u}_o' + f)(\mathbf{P})$ *is an n-plane.*

(c) *If* **T** *is an n-parallelotope and* $\boldsymbol{u}_o' + f$ *is affine and injective, then* $(\boldsymbol{u}_o' + f)(\mathbf{T})$ *is an n-parallelotope.*

A proof of (a) follows immediately from Proposition 3.1, which implies that if f is injective and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of S, then $\{f(\mathbf{u}_1), \ldots, f(\mathbf{u}_n)\}$ is a basis of $f(\mathbf{S})$. For the proof of (b), if $\mathbf{P} = \mathbf{u}_o + \mathbf{S}$, then

$$(\boldsymbol{u}_o' + f)(\mathbf{P}) = (\mathbf{u}_o' + f(\mathbf{u}_o)) + f(\mathbf{S}),$$

and the desired conclusion follows from (a). Part (c) is proved similarly (see Proofs, exercise 1).

Questions

1. A linear function preserves dimension if it is ______.
2. If f is linear, then the f-image of $\mathrm{Sp}\{\mathbf{u}_1, \mathbf{u}_2\}$ is ______.
3. The affine function $\boldsymbol{u}_o' + f$ is injective provided ______.
   (a) f is injective,
   (b) $\mathbf{u}_o' = \mathbf{0}'$,
   (c) $f(\mathbf{0}) = \mathbf{0}'$.
4. If f is affine and injective, then the f-image of an independent set ______.
   (a) is an independent set,
   (b) is an n-plane,
   (c) contains the zero vector.

Exercises

1. Determine which of the following functions are injective.
   (a) $f(x) = x^3, -1 \leq x \leq 1$, (b) $f(x) = \sin x, 0 \leq x \leq 4$,
   (c) $f(x) = e^x, -1 \leq x \leq 1$, (d) $f(x) = x^2 - 2x, 0 \leq x \leq 2$.

Proofs

1. Prove that if **T** is a 3-parallelotope and $\boldsymbol{u}_o' + f$ is affine and injective, then $(\boldsymbol{u}_o' + f)(\mathbf{T})$ is a 3-parallelotope.

2. If f is linear from $\mathbf{V}$ to $\mathbf{V}'$, then the *kernel* $\mathbf{K}_f$ is defined to be the set $\{\mathbf{u}: f(\mathbf{u}) = \mathbf{0}'\}$. Prove that
   (a) $\mathbf{K}_f$ is a subspace of $\mathbf{V}$,
   (b) f is injective if and only if $\mathbf{K}_f = \{\mathbf{0}\}$.
3. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of $\mathbf{V}$ and $\{f(\mathbf{u}_1), f(\mathbf{u}_2), f(\mathbf{u}_3)\}$ is independent, then f is injective.

4. Rank

We shall continue the study of image dimension and obtain, in the process, results related to the composition operation on functions. A corresponding analysis will also be made for matrices. If f is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$, then the *rank*, $\mathrm{r}(f)$, of f is defined by the expression,

$$\mathrm{r}(f) = \dim f(\mathbf{R}^n).$$

Thus, the rank of f is the dimension of the image of f. If $A = \langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle$, then the rank $\mathrm{r}(A)$ of A is defined by

$$\mathrm{r}(A) = \dim \mathrm{Sp}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}.$$

Since $\mathrm{Mat}\, f = \langle f(\mathbf{e}_1), \dots f(\mathbf{e}_n) \rangle$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbf{R}^n$, it follows from Proposition 3.1(a) that if $\mathrm{Mat}\, f = A$, then $\mathrm{r}(f) = \mathrm{r}(A)$. We now state a result relating rank and composition.

Proposition 4.1 If g and f are linear, then
(a) $\mathrm{r}(g \circ f) \le \mathrm{r}(f)$ and $\mathrm{r}(g \circ f) = \mathrm{r}(f)$ if g is injective,
(b) $\mathrm{r}(g \circ f) \le \mathrm{r}(g)$ and $\mathrm{r}(g \circ f) = \mathrm{r}(g)$ if f is injective.

A proof of (a) follows from the invariance theorem for sets and the chain of inequalities shown below.

$$\mathrm{r}(g \circ f) = \dim (g \circ f)(\mathbf{R}^n) = \dim g(f(\mathbf{R}^n)) \le \dim f(\mathbf{R}^n) = \mathrm{r}(f).$$

If g is injective, then equality holds throughout by the invariance theorem for dimension. For (b) see Proofs, exercise 1.

From the correspondence between function composition and matrix product, we have the following result.

Proposition 4.2

(a) $\mathrm{r}(BA) \le \mathrm{r}(A)$; $\mathrm{r}(BA) = \mathrm{r}(A)$ if the columns of B are independent.
(b) $\mathrm{r}(BA) \le \mathrm{r}(B)$; $\mathrm{r}(BA) = \mathrm{r}(B)$ if the columns of A are independent.

The rank of a matrix A, as defined here, is often called the *column rank* of A. The *row rank* of A is then the dimension of the span space of the row vectors. The row rank and column rank can be shown to be equal, so that the distinction is unnecessary. In order to prove this equality, it suffices to show that $\mathrm{r}(A^*) = \mathrm{r}(A)$ for any matrix A, since the row vectors of A are the column vectors of A^*. This is a consequence of the corresponding result for functions which follows (see Proofs, exercises 2 and 3).

Proposition 4.3 If f is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$, then

$$\mathrm{r}(f) = \mathrm{r}(f^* \circ f) = \mathrm{r}(f^*).$$

The results on rank enable us to specify conditions on the existence of the inverse of a linear function f from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Proposition 4.4 If f is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$, then f is invertible if and only if $m = n$ and f is injective.

We first assume that f is invertible. The proof of $m = n$ follows from the properties of rank. Thus from

$$f^{-1} \circ f = \mathrm{id}_{\mathbf{R}^n} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}_{\mathbf{R}^m},$$

we have

$$n = \mathrm{r}(\mathrm{id}_{\mathbf{R}^n}) = \mathrm{r}(f^{-1} \circ f) \leq \mathrm{r}(f) \leq m$$

and

$$m = \mathrm{r}(\mathrm{id}_{\mathbf{R}^m}) = \mathrm{r}(f \circ f^{-1}) \leq \mathrm{r}(f^{-1}) = n,$$

from which $m = n$. If f were not injective, then $f(\mathbf{u}) = f(\mathbf{v})$ for some distinct $\mathbf{u}$ and $\mathbf{v}$, and the equality chain

$$\mathbf{u} = (f^{-1} \circ f)(\mathbf{u}) = (f^{-1} \circ f)(\mathbf{v}) = \mathbf{v},$$

gives a contradiction. We next assume that f is injective from $\mathbf{R}^n$ to $\mathbf{R}^n$ and show that f^{-1} exists. By the invariance theorem for dimension, $\dim f(\mathbf{R}^n) = n$, and hence, $f(\mathbf{R}^n) = \mathbf{R}^n$. Thus, each vector in $\mathbf{R}^n$ is the f-image of exactly one vector in $\mathbf{R}^n$, and if we define g by the rule.

$$g(\mathbf{u}') = \mathbf{u} \quad \text{if and only if} \quad f(\mathbf{u}) = \mathbf{u}',$$

then it is easily verified that g satisfies the desired properties of f^{-1}.

Since Mat f is nonsingular if and only if f is invertible, there is the following consequence of Propositions 4.3 and 4.4.

Proposition 4.5 If A is an $n \times n$ matrix, then A is nonsingular if and only if $\mathrm{r}(A) = n$.

Questions

1. The dimension of the image space of f is called the ______ of f.
2. The rank of a matrix is ______ the rank of its corresponding function.
   (a) equal,
   (b) less than,
   (c) greater than.
3. The rank of BA cannot be ______ the rank of A.
   (a) equal,
   (b) less than,
   (c) greater than.
4. The rank of A could be defined as the dimension of the space spanned by the ______ or ______ vectors of A.
5. For square matrices, the linear function property corresponding to nonsingularity is ______.

Exercises

1. Given that f is linear from $\mathbf{R}^3$ to $\mathbf{R}^3$, that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbf{R}^3$, and that $\{\mathbf{u}', \mathbf{v}'\}$ is independent; find $\mathrm{r}(f)$ if
   (a) $f(\mathbf{u}) = \mathbf{u}', f(\mathbf{v}) = 2\mathbf{u}', f(\mathbf{w}) = \mathbf{u}' + \mathbf{v}'$;
   (b) $f(\mathbf{u}) = \mathbf{0}', f(\mathbf{v}) = \mathbf{v}', f(\mathbf{w}) = -\mathbf{v}'$;
   (c) $f(\mathbf{u}) = \mathbf{u}' + \mathbf{v}', f(\mathbf{v}) = \mathbf{u}' - \mathbf{v}', f(\mathbf{w}) = 3\mathbf{u}' + 2\mathbf{v}'$.
2. Using the obvious extensions of rank of linear functions on Euclidean vector spaces, find $\mathrm{r}(\mathrm{D})$ if D is the derivative function from $\mathbf{V}$ to $\mathbf{V}$, where $\mathbf{V}$ is the vector space of polynomials on $[0, 1]$ of degree ≤ 5.

Proofs

1. Justify the steps of the following proof, where f is linear from $\mathbf{R}^n$ to $\mathbf{R}^m$.
$$\mathrm{r}(g \circ f) = \dim(g(f(\mathbf{R}^n))) \leq \dim g(\mathbf{R}^m) = \mathrm{r}(g).$$
2. Given that f is linear from $\mathbf{R}^n$ to $\mathbf{R}^m$, prove that if $\mathbf{u}$ is in $f^*(\mathbf{R}^m)$, then $\mathbf{u}$ is also in $f^*(f(\mathbf{R}^n))$ as follows.
   (a) Let $\mathbf{u}$ (arbitrary) $= f^*(\mathbf{u}') = f^*(\mathbf{u}_1' + \mathbf{u}_2')$, where $\mathbf{u}_1'$ is in $f(\mathbf{R}^n)$ and $\mathbf{u}_2'$ is in $(f(\mathbf{R}^n))^\perp$;
   (b) show that $f^*(\mathbf{u}_2') \cdot f^*(\mathbf{u}_2') = f(f^*(\mathbf{u}_2')) \cdot \mathbf{u}_2' = 0$, and hence, $f^*(\mathbf{u}_2') = \mathbf{0}$;
   (c) conclude that $\mathbf{u} = f^*(\mathbf{u}_1')$ is in $f^*(f(\mathbf{V}))$.
3. Prove that $\mathrm{r}(f) = \mathrm{r}(f^*)$. (*Hint*: Using Proofs, exercise 2, deduce $\mathrm{r}(f^*) \leq \mathrm{r}(f)$ for any linear f. Replace f by f^* in this inequality and obtain the reverse inequality.)

5. Elementary Operations

It is not generally possible to determine by inspection the rank of a given matrix. With certain matrices, such as those in *diagonal* or *echelon form*, this is possible. In this section we shall see how a given matrix can be converted to a diagonal or echelon form matrix having the same rank.

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *diagonal* of A consists of all entries a_{ij}, where $i = j$. A is a *diagonal matrix* if all entries not on the diagonal are zero.

Example 5.1 Given the matrices,

$$A = \begin{bmatrix} 1 & 4 \\ 6 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$

the diagonal entries of A are 1, 2 and of B are 4, 3. Matrix B is a diagonal matrix.

We now observe an elementary property of diagonal matrices.

Proposition 5.1 The rank of a diagonal matrix is the number of nonzero diagonal entries.

A proof will be indicated for the case where

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is the maximum number of independent column vectors of A. Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ denote the columns of A. If a_{11}, a_{22}, and a_{33} are all nonzero, then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \langle c_1a_{11}, c_2a_{22}, c_3a_{33}, 0\rangle$$

is the zero vector only if $c_1 = c_2 = c_3 = 0$. Hence, in this case,

$$3 = \mathrm{r}(A) = \text{the number of nonzero diagonal entries.}$$

If $a_{11} = 0$ and a_{22}, a_{33} are nonzero, then $\{\mathbf{u}_2, \mathbf{u}_3\}$ is independent, since $c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \langle 0, c_2a_{22}, c_3a_{33}, 0\rangle = \mathbf{0}$ implies that $c_2 = c_3 = 0$, whereas $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is dependent, since $\mathbf{u}_1 = \mathbf{0}$. Thus,

$$2 = \mathrm{r}(A) = \text{number of nonzero diagonal entries.}$$

The argument generalizes to all cases.

The tuple $\langle a_1, a_2, \ldots, a_n \rangle$ has k *initial zeros* if $a_1 = a_2 = \cdots = a_k = 0$ and $a_{k+1} \neq 0$. An $m \times n$ matrix is in *echelon form* if for each $i = 1, 2, \ldots, (m-1)$ the $(i+1)$ row vector is either the zero vector or has more initial zeros than the ith row vector. A similar proof to that of Proposition 5.1, this time using the row vectors, gives the following result.

Proposition 5.2 The rank of an echelon form matrix is the number of nonzero row vectors in the matrix.

Example 5.2 The echelon form matrix

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has two nonzero row vectors, and, therefore, its rank is 2.

We shall now show that any matrix can be converted to diagonal or echelon form without changing its rank. Associated with each ordered finite set of vectors are three *elementary vector operations.*

(I) *Interchange of two vectors.*
(II) *Addition of a scalar multiple of one vector to another.*
(III) *Multiplication of a vector by a nonzero scalar.*

These will be illustrated by the following example.

Example 5.3 Given $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$,
(a) a type-I operation gives $\{\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_2\}$,
(b) a type-II operation gives $\{\mathbf{u}_1 + 4\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_3\}$,
(c) a type-III operation gives $\{\mathbf{u}_1, 6\mathbf{u}_2, \mathbf{u}_3\}$.

Our fundamental result says that the span set is invariant under elementary operations.

Proposition 5.3 If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is obtained from $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ by an elementary operation, then

$$\mathrm{Sp}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \mathrm{Sp}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}.$$

The proof is obvious for operations of type I. For the type-II operations, consider the case where

$$\mathbf{v}_1 = \mathbf{u}_1 + c\mathbf{u}_2, \ \mathbf{v}_2 = \mathbf{u}_2, \ \mathbf{v}_3 = \mathbf{u}_3.$$

Given the arbitrary vector in $\mathrm{Sp}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$,

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3,$$

then, by substitution, we have

$$\begin{aligned} \mathbf{w} &= a_1(\mathbf{u}_1 + c\mathbf{u}_2) + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 \\ &= a_1\mathbf{u}_1 + (a_1 c + a_2)\mathbf{u}_2 + a_3\mathbf{u}_3, \end{aligned}$$

and this shows that $\mathrm{Sp}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a subset of $\mathrm{Sp}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. The reverse inclusion is done similarly using the equalities

$$\mathbf{u}_1 = \mathbf{v}_1 - c\mathbf{u}_2 = \mathbf{v}_1 - c\mathbf{v}_2 \quad \text{and} \quad \mathbf{u}_2 = \mathbf{v}_2, \ \mathbf{u}_3 = \mathbf{v}_3.$$

The type-III proof is also similar (see Proofs, exercise 1).

The elementary operations of types I, II, and III applied to the columns of a matrix are called *elementary column operations* of types I, II, and III, respectively. When applied to the rows they are called *elementary row operations.* Since elementary operations preserve the span set, they necessarily preserve the dimension of the span set. This gives an important theorem.

Invariance of Rank Theorem

If B is an $m \times n$ matrix obtained from A by elementary row and column operations, then $\mathrm{r}(A) = \mathrm{r}(B)$.

The usefulness of the invariance of rank theorem is realized by a proposition stated in the next section, which says that any matrix can be converted to diagonal form by elementary operations.

We shall now correspond to each elementary vector operation performed on n-tuples, the matrix obtained by applying that operation to the column vectors of the $n \times n$ identity matrix I^n. Thus, there are the following elementary matrix types.

(I) E^n_{ij} *obtained from I^n by interchanging the ith and jth columns.*

(II) $E^n_{i+(c)j}$ *obtained from I^n by adding c times the jth column to the ith column.*

(III) $E^n_{(c)i}$ *obtained from I^n by multiplying the ith column by c, $c \neq 0$.*

Elementary matrix superscripts are often omitted.

Example 5.4 (a) For $n=2$,

$$E_{12}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad E_{1+(4)2}=\begin{bmatrix}1&0\\4&1\end{bmatrix},\quad\text{and}\quad E_{(5)2}=\begin{bmatrix}1&0\\0&5\end{bmatrix};$$

(b) for $n=3$,

$$E_{12}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad E_{2+(4)3}=\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix},\quad\text{and}\quad E_{(3)1}=\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

The results of premultiplication or postmultiplication of matrices by an elementary matrix may be verified by considering various cases. The following propositions can be shown to be valid.

Proposition 5.4

(a) AE_{ij} is obtained from A by interchanging the ith and jth columns.
(b) $AE_{i+(c)j}$ is obtained from A by adding c times the jth column to the ith column.
(c) $AE_{(c)i}$ is obtained from A by multiplying the ith column by c.

Proposition 5.5
(a) $E_{ij}A$ is obtained from A by interchanging the ith and jth rows.
(b) $E_{i+(c)j}A$ is obtained from A by adding c times the ith row to the jth row.
(c) $E_{(c)i}A$ is obtained from A by multiplying the ith row by c.

The reversal of i and j should be noted in the comparison of Propositions 5.4(b) and 5.5(b). A related reversal occurs in the next result, which says that the transpose of an elementary matrix is an elementary matrix of the same type.

Proposition 5.6 (a) ${E_{ij}}^*=E_{ij}$,
(b) ${E_{i+(c)j}}^*=E_{j+(c)i}$,
(c) ${E_{(c)i}}^*=E_{(c)i}$.

Also, each elementary matrix is nonsingular and its inverse is an elementary matrix of the same type.

Proposition 5.7 (a) ${E_{ij}}^{-1}=E_{ij}$,
(b) ${E_{i+(c)j}}^{-1}=E_{i+(-c)j}$,
(c) ${E_{(c)i}}^{-1}=E_{(1/c)i}$.

A proof of Proposition 5.7 may be made using the corresponding elementary vector operations. If the ith and jth vectors are interchanged twice, then the original ordered set of vectors results, and this proves (a). If c times the jth vector is added to the ith and $(-c)$ times the jth vector is added to the ith, then the original set of vectors results, proving (b). A similar analysis proves (c).

Example 5.5 (a) Given $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$, then

$$AE_{12} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 5 & 3 \end{bmatrix}; \qquad AE_{2+(4)1} = \begin{bmatrix} 1 & 2+4 \\ 0 & 1+0 \\ 3 & 5+12 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \\ 3 & 17 \end{bmatrix};$$

$$E_{(4)2}A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}; \qquad E_{2+(4)1}A = \begin{bmatrix} 1+0 & 2+4 \\ 0 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}.$$

(b) $E_{13}{}^* = E_{13}$; $\quad E_{2+(4)1}{}^* = E_{1+(4)2}$.

(c) $E_{3+(5)2}{}^{-1} = E_{3+(-5)2}$; $\quad E_{(4)1}{}^{-1} = E_{(1/4)1}$.

Questions

1. A diagonal matrix necessarily ______.
   (a) is square,
   (b) has 1's on the diagonal,
   (c) has 0's off the diagonal.

2. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in echelon form, then ______ $= 0$.

3. The tuple, $\langle 0, 1, 0, 0, 2\rangle$, has ______ initial zero(s).

4. The rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is ______.

5. Elementary column operations may change the ______ of a matrix.
   (a) rank,
   (b) size,
   (c) number of zero column vectors.

6. An elementary matrix must ______.
   (a) be nonsingular,
   (b) have rank 2 or 3,
   (c) be in echelon form.

Problems

1. Do Problem Set B at the end of the chapter.

Proofs

1. Prove that if $c \neq 0$, then Sp $\{c\mathbf{u}_1, \mathbf{u}_2\} = \text{Sp}\{\mathbf{u}_1, \mathbf{u}_2\}$.

6. Canonical Forms

Various relationships may be defined in a natural way on the set of $m \times n$ matrices. As an instance, let A be related to B, written $A \sim B$, provided there is a nonsingular matrix C such that $A = CB$. Certain relations called *equivalence relations* are of special interest and importance. An equivalence relation satisfies the three properties below.

---

(1) *Reflexive:* $A \sim A$,
(2) *Symmetric: If* $A \sim B$, *then* $B \sim A$,
(3) *Transitive: If* $A \sim B$ *and* $B \sim C$, *then* $A \sim C$.

If $\sim$ is an equivalence relation and $A \sim B$, then A is said to be *equivalent* to B. An equivalence relation on a set decomposes the set into a collection of disjoint subsets called *equivalence classes.* Two elements are in the same equivalence class if and only if they are equivalent. A set of *canonical forms* for an equivalence relation is a representative set consisting of one element from each equivalence class. Canonical forms are usually selected for their simplicity.

Before applying these concepts to the study of matrices, we shall consider a simple illustration. Let $\sim$ denote the relation on the set of positive integers defined by the condition

$$a \sim b \text{ if and only if } (a - b) \text{ is divisible by 3.}$$

It may be verified by simple arithmetic that $\sim$ is an equivalence relation. There are three equivalence classes in this case.

(1) $1, 4, 7, 10, \ldots,$
(2) $2, 5, 8, 11, \ldots,$
(3) $3, 6, 9, 12, \ldots,$

A natural set of canonical forms is 1,2,3; a less-natural choice of canonical forms would be 1,11,36.

Examples of important equivalence relations in matrix theory are given by $A \sim B$ if and only if there exist(s)

(1) a nonsingular matrix C such that $B = AC$,
(2) two nonsingular matrices C and D such that $B = CAD$,
(3) a matrix C such that $B = C^{-1}AC$,
(4) a nonsingular matrix C such that $B = C^*AC$, and
(5) an orthogonal matrix C such that $B = C^*AC$.

Some of these are meaningful only for square ($n \times n$) matrices, and others have special interest for particular subclasses of $n \times n$ matrices. In this section we shall consider only (2) and (4). We define two $m \times n$ matrices A and B to be *equivalent* if there are nonsingular matrices C and D such that $B = CAD$. It should be noted that the term "equivalent" is used here in a more restricted sense than in the beginning of this section. This restricted meaning will be assumed in the remainder of this section. Two $n \times n$ matrices A, B are said to be *congruent* if $B = C^*AC$ for some nonsingular C. Equivalence and congruence each give an equivalence relation (see Proofs, exercises 1 and 2). Our first goal is to express canonical forms for the equivalence relation. We consider the $m \times n$ matrix

$$D_r^{m,n} = \langle \mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{0}, \ldots, \mathbf{0} \rangle,$$

which has a 1 in the first r diagonal entries and 0 elsewhere. The next result shows the $D_r^{m,n}$ form a set of canonical forms.

Theorem on Canonical Forms under Equivalence

A $m \times n$ matrix A is equivalent to $D_r^{m,n}$ if and only if it has rank r.

The sufficiency condition of this theorem follows, since multiplication by a nonsingular matrix does not alter rank. For the necessity part, we assume A has rank r and seek a nonsingular C and D such that $CAD = D_r^{m,n}$. The idea of the proof is to convert A to $D_r^{m,n}$ by elementary row and column operations and use the correspondence between these operations and pre- and postmultiplication by elementary matrices. The following example illustrates the idea.

Example 6.1 The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{bmatrix}$$

may be converted to $D_2^{3,3}$ by the following sequence of matrices, each of which is obtained from the preceding matrix by a row or column operation.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 4 & 7 & 10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_2^{3,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using elementary operations, this conversion may be written

$$\begin{array}{llll} A, & A_1 = E_{12}A, & A_2 = E_{1+(-2)3}A_1, & A_3 = E_{2+(-1)3}A_2, \\ & A_4 = A_3E_{2+(-3/2)1}, & A_5 = A_4E_{3+(-2)1}, & A_6 = A_5E_{3+(-2)2}, \\ & & D_2^{3,3} = A_6E_{(1/2)1}. & \end{array}$$

If we let

$$C = E_{2+(-1)3}E_{1+(-2)3}E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

and

$$D = E_{2+(-3/2)1}E_{3+(-2)1}E_{3+(-2)2}E_{(1/2)1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $CAD = D_2^{3,3}$.

The matrices $D_r^{m,n}$ are called the *canonical forms under equivalence*. Each $m \times n$ matrix is equivalent to exactly one canonical form under equivalence. The rank of a matrix is an *invariant* under equivalence, meaning that any two equivalent matrices have the same rank.

The canonical form under equivalence of a nonsingular $n \times n$ matrix is the identity I^n. Using the foregoing methods we can also show the next proposition.

Proposition 6.1 Every nonsingular $n \times n$ matrix, A, is a product of elementary matrices.

The proof follows from the previous observation that there exist C and D, which are products of elementary matrices, such that

$$CAD = I^n.$$

Considering the case

$$C = E_1E_2 \quad \text{and} \quad D = E_3E_4E_5,$$

where the E_i are elementary matrices, then

$$C^{-1} = E_2^{-1}E_1^{-1}; \qquad D^{-1} = E_5^{-1}E_4^{-1}E_3^{-1},$$

from which we see that

$$A = C^{-1}CADD^{-1} = C^{-1}I^nD^{-1} = E_2^{-1}E_1^{-1}E_5^{-1}E_4^{-1}E_3^{-1}.$$

The desired conclusion follows, since the inverse of an elementary matrix is again an elementary matrix.

The congruence relation is meaningful only for square matrices, and it has special interest on a subclass of these called *symmetric* matrices. A linear function f from $\mathbf{R}^n$ to $\mathbf{R}^n$ is *self-adjoint* provided $f = f^*$, and an $n \times n$ matrix A is *symmetric* if $A = A^*$. Thus, self-adjoint and symmetric are corresponding terms. The ij and ji entries of a symmetric matrix are equal for all i and j; hence, entries placed symmetrically with the diagonal are equal. The study of congruences of symmetric matrices has application to the study of quadratic forms, which in turn are useful in the analysis of maxima and minima of functions of several variables.

Example 6.2 Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \qquad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then A is symmetric provided that $a_{12} = a_{21}$; B is symmetric if $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$.

Our goal is to establish canonical forms for the set of symmetric $n \times n$ matrices relative to the congruence relation. We shall modify the procedure for finding canonical forms under equivalence. Consider first the matrix,

$$A_1 = E_1^*AE_1,$$

where A is symmetric and E_1 is an elementary matrix. From

$$A_1^* = (E_1^*AE_1)^* = E_1^*A^*(E_1^*)^* = E_1^*AE_1 = A_1,$$

it follows that A_1 is also symmetric. From the study of elementary matrices, postmultiplication by E_1 produces a certain operation on the column vectors and premultiplication by E_1^* produces the same operation on the row vectors. If

$$A_2 = E_2^*A_1E_2,$$

where E_2 is an elementary matrix, then

$$A_2 = E_2^*E_1^*AE_1E_2 = (E_1E_2)^*AE_1E_2,$$

and hence, A and A_2 are congruent, symmetric matrices. Continuing the process gives the following result.

Proposition 6.2 If B is obtained from A by successively applying elementary operations in pairs, a pair consisting of the same operation on both the column and row vectors, then B is congruent to A.

The operations within a pair may be applied in either order. Using such pairs of operations we can, in a manner similar to that of obtaining the canonical form under equivalence, convert any symmetric matrix to diagonal form. It is not possible, however, to obtain only 1's and 0's on the diagonal, since a pair of elementary operations cannot change the sign of an entry of a diagonal matrix. Letting

$$D^n_{\mathrm{rs}} = \langle \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{\mathrm{s}}, (-1)\mathbf{e}_{\mathrm{s}+1}, \ldots, (-1)\mathbf{e}_{\mathrm{r}}, \mathbf{0}, \ldots, \mathbf{0} \rangle$$

be the $n \times n$ diagonal matrix with $+1$ in the first s diagonal positions, -1 in the next $(\mathrm{r}-\mathrm{s})$ diagonal positions, and 0 elsewhere, the following result can be shown.

Theorem on Canonical Form under Congruence

An $n \times n$ symmetric matrix, A, is congruent to exactly one $D^{\mathrm{n}}_{\mathrm{rs}}$, where $0 \le \mathrm{s} \le \mathrm{r} \le n$.

A formal proof of this theorem will not be considered. The number s is called the *signature of A*; r is, of course, the rank of A. The signature and rank are both *invariants* with respect to the congruence relation. The conversion process to canonical form under congruence will be illustrated by the next example.

Example 6.3 The following sequence gives a conversion to canonical form under congruence.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 \\ 2 & -4 & 6 \\ 3 & 0 & 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -4 & 0 \\ 3 & 0 & 8 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$D^3_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Thus A has rank 3 and signature 1. The process may be described using elementary matrices.

$$\begin{aligned} &A, \\ A_1 &= E_{2+(-2)1}{}^*AE_{2+(-2)1}, \\ A_2 &= E_{3+(-3)1}{}^*A_1E_{3+(-3)1}, \\ D^3_{3,1} &= E_{1/2(2)}{}^*A_2E_{1/2(2)}. \end{aligned}$$

Letting $C = E_{2+(-2)1}E_{3+(-3)1}E_{(1/2)2} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

we have $D^3_{3,1} = C^*AC$.

Questions

1. An equivalence relation satisfies the _______, _______, and _______ properties.
2. A representative set, one from each equivalence class, is called a set of _______.
3. A and B are equivalent if $B = CAD$ where C and D are _______.
4. An invariance under equivalence is _______.
   (a) rank,
   (b) signature,
   (c) symmetry.
5. The signature is a congruent invariance for _______ matrices.
   (a) nonsingular,
   (b) symmetric,
   (c) elementary.
6. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank _______ and signature _______.
7. If A is symmetric then C^*AC is _______ matrix.
   (a) a nonsingular,
   (b) a symmetric,
   (c) an elementary.

Problems

1. Do Problem Set C at the end of the chapter.
2. Do Problem Set D at the end of the chapter.

Exercises

1. Write as a product of elementary matrices the following matrices.

$$\text{(a)}\quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \text{(b)}\quad \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

2. Find a matrix C so that $C^*AC = D^3_{3,2}$ if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Proofs

1. Prove that the relation $B \sim A$, provided $B = CAD$ for some nonsingular C and D, is an equivalence relation.
2. Prove that the congruence relation is an equivalence relation on symmetric $n \times n$ matrices.

7. Systems of Linear Equations

In this section we apply the theory of matrices and vector spaces to the study of linear equations. A system of m linear equations in n unknowns has the form, where a_{ij} and b_i are real numbers,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m . \end{aligned}$$

The *coefficient matrix* and the *augmented matrix* of this system are, respectively,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

The *constant vector* of the system is $\mathbf{u} = \langle b_1, b_2, \ldots, b_m \rangle$. The symbols $\mathbf{u}_i$ and $\mathbf{u}_j'$

will respectively denote the ith row vector and the jth column vector of the coefficient matrix. The system has the corresponding vector equation,

$$\mathbf{u} = x_1\mathbf{u}_1' + x_2\mathbf{u}_2' + \cdots + x_n\mathbf{u}_n'.$$

Equating the ith coordinates in the vector equation gives the ith equation of the linear system. Hence, a set of numbers $x_1, \ldots, x_n$ satisfies the given system if and only if it satisfies the corresponding vector equation. We define the *solution set* **P** of the vector equation by the expression,

$$\mathbf{P} = \{\langle x_1, x_2, \ldots, x_n\rangle; \mathbf{u} = x_1\mathbf{u}_1' + x_2\mathbf{u}_2' + \cdots + x_n\mathbf{u}_n'\}.$$

Thus, $\langle x_1, x_2, \ldots, x_n\rangle$ is in **P** provided $x_1, x_2, \ldots, x_n$ satisfy the given system. We now seek to determine the nature of **P**. The three systems

$$\begin{array}{lll} x+y=0 & \quad x+y=1 & \quad x+y=1 \\ x+y=1; & \quad x-y=1; & \quad 2x+2y=2 \end{array}$$

show that, in general, **P** may have no elements, one element, or infinitely many elements. The given system is *inconsistent* if **P** has no elements (**P** is the empty set), and *consistent* if **P** has at least one element. From the corresponding vector equation, we observe that the given system is consistent provided the constant vector is a linear combination of the column vectors of the coefficient matrix. This is true if the column vectors of the coefficient and augmented matrices span the same space. We therefore conclude the following result.

Proposition 7.1 A system of linear equations is consistant if and only if its coefficient and augmented matrices have the same rank.

In order to get a description of **P**, it is convenient to introduce the *associated-homogeneous system* which is obtained from the given system by replacing the constant vector by **0**. The homogeneous system has the same coefficient matrix as the given system. Letting **S** denote the solution set of the homogeneous system, then $\langle x_1, x_2, \ldots, x_n\rangle$ is in **S** provided

$$\mathbf{0} = x_1\mathbf{u}_1' + x_2\mathbf{u}_2' + \cdots + x_n\mathbf{u}_n'.$$

The homogeneous system is necessarily consistent, since **S** contains the zero vector. In fact, $\langle x_1, x_2, \ldots, x_n\rangle$ is in **S** if and only if it is orthogonal to each row vector of the coefficient matrix. This is seen by observing that the ith equation of the homogeneous system may be written

$$\langle x_1, x_2, \ldots, x_n\rangle \cdot \mathbf{u}_i = 0,$$

where we recall that $\mathbf{u}_i$ is the ith row vector of the coefficient matrix. We may therefore conclude our next proposition.

Proposition 7.2 The solution set of the associated homogeneous system is the subspace of $\mathbf{R}^n$,

$$\mathbf{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}^{\perp}.$$

We shall now show that the solution set $\mathbf{P}$ of any consistent system of linear equations is a translate of the solution set $\mathbf{S}$ of its associated homogeneous system; hence, the solution set is a k-plane for some k.

Proposition 7.3 If $\mathbf{v} = \langle c_1, c_2, \ldots, c_n \rangle$ is a solution of a given system of linear equations and $\mathbf{S}$ is the solution set of its associated homogeneous system, then the solution set of the given system is $\mathbf{P} = \mathbf{v} + \mathbf{S}$.

A proof will be made for the case where $n = 3$. It will first be shown that $\mathbf{v} + \mathbf{S}$ is a subset of $\mathbf{P}$. Letting $\mathbf{w} = \langle d_1, d_2, d_3 \rangle$ (arbitrary) be in $\mathbf{S}$, then the equality chain

$$\begin{aligned} \mathbf{u} &= \mathbf{u} + \mathbf{0} \\ &= (c_1\mathbf{u}_1' + c_2\mathbf{u}_2' + c_3\mathbf{u}_3') + (d_1\mathbf{u}_1' + d_2\mathbf{u}_2' + d_3\mathbf{u}_3') \\ &= (c_1 + d_1)\mathbf{u}_1' + (c_2 + d_2)\mathbf{u}_2' + (c_3 + d_3)\mathbf{u}_3' \end{aligned}$$

proves that $\mathbf{v} + \mathbf{w} = \langle c_1 + d_1, c_2 + d_2, c_3 + d_3 \rangle$ is in $\mathbf{P}$ as desired. For the reverse inclusion, if $\mathbf{v}' = \langle c_1', c_2', c_3' \rangle$ is also in $\mathbf{P}$, then

$$\begin{aligned} \mathbf{0} &= \mathbf{u} - \mathbf{u} \\ &= (c_1'\mathbf{u}_1' + c_2'\mathbf{u}_2' + c_3'\mathbf{u}_3') - (c_1\mathbf{u}_1' + c_2\mathbf{u}_2' + c_3\mathbf{u}_3') \\ &= (c_1' - c_1)\mathbf{u}_1' + (c_2' - c_2)\mathbf{u}_2' + (c_3' - c_3)\mathbf{u}_3' \end{aligned}$$

shows that $\mathbf{v}' - \mathbf{v} = \langle c_1' - c_1, c_2' - c_2, c_3' - c_3 \rangle$ is in $\mathbf{S}$. Therefore, $\mathbf{v}'$ is in $\mathbf{v} + \mathbf{S}$ and hence $\mathbf{P}$ is a subset of $\mathbf{v} + \mathbf{S}$. This completes the proof.

Example 7.1 The system

$$\begin{aligned} 2x_1 + x_2 - 4x_3 &= 8 \\ 3x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

has a particular solution, $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, as may be easily verified. The associated homogeneous system is

$$\begin{aligned} 2x_1 + x_2 - 4x_3 &= 0 \\ 3x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

and its solution set consists of all vectors orthogonal to both $\langle 2, 1, -4 \rangle$ and $\langle 3, -1, 2 \rangle$. This is the set of scalar multiples of

$$\langle 2, 1, -4 \rangle \times \langle 3, -1, 2 \rangle = \langle -2, -16, -5 \rangle.$$

Hence, the solution set of the given system is $\langle 1, 2, 1 \rangle + \mathrm{Sp}\{\langle -2, -16, -5 \rangle\}$.

Thus far in this section we have been concerned with the form of the solution set, rather than techniques of finding it. A standard method of solving linear systems, called the *Gaussian elimination* method, uses the principle that elementtary operations on the row vectors of the augmented matrix do not alter the solution set (see Proofs, exercise 1). Conversion of the augmented matrix to echelon form yields the augmented matrix of a system which has the same solution set and can be easily solved.

Example 7.2 The system

$$\begin{aligned} x_1 + 2x_2 \qquad\quad + x_4 &= 2 \\ 2x_1 + 7x_2 + 6x_3 + 2x_4 &= 13 \\ 3x_1 + 6x_2 + 2x_3 + 7x_4 &= 8 \end{aligned}$$

has the augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 7 & 6 & 2 & 13 \\ 3 & 6 & 2 & 7 & 8 \end{bmatrix},$$

which has an echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 6 & 0 & 9 \\ 0 & 0 & 2 & 4 & 2 \end{bmatrix}.$$

An equivalent system is, therefore,

$$\begin{aligned} x_1 + 2x_2 + \qquad + x_4 &= 2 \\ 3x_2 + 6x_3 + \qquad &= 9 \\ 2x_3 + 4x_4 &= 2. \end{aligned}$$

Setting $x_4 = 1$ gives, solving from the bottom equation up,

$$\begin{aligned} & & x_4 &= 1, \\ 2x_3 &= -2, & x_3 &= -1, \\ 3x_2 - 6 &= 9, & x_2 &= 5, \\ x_1 + 10 + 1 &= 2, & x_1 &= -9, \end{aligned}$$

as a particular solution. The homogeneous system

$$\begin{aligned} x_1 + 2x_2 + \qquad + x_4 &= 0 \\ 3x_2 + 6x_3 \qquad &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

has the solution

$$\begin{aligned} & & x_4 &= c \text{ (c arbitrary)}, \\ 2x_3 + 4c &= 0, & x_3 &= -2c, \\ 3x_2 - 12c &= 0, & x_2 &= 4c, \\ x_1 + 8c + c &= 0, & x_1 &= -9c. \end{aligned}$$

An arbitrary vector in the solution set has the form

$$\langle -9, 5, -1, 1\rangle + c\langle -9, 4, -2, 1\rangle,$$

and hence the solution set is the 1-plane,

$$\langle -9, 5, -1, 1\rangle + \mathrm{Sp}\{\langle -9, 4, -2, 1\rangle\}.$$

Questions

1. A(n) ______ system of equations has no solution.
2. A system of equations is consistent if its ______ and ______ matrices have the same rank.
3. Replacing the constant vector of a system by **0** gives the ______ system.
4. The solution set of a linear system is a ______ of the solution set of the associated homogeneous system.
   (a) translate,
   (b) subspace,
   (c) scalar multiple.

Exercises

1. For the following systems, give the coefficient matrix, augmented matrix, constant vector, and associated homogeneous system.

   (a) $$\begin{aligned} x - 2y + z &= 3 \\ 2x + 4y + 3z &= 0, \end{aligned}$$

   (b) $$\begin{aligned} x + y &= 16 \\ x - y &= 4 \\ 2x - 3y &= 8. \end{aligned}$$

2. For the following systems, find (1) a particular solution, (2) a general solution of the associated homogeneous systems, involving arbitrary constants if necessary, and (3) the solution set as a suitable k-plane.

   (a) $$\begin{aligned} x + y - z &= 1 \\ x + 2y + z &= 0, \end{aligned}$$

   (b) $$\begin{aligned} w - x + y + z &= 0 \\ w + x - y - z &= 1 \\ w - 3x + 3y + 3z &= -1 \\ 3w - 3x + 3y + 3z &= 0. \end{aligned}$$

Proofs

1. Prove that the solution set of the system

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

is not changed by the following operations on the augmented matrix of the system:
(a) addition of c times the first row to the second row,
(b) multiplication of the first row by $c \neq 0$.

Problems

A. *Affine Images of Sets*

The affine image of a line in the Cartesian plane is given in the following manner.

A.1 Given the line in $\mathbf{R}^2$ with matrix form $C_1 + rA$ and the affine function f from $\mathbf{R}^2$ to $\mathbf{R}^2$ with constant matrix C_2 and linear matrix B, the f-image of the line is

$$(C_2 + BC_1) + rBA = C_2 + B(C_1 + rA),$$

provided $BA \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

If

$$BA = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in A.1, then the f-image consists of the single element $C_2 + BC_1$. The image formula in A.1 should be compared with the formula in C.2 in Chapter V.

1. Find the f-image of $\langle 2, 3 \rangle + r\langle 3, -1 \rangle$ if f is an affine function with respective constant and linear matrices,

   (a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix}$; (b) $\begin{bmatrix} 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$.

A similar formula holds if f is an affine function from $\mathbf{R}^2$ to $\mathbf{R}^3$, $\mathbf{R}^3$ to $\mathbf{R}^2$, or $\mathbf{R}^3$ to $\mathbf{R}^3$.

2. Find the f-image of $\langle 1, 0\rangle + r\langle 2, -4\rangle$ if f is an affine function with respective constant and linear matrices.

(a) $\begin{bmatrix}3\\1\\6\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 3\\-5 & 1\end{bmatrix}$; (b) $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1 & 2\\2 & 0\\3 & 1\end{bmatrix}$.

3. Find the f-image of $\langle 1, 0, 2\rangle + r\langle 3, 4, 1\rangle$ if f is an affine function with respective constant and linear matrices,

(a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1 & 2 & 3\\0 & 4 & -6\end{bmatrix}$; (b) $\begin{bmatrix}3\\4\\-2\end{bmatrix}, \begin{bmatrix}2 & -4 & 6\\-1 & 0 & 1\\3 & 2 & -7\end{bmatrix}$.

The affine image of a plane now follows.

A.2 Given the plane in $\mathbf{R}^3$ with matrix form $C_1 + rA_1 + sA_2$ and affine function f from $\mathbf{R}^3$ to $\mathbf{R}^3$ with constant matrix C_2 and linear matrix B, then the f-image of the plane is

$$C_2 + BC_1 + rBA_1 + sBA_2 = C_2 + B(C_1 + rA_1 + sA_2)$$

provided $\{BA_1, BA_2\}$ is independent.

If $\{BA_1, BA_2\}$ is dependent, then the f-image is a line or point.

4. Find the f-image of $\langle 3, -1, 2\rangle + r\langle 4, 0, 6\rangle + s\langle 1, -1, 2\rangle$, if f is an affine function with respective constant and linear matrices,

(a) $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}6 & 0 & 1\\-1 & 2 & 4\\3 & 5 & 7\end{bmatrix}$; (b) $\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2 & 6 & 0\\1 & 3 & 7\\-1 & 4 & 3\end{bmatrix}$.

The affine images of segments, parallelograms, and parallelepipeds are found in a similar manner by restricting values of r, s, and t to the interval $[0, 1]$.

5. Find the f-image of the segment $\langle 1, 3\rangle + [r\langle 2, -4\rangle]$ if f is an affine function with respective constant and linear matrices,

(a) $\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}1 & 2\\-1 & 5\end{bmatrix}$; (b) $\begin{bmatrix}1\\-2\\4\end{bmatrix}, \begin{bmatrix}2 & 3\\3 & -1\\0 & 5\end{bmatrix}$.

6. Find the f-image of the parallelogram, $\langle 1, 2\rangle + [r\langle 3, -2\rangle + s\langle 2, 9\rangle]$, given that f is an affine function with respective constant and linear matrices.

(a) $\begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}3 & 4\\-5 & 2\end{bmatrix}$; (b) $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3 & 4\\0 & -1\\2 & -5\end{bmatrix}$.

7. Find the f-image of the parallelepiped,

$$\langle 1, 6, 3\rangle + [r\langle 2, 0, 1\rangle + s\langle -1, 3, 5\rangle + t\langle 1, 1, 0\rangle],$$

given that f is an affine function with respective constant and linear matrices.

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 0 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{bmatrix}.$$

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8. For each of the following sets, find the f-image.

Set	Constant Matrix	Linear Matrix
(a) $\langle 0, 1\rangle + r\langle 2, -3\rangle$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$,
(b) $\langle 3, 0, 2\rangle + r\langle 1, 1, 5\rangle$	$\begin{bmatrix} 1 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$,
(c) $\langle 2, 0, 3\rangle + r\langle 1, 0, 2\rangle$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 0 \\ 0 & -1 & 6 \end{bmatrix}$,
(d) $\langle 1, 1, 0\rangle + r\langle 1, 1, 2\rangle + s\langle 0, 1, 4\rangle$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$,
(e) $\langle 1, 3\rangle + [r\langle 2, -1\rangle]$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & -1 \end{bmatrix}$,
(f) $\langle 1, 0\rangle + [r\langle 2, 0\rangle + s\langle 1, 1\rangle]$	$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ -1 & 1 \end{bmatrix}$,
(g) $\langle 1, 1, -1\rangle + [r\langle 1, 0, 0\rangle + s\langle 1, 0, -1\rangle + t\langle 1, 1, 5\rangle]$	$\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$.

B. Elementary Matrices

An *elementary matrix* is a matrix that can be obtained from an identity matrix by an elementary operation on the column vectors. We now introduce some special notation.

B.1 (I) E_{ij} is the matrix obtained from the identity by interchanging the ith and jth columns.
(II) $E_{i+(c)j}$ is the matrix obtained from the identity by adding c times the jth column to the ith column.
(III) $E_{(c)i}$ is the matrix obtained from the identity by multiplying the ith column by c, $c \neq 0$.

1. For 2×2 matrices find the following matrices.

(a) E_{12}, (b) $E_{2+(4)1}$, (c) $E_{(7)2}$.

2. For 3×3 matrices find the indicated elementary matrix.

(a) E_{13}, (b) $E_{3+(5)2}$, (c) $E_{(5)1}$.

The inverse of an elementary matrix is again an elementary matrix. There are the following special cases.

B.2 (I) ${E_{ij}}^{-1} = E_{ij}$,
(II) ${E_{i+(c)j}}^{-1} = E_{i+(-c)j}$,
(III) ${E_{(c)i}}^{-1} = E_{(1/c)i}$.

3. Write as an elementary matrix symbol the following inverses.

(a) ${E_{12}}^{-1}$, (b) ${E_{2+(5)3}}^{-1}$, (c) ${E_{(6)2}}^{-1}$.

Postmultiplication by an elementary matrix produces an elementary operation on the column vectors, and premultiplication produces an elementary operation on the row vectors.

B.3 (I) AE_{ij} is obtained from A by interchanging the ith and jth columns.
(II) $AE_{i+(c)j}$ is obtained from A by adding c times the jth column to the ith column.
(III) $AE_{(c)i}$ is obtained from A by multiplying the ith column by c.

B.4 (I) $E_{ij}A$ is obtained from A by interchanging the ith and jth rows.
(II) $E_{i+(c)j}A$ is obtained from A by adding c times the ith row to the jth row.
(III) $E_{(c)i}A$ is obtained from A by multiplying the ith row by c.

The reversal of i and j should be noted in the comparison of B.3 (II) and B.4 (II).

4. Given

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix},$$

find by the rules set out in B.3 and B.4 the following products.

(a) $E_{12}A$, (b) AE_{12}, (c) $E_{1+(4)2}A$,
(d) $AE_{1+(4)2}$, (e) $E_{(6)2}A$, (f) $AE_{(6)2}$.

5. Given

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix},$$

find

(a) $E_{13}A$, (b) AE_{12}, (c) $E_{1+(3)3}A$,
(d) $AE_{2+(3)1}$, (e) $E_{(4)3}A$, (f) $AE_{(5)2}$.

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6. For 3×3 matrices give the following elementary matrices.

(a) E_{23}, (b) $E_{1+(5)3}$, (c) $E_{(-1)2}$,
(d) E_{13}^{-1}, (e) $E_{2+(4)1}^{-1}$, (f) $E_{(2)3}^{-1}$.

7. Given

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix},$$

find the indicated products.

(a) $E_{12}A$, (b) AE_{13}, (c) $E_{2+(4)1}A$,
(d) $AE_{3+(5)2}$, (e) $E_{(6)1}A$, (f) $AE_{(2)3}$.

C. *Canonical Forms under Equivalence*

A matrix has *canonical form under equivalence* if it has 1 in an initial number of diagonal entries and 0 elsewhere. The canonical forms of 2×2 matrices are shown below.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. Give the canonical forms under equivalence for

(a) 2×3 matrices, (b) 3×2 matrices.

Each matrix has a unique canonical form under equivalence to which it can be converted by elementary operations on its row and column vectors. A successful procedure for finding this canonical form is to first convert to echelon form by row operations, then convert to diagonal form by column operations, and finally, get canonical form by multiplying each column by a suitable constant.

2. Convert the following matrices to canonical form under equivalence.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$.

The canonical form of a matrix A can be expressed as CAD where C and D are products of elementary matrices corresponding to the elementary operations used in the conversion to canonical form.

3. Using the conversions in 2(a), (b), and (c), write the canonical forms as a product of the given matrices and elementary matrices.
4. From 3, write each of the matrices in 2(a), (b), and (c) in the form CAD where A is the given matrix and C and D are products of elementary matrices.

We shall need to know the following fact.

C.1 If A and B are square matrices of the same size with respective inverses A^{-1} and B^{-1}, then $(AB)^{-1} = B^{-1}A^{-1}$.

If A is a square matrix with an inverse, then its canonical form is the identity I. From $CAD = I$ we can write

$$AD = C^{-1}CAD = C^{-1}I = C^{-1}, \quad \text{and} \quad A = ADD^{-1} = C^{-1}D^{-1}.$$

If we combine this conclusion with C.1 and the property—from Problem Set B—that the inverse of an elementary matrix is again an elementary matrix, then it follows that we can write A as a product of elementary matrices.

5. From

$$E_{1+(-1)2}\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}E_{2+(-1)1}E_{(1/2)1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

write

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

as a product of elementary matrices in symbolic form.

6. Write the following matrices as a product of elementary matrices in symbolic form.

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$.

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7. Write the canonical form of A as a product of A and elementary matrices.

(a) $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

8. Write each matrix as a product of elementary matrices.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

D. *Canonical Forms under Congruence*

The transpose A^* of a matrix A is obtained from A by interchanging its rows and columns. Thus, the row vectors of A^* are the column vectors of A.

1. Find the transpose of the following matrices.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 6 \\ 2 & 3 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

A square matrix A is *symmetric* if $A^* = A$. Thus A is symmetric provided each entry equals the entry positioned symmetrically opposite the diagonal.

2. Complete the lettered entries to make the matrix symmetric.

(a) $\begin{bmatrix} 1 & 0 & c \\ a & 3 & 5 \\ 2 & b & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 1 & 5 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & a & d & f \\ 1 & 2 & 5 & 7 \\ 0 & b & 3 & 6 \\ 2 & c & e & 4 \end{bmatrix}$.

Associated with each symmetric matrix is a unique *canonical form under congruence.* The canonical form under congruence has initially $+1$'s and then (-1)'s on the diagonal. The remaining entries on the diagonal and elsewhere are 0. The number of $+1$'s is called the *signature* and denoted s; the total number of $+1$'s and -1's is the *rank*, denoted r. For 2×2 matrices there are the following possibilities.

$$\underset{r=s=2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad \underset{r=2,\ s=1}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \quad \underset{r=s=1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad \underset{r=2,\ s=0}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}, \quad \underset{r=1,\ s=0}{\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad \underset{r=s=0}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}.$$

3. For 3×3 matrices give all canonical forms under congruence, together with the rank and signature, which have $+1$ as the upper left entry.

The canonical form under congruence may be found for a given symmetric matrix A by applying elementary operations in pairs, a pair consisting of the same operation applied on both the column and row vectors of A. The technique resembles that for reduction to echelon form.

4. Convert

$$\begin{bmatrix} 1 & 6 \\ 6 & 100 \end{bmatrix}$$

to canonical form under congruence using the following procedure.
(a) add (-6) times the first row to the second row,
(b) add (-6) times the first column to the second column,
(c) multiply the second row by $1/8$,
(d) multiply the second column by $1/8$.

5. Convert to canonical form under congruence and give the rank and signature of each matrix below.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

We need to be aware that the operations must be applied in pairs. At times the upper left entry cannot be made nonzero by a pair of interchanges. In that case a suitable multiple of another row and column must be added to the first.

6. Convert to canonical form under congruence and find the rank and signature of each matrix.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$.

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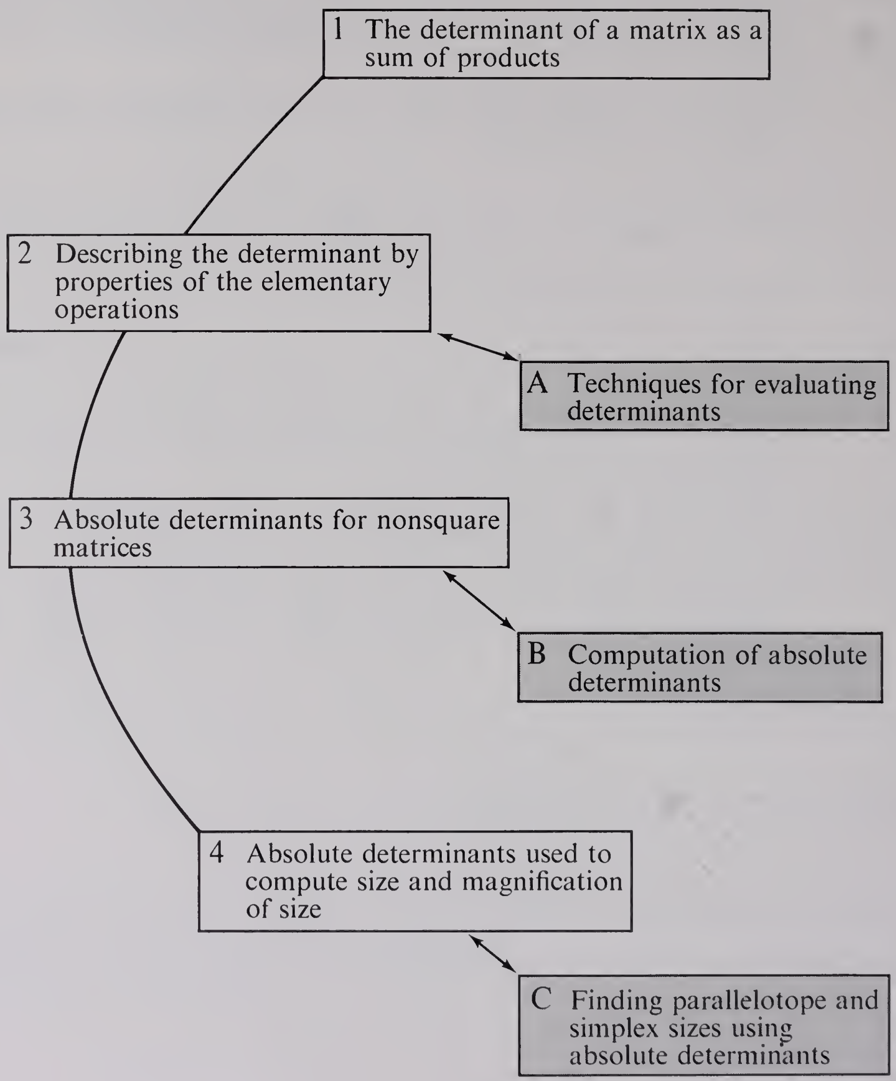
7. Convert to canonical form under congruence and find the rank and signature for each matrix below.

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 4 & 6 & 8 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, (f) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.


1 The determinant of a matrix as a sum of products
2 Describing the determinant by properties of the elementary operations
A Techniques for evaluating determinants
3 Absolute determinants for nonsquare matrices
B Computation of absolute determinants
4 Absolute determinants used to compute size and magnification of size
C Finding parallelotope and simplex sizes using absolute determinants

VII

Determinants

In this chapter we associate with each square matrix a number called its *determinant*. This number is often used to give information about the matrix, or information about a mathematical or physical quantity from which the matrix was derived. Historically, the application of determinants preceded the systematic study of matrices. In 1750 Gabriel Cramer of Switzerland (1704–1752) employed determinants in the solution of systems of equations. During the last half of the 19th century there was an enormous amount of research on determinants, much of it having little interest today.

The determinant of a matrix may be defined in the following conventional ways.

(I) The determinant is a sum of products of the entries of the matrix.

(II) It is a number characterized by values assigned to special matrices and changes produced by elementary matrix operations.

These two definitions, of course, lead to the same number. For $n \times n$ matrices with $n \geq 4$, definition (II) gives a much less laborious method for computing determinants. On the other hand, the relationship of the determinant to a particular mathematical problem is usually first noted in the summation form of (I). We shall begin with (I). then introduce (II), and show that they lead to the same value.

1. Determinant of a Square Matrix

Preliminary to the definition of the determinant as a sum of products, it is necessary to study permutations briefly. A *permutation* generally denotes an ordering on a finite set. It may be more precisely defined as an injective function from a finite set X to itself. On the set $\{1, 2\}$ there are two permutations,

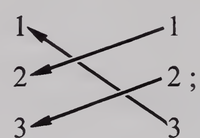
$$1, 2 \quad \text{and} \quad 2, 1.$$

On the set $\{1, 2, 3\}$ there are six permutations

$$1, 2, 3; \quad 1, 3, 2; \quad 2, 1, 3; \quad 2, 3, 1; \quad 3, 1, 2; \quad 3, 2, 1.$$

It may be shown by induction that there are $n!$ permutations of $\{1, 2, \ldots, n\}$. (It may be recalled that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n$.) There are various ways to symbolize a permutation, each having particular merits. The permutation p: 2, 3, 1 may be written in any of the following ways.

$$\begin{aligned} p(1) &= 2 \\ p(2) &= 3; \\ p(3) &= 1 \end{aligned}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}; \quad \text{or} \quad [2, 3, 1].$$

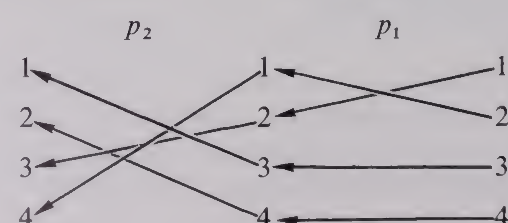
The simpler symbol [2, 3, 1] will be most commonly used here, although others will be employed when convenient.

The *identity permutation* is the natural ordering, or identity function. Thus,

$$[1, 2], \qquad [1, 2, 3], \quad \text{and} \quad [1, 2, 3, 4]$$

are examples of the identity permutation. This is the identity with respect to the *product operation* ($\circ$). When permutations are regarded as orderings or rearrangements, then the product of two permutations describes the process of following one by the other. If permutations are viewed as functions, then this is the composite operation.

Example 1.1 The operation described by the composition $p_2 \circ p_1 = [4, 3, 1, 2] \circ [2, 1, 3, 4]$ is achieved by the following diagram.



Reading from right to left as is customary with the composite, we see $3 \leftarrow 2 \leftarrow 1$. Hence, the first digit of the product is 3. From $4 \leftarrow 1 \leftarrow 2$, $1 \leftarrow 3 \leftarrow 3$, $2 \leftarrow 4 \leftarrow 4$, we obtain the product $p_2 \circ p_1 = [3, 4, 1, 2]$.

Each permutation p has an inverse, p^{-1}, with respect to the product operation. This is evident when permutations are viewed as injective functions. The inverse p^{-1} of p satisfies $p^{-1}(j) = i$ provided $p(i) = j$, and this may be used to find the inverse of p.

Example 1.2 The inverse of $p = [5, 1, 3, 2, 4]$ may be found by inspection from the forms shown below.

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix} \quad \text{and} \quad p^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix}.$$

Since $p(1) = 5$, it follows that $p^{-1}(5) = 1$, and, hence, $e = 1$. Similarly $p(2) = 1$ implies $a = p^{-1}(1) = 2$. In general if j is in the bottom row of the symbol for p, then the number lying above j is $p^{-1}(j)$. Therefore $p^{-1} = [2, 4, 3, 5, 1]$.

An *interchange* is a permutation which leaves all but two integers fixed.

Example 1.3 (a) The three interchanges of $\{1, 2, 3\}$ are

$$[2, 1, 3];\ [3, 2, 1];\ [1, 3, 2].$$

(b) Permutations of $\{1, 2, 3\}$ which are not interchanges may be written as a product of interchanges.

$$\begin{aligned} [1, 2, 3] &= [2, 1, 3] \circ [2, 1, 3], \\ [3, 1, 2] &= [3, 2, 1] \circ [1, 3, 2], \\ [2, 3, 1] &= [1, 3, 2] \circ [3, 2, 1]. \end{aligned}$$

Each permutation is a product of interchanges, which means that each can be obtained by beginning with the identity and successively interchanging two digits. This can be done in many ways but it can be shown that, for a given permutation, either all such ways involve an even number of interchanges or all such ways involve an odd number of interchanges. The proof is found in most modern algebra texts. We may therefore define a permutation to be *even* if it is the product of an even number of interchanges, and *odd* if it the product of an odd number of interchanges.

Example 1.4 We shall show that [2, 6, 5, 1, 3, 4] is even. Beginning with the identity [1, 2, 3, 4, 5, 6], there is the following succession of interchanges.

(i) Interchange 1 and 2 (1 interchange) gives [2, 1, 3, 4, 5, 6].
(ii) Move 6 past 1 (4 interchanges) gives [2, 6, 1, 3, 4, 5].
(iii) Move 5 past 1 (3 interchanges) gives [2, 6, 5, 1, 3, 4].

Adding gives $1+4+3=8$ interchanges, and, hence, the given permutation is even.

Some useful properties of the concept of even and odd are given next.

Proposition 1.1

(a) The identity permutation is even.
(b) An interchange is odd.
(c) If p_1 and p_2 are both even or both odd, then $p_2 \circ p_1$ is even.
(d) If one of p_1, p_2 is even and the other odd, then $p_2 \circ p_1$ is odd.

The proofs of (c) and (d) follow from the observation that if p_1 is a composite of k_1 interchanges and p_2 of k_2 interchanges, then $p_2 \circ p_1$ is a composite of (k_1+k_2) interchanges. The desired conclusion is then deduced from the property of integers that the sum of two positive integers is even if and only if both integers are even or both are odd.

We next associate with each permutation a number, called its *signum* and denoted sgn p, which satisfies the following rule.

$$\operatorname{sgn} p = \begin{cases} +1 & \text{if } p \text{ is even,} \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

An immediate corollary of Proposition 1.1 (c) and (d) is set forth in our next result.

Proposition 1.2 $\operatorname{sgn}(p_2 \circ p_1) = (\operatorname{sgn} p_2)(\operatorname{sgn} p_1)$.

The subject area of permutations is now sufficiently developed for our study of determinants. If $A=[a_{ij}]$ is an $n \times n$ matrix and

$$p=[p(1), p(2), \ldots, p(n)]$$

is a permutation of $\{1,2,\ldots,n\}$, then the symbol A_p will denote the real number

$$A_p = a_{p(1)1}a_{p(2)2}\cdots a_{p(n)n}.$$

Thus A_p is the product of n entries of A, each row and column of A being represented exactly once.

Example 1.5 Given the 4×4 matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 8 \\ 0 & -4 & 7 & -6 \\ 2 & 6 & 9 & -5 \\ 3 & -1 & -3 & 4 \end{bmatrix},$$

(a) if $p = [1, 4, 3, 2]$, then $A_p = (1)(-1)(9)(-6) = 54$;
(b) if $p = [3, 2, 4, 1]$, then $A_p = (2)(-4)(-3)(8) = 192$.

We are now ready for the basic definition.

Definition of Determinant

The determinant, det A, *of an* $n \times n$ *matrix* A *is the sum of all numbers of the form*

$$(\operatorname{sgn} p)A_p,$$

where p *is a permutation of* $\{1, 2, \ldots, n\}$.

A simple formula for the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can easily be found, since there are only two permutations of $\{1, 2\}$. A table will display the computation.

p	sgn p	A_p	$(\operatorname{sgn} p)A_p$
[1, 2]	+1	ad	ad
[2, 1]	-1	bc	$-bc$
			$ad - bc$

Thus, det $A = ad - bc$. It is possible, but impractical, to obtain formulas for the determinants of larger size square matrices. Superior methods of computation will be found in the next section. An example will illustrate the computation of the determinant of a 3×3 matrix by the definition.

Example 1.6 The following is a determinant table for

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & -3 \\ 0 & -2 & 5 \end{bmatrix}.$$

p	sgn p	A_p	(sgn p) A_p
[1, 2, 3]	+1	(1)(−1)(5)	−5
[1, 3, 2]	−1	(1)(−2)(−3)	−6
[2, 1, 3]	−1	(2)(3)(5)	−30
[2, 3, 1]	+1	(2)(−2)(4)	−16
[3, 1, 2]	+1	(0)(3)(−3)	0
[3, 2, 1]	−1	(0)(−1)(4)	0
			−57 = det A

Questions

1. A permutation may be defined as a(n) ______ function.
2. A(n) ______ permutes exactly two integers.
3. The inverse of an even permutation is ______.
   (a) even,
   (b) odd,
   (c) possibly either even or odd.
4. A determinant is ______.
   (a) a matrix,
   (b) a number,
   (c) a vector,
   (d) a permutation.
5. Every permutation ______.
   (a) has an inverse,
   (b) is an interchange,
   (c) is even,
   (d) has a determinant.

Exercises

1. Give all possible interchanges on $\{1, 2, 3, 4\}$.
2. Determine whether each permutation is even or odd.
   (a) [1, 6, 5, 3, 2, 4], (b) [3, 8, 7, 1, 6, 5, 2, 4].
3. Find the inverse of the permutations in Exercise 2.
4. Find $p_1 \circ p_2$ and $p_2 \circ p_1$, given $p_1 = [2, 4, 3, 1]$ and $p_2 = [3, 1, 4, 2]$.
5. Given

$$A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 1 & 7 \\ 3 & 5 & 6 & 2 \\ 2 & 2 & 3 & 5 \end{bmatrix},$$

find (a) $A_{[1,3,4,2]}$, (b) $A_{[3,2,4,1]}$.

6. Construct a determinant table for the following matrices.

(a) $\begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & -3 \\ 2 & -1 & -4 \end{bmatrix}$.

Proofs

1. Using a determinant table, prove that if A is the 3×3 matrix $[a_{ij}]$, then

det $A =$

$$a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

2. Prove that $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33}$.

2. *Characterization of the Determinant*

The computation of the determinant of a 100×100 matrix according to the definition would require more than a lifetime of effort from the fastest of our modern computers. Thus, other approaches must be found for evaluating determinants of large size matrices. To this end we begin with the concept of a *triangular* square matrix. An $n \times n$ matrix $A = [a_{ij}]$ is

(a) *lower triangular* if $a_{ij} = 0$ for $i < j$,
(b) *upper triangular* if $a_{ij} = 0$ for $j < i$,
(c) *triangular* if it is either lower or upper triangular.

Thus A is lower triangular if all entries above the diagonal are 0 and upper triangular if all entries below the diagonal are 0. We are now in position for the fundamental result for evaluating determinants.

Characterization Theorem of the Determinant

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(0) If *A is triangular, then* det *A is the product of the diagonal entries of A.*
(I) *An interchange of two row or columns of A causes* det *A to be multiplied by* (-1).
(II) *Adding a multiple of one row or column of A to another does not change* det *A.*
(III) *Multiplication of a row or column of A by c causes* det *A to be multiplied by c.*

Since every square matrix can be converted to triangular form by elementary operations, it is a consequence of this theorem that det A can be evaluated by the use of statements (0)–(III). This, then, implies that properties (0)–(III) completely determine det A and, hence, could serve as an alternate definition of the determinant of A. An example will precede a consideration of the proof.

Example 2.1 Let A be the 4×4 matrix

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

We evaluate A using the previous theorem. Successive matrices are obtained by elementary row operations.

$$\det A = \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Hence, $\det A = (-1)[(1)(1)(1)(-2)] = 2$.

The proof of property (0) is a consequence of the fact that if A is triangular and p is not the identity permutation, then $A_p = 0$. For instance, if A is lower triangular and p is not the identity, then $p(j) < j$ for some j, $a_{p(j)j} = 0$, and hence, $A_p = 0$, since it is the product of a set of numbers, one of which is 0. Therefore, if A is triangular, the only nonzero term in the defining determinant sum is produced by the identity permutation, and this term is

$$\det A = a_{11}a_{22} \ldots a_{nn} = \text{product of diagonal entries.}$$

A proof of (I) will be indicated for the case $n = 3$, and the first two columns of A are interchanged. Then

$$B = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

is the resulting matrix and

$$B_{[2,1,3]} = a_{22}a_{11}a_{33} = a_{11}a_{22}a_{33} = A_{[1,2,3]},$$
$$B_{[3,1,2]} = a_{32}a_{11}a_{23} = a_{11}a_{32}a_{23} = A_{[1,3,2]}.$$

Every B_p equals a corresponding $A_{p'}$, where p' results from p by an interchange of its first two integers, and hence $\operatorname{sgn} p' = -\operatorname{sgn} p$. The desired conclusion follows from the definition of the determinant.

Preliminary to the proof of (II) we make the following observation.

Proposition 2.1 If two rows or columns of A are equal, then $\det A = 0$.

For the proof of Proposition 2.1, it is seen that interchanging equal rows or columns of A does not change A, and yet it multiplies its determinant by (-1). Hence, by (I), $\det A = (-1) \det A$, from which $\det A = 0$.

We now consider the proof of (II) for the case where $n = 3$ and B results from A by adding c times the first column to the second; then

$$B = \begin{bmatrix} a_{11} & a_{12} + ca_{11} & a_{13} \\ a_{21} & a_{22} + ca_{21} & a_{23} \\ a_{31} & a_{32} + ca_{31} & a_{33} \end{bmatrix}.$$

Introducing the auxiliary matrix

$$C = \begin{bmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{bmatrix},$$

we see that

$$\begin{aligned} B_{[1,2,3]} &= a_{11}(a_{22} + ca_{21})a_{33} \\ &= a_{11}a_{22}a_{33} + ca_{11}a_{21}a_{33} \\ &= A_{[1,2,3]} + cC_{[1,2,3]}. \end{aligned}$$

Similarly, $A_p = B_p + cC_p$ for every p, and consequently,

$$\det A = \det B + c \det C = \det B,$$

where it is observed that $\det C = 0$, since the first two columns of C are equal.

For the proof of (III), if a column of A is multiplied by c, then each A_p, and hence also $\det A$, is multiplied by c.

Additional properties of the determinant will now be established.

Proposition 2.2 $\det A^* = \det A$.

A proof will be indicated for 3×3 matrices. Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix},$$

then

$$A^*_{[2,3,1]} = a_{12}a_{23}a_{31} = a_{31}a_{12}a_{23} = A_{[3,1,2]}.$$

It may be noted that $[3, 1, 2]^{-1} = [2, 1, 3]$ and in general

$$A^*_p = a_{1p(1)}a_{2p(2)}a_{3p(3)} = a_{p^{-1}(1)1}a_{p^{-1}(2)2}a_{p^{-1}(3)3} = A_{p^{-1}}.$$

From $\operatorname{sgn} p = \operatorname{sgn} p^{-1}$ it may be seen that the defining sums for $\det A$ and $\det A^*$ are composed of the same terms.

Proposition 2.3 det $BA = (\det B)(\det A)$.

The proof will be made by stages. First we consider the case in which A is an elementary matrix. The characterization theorem implies that

$$\det I = 1; \quad \det E_{ij} = -1; \quad \det E_{i+(c)j} = 1; \quad \text{and} \quad \det E_{(c)i} = c.$$

From this, it easily follows that Proposition 2.3 is valid if A is an elementary matrix. For instance, if $A = E_{(c)i}$, then by the property that postmultiplication by $E_{(c)i}$ multiplies the ith column by c, we have the equation chain

$$\det (BE_{(c)i}) = c \det B = (\det B)c = (\det B)(\det E_{(c)i}).$$

The next step is to show that Proposition 2.3 holds when A is nonsingular. This may be done by using the property that A is a product of elementary matrices. For example, if $A = E_2E_1$, where E_1 and E_2 are elementary matrices, then

$$\begin{aligned} \det BA &= \det B(E_2E_1) = \det (BE_2)E_1 = \det (BE_2) \det E_1 \\ &= (\det B)(\det E_2)(\det E_1) = (\det B) \det (E_2E_1) \\ &= (\det B)(\det A). \end{aligned}$$

The idea may be extended by induction to the general case in which A is a product of n elementary matrices. We conclude the proof of Proposition 2.3 by showing that each side of the equality is 0 when A is singular. This is a consequence of the next result. It should be recalled that an $n \times n$ matrix A is nonsingular if and only if $\mathrm{r}(A) = n$ (Proposition 4.5 in Chapter VI).

Proposition 2.4 If A is an $n \times n$ matrix, then $\mathrm{r}(A) = n$ if and only if $\det A \neq 0$.

As for the proof, if $\mathrm{r}(A) = n$, then A is nonsingular and therefore a product of elementary matrices. We have already seen that det A is then the product of the determinants of these elementary matrices; hence, $\det A \neq 0$, since no elementary matrix has a zero determinant. Conversely, if $\mathrm{r}(A) < n$, then the columns of A are a dependent set, and some column vector is a linear combination of the others. A may then be converted by type-II column operations to a matrix B which has the zero vector as a column vector (see Proofs, exercise 1). Then $B_p = 0$ for each p and, hence, $\det A = \det B = 0$, which concludes the proof.

Questions

1. If p is the identity permutation, then A_p is ______.
   (a) 0,
   (b) det A,
   (c) the product of the diagonal entries of A,
   (d) the product of the entries of the first column of A.

2. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is lower triangular, then ________ $= 0$.
3. If $A = \langle \mathbf{u}, \mathbf{u} \rangle$ is a 2×2 matrix, then $\det A =$ ________.
4. If $\det A = 0$, then ________.
   (a) the rows of A are dependent,
   (b) two columns of A are equal,
   (c) some diagonal entry of A is 0.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. Given $\det\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \rangle = a$, find

   (a) $\det \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_3 \rangle$, (b) $\det \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1, \mathbf{u}_4 \rangle$,
   (c) $\det \langle 2\mathbf{u}_1, \mathbf{u}_2, 2\mathbf{u}_3, \mathbf{u}_4 - \mathbf{u}_1 \rangle$, (d) $\det \langle \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2, 3\mathbf{u}_3, \mathbf{u}_4 - 2\mathbf{u}_2 \rangle$.
2. Find $\det cA$ if A is a 3×3 matrix and $\det A = a$.

Proofs

1. Given $A = \langle \mathbf{u}_1, \mathbf{u}_2, c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \rangle$, prove that $\det A = 0$.
2. Prove that $\det A^*A = (\det A)^2$.
3. Prove that if A is nonsingular, then

   (a) $\det A^{-1} = \dfrac{1}{\det A}$, (b) $\det B^{-1}AB = \det A$.
4. Prove that if A is orthogonal, then $\det A = \pm 1$.

3. *Absolute Determinant*

The definition of determinant applies only to square matrices. In this section we extend this concept to other matrices. The extension is called the *absolute determinant*, and has applications in the study of areas and volumes related to the development of the integral on curves and surfaces. The two standard ways to define the absolute determinant of an $m \times n$ matrix A, where $n \leq m$, are presented below,

---

(I) *The absolute determinant is the square root of* det A^*A, *or*
(II) *it is the square root of the sum of the squares of the determinants of all $n \times n$ submatrices of A.*

Definition (I) is used in this section and leads to a simpler development of the properties of the absolute determinant. Definition (II) is useful for computations in dimensions less than or equal to 3.

If A is an $m \times n$ matrix, then A^*A is an $n \times n$ matrix and, hence, has a determinant. The following definition is therefore meaningful.

Definition of Absolute Determinant

The absolute determinant $|A|$ of an $m \times n$ matrix A, $n \leq m$, is given by the equation,

$$|A| = (\det A^*A)^{1/2}.$$

The absolute determinant formula is not applied to the case $m < n$, because it then always gives a value of zero. This follows from the inequalities

$$\mathrm{r}(A^*A) \leq \mathrm{r}(A^*) \leq m < n$$

and Proposition 2.4, which says that the determinant of an $n \times n$ matrix is zero if the rank is less than n.

Example 3.1 (a) If $A = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, then

$$A^*A = [a_{11} \quad a_{21}] \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = a_{11}{}^2 + a_{21}{}^2$$

and, hence, $|A| = (a_{11}{}^2 + a_{21}{}^2)^{1/2}$.

(b) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}$, then

$$A^*A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 7 & 25 \end{bmatrix},$$

and hence, $|A| = (\det A^*A)^{1/2} = \sqrt{150 - 49} = \sqrt{101}$.

Our first property shows that the absolute determinant is an extension of the absolute value of a determinant.

Proposition 3.1 If A is an $n \times n$ matrix, then $|A| = |\det A|$.

The proof follows from the equalities shown here.

$$\det A^*A = (\det A^*)(\det A) = (\det A)(\det A) = (\det A)^2,$$
$$|A| = (\det A^*A)^{1/2} = [(\det A)^2]^{1/2} = |\det A|.$$

We next extend the property $\det BA = (\det B)(\det A)$, whenever the matrix A in the extension is a square matrix.

Proposition 3.2 If A is an $n \times n$ matrix and B an $m \times n$ matrix, then $|BA| = |B|\,|A|$.

A proof comes from taking square roots of the equality chain

$$\begin{aligned} |BA|^2 &= \det (BA)^*(BA) = \det A^*(B^*BA) = (\det A^*)(\det B^*BA) \\ &= (\det A^*)(\det B^*B)(\det A) = (\det A)\,|B|^2(\det A) \\ &= |B|^2|A|^2. \end{aligned}$$

As a corollary of Proposition 3.2, we can deduce the effect on $|A|$ of elementary column operations.

Proposition 3.3

(a) The interchange of two columns of A does not change $|A|$.
(b) Adding a multiple of one column of A to another does not change $|A|$.
(c) Multiplication of a column of A by c causes $|A|$ to be multiplied by $|c|$.

The proof of (a) is

$$|AE_{ij}| = |A|\,|E_{ij}| = |A|\,|\det E_{ij}| = |A|.$$

We next extend the property of an $n \times n$ matrix A that $\det A \neq 0$ if and only if $\mathrm{r}(A) = n$.

Proposition 3.4 If A is an $m \times n$ matrix, then $|A| \neq 0$ if and only if $\mathrm{r}(A) = n$.

The proof uses Proposition 4.3 in Chapter VI, which says $\mathrm{r}(f^* \circ f) = \mathrm{r}(f)$. The corresponding result for matrices is $\mathrm{r}(A^*A) = \mathrm{r}(A)$, and an application of Proposition 2.4 implies that the following statements are equivalent.

$$(\det A^*A)^{1/2} \neq 0, \qquad \mathrm{r}(A^*A) = n, \qquad \mathrm{r}(A) = n.$$

One final result on the absolute determinant will now be developed. It is the most important for our purposes, since it lays the groundwork for the application of the absolute determinant to the study of length, area, and volume and their relation to the development of the integral in the calculus of several variables. This property has no analogue for ordinary determinants: it is concerned with the effect on the absolute determinant of a matrix produced by augmenting the matrix with an additional column vector. Its meaning will be enhanced by geometric discussions in the next section. The proof is found in the appendix.

Proposition 3.5 The augmenting of an $m \times n$ matrix A, $n < m$, by a column vector $\mathbf{u}$ causes $|A|$ to be multiplied by $|\mathbf{u}^\perp|$, where $\mathbf{u}^\perp$ is the component of $\mathbf{u}$ orthogonal to the span space of the column vectors of A.

Example 3.2 Let

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

be augmented by $\mathbf{u} = \langle 0, 3 \rangle$. The resulting matrix is

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

It may be verified that

$$\langle 0, 3 \rangle = \tfrac{6}{5}\langle 1, 2 \rangle + \tfrac{3}{5}\langle -2, 1 \rangle,$$

and therefore if $\mathbf{u}^\perp$ is the component of $\mathbf{u}$ orthogonal to Sp $\{\langle 1, 2 \rangle\}$, then

$$|\mathbf{u}^\perp| = |\tfrac{3}{5}\langle -2, 1 \rangle| = \frac{3}{\sqrt{5}}.$$

From

$$|A| = \sqrt{5}, \qquad |B| = 3, \qquad |\mathbf{u}^\perp| = \frac{3}{\sqrt{5}},$$

we may easily verify Proposition 3.5 for this case.

Questions

1. $|BA| = |B|\,|A|$ provided ______.
   (a) A is square,
   (b) B is square,
   (c) BA is square.
2. If A is an $m \times n$ matrix, then A^*A is ______.
   (a) $n \times m$,
   (b) $n \times n$,
   (c) $m \times m$.
3. If A is an $m \times n$ matrix and $|A| > 0$, then ______.
   (a) $m = n$,
   (b) the columns of A are independent,
   (c) $r(A^*) = m$.
4. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal and $A = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$, $B = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$, then
   (a) $|B| = |A|$,
   (b) $|B| = |A|\,|\mathbf{u}_3|$,
   (c) $|B| = 0$.

Problems

1. Do Problem Set B at the end of the chapter.

Exercises

1. Given $|\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle| = a$, find

   (a) $|\langle \mathbf{u}_2, \mathbf{u}_1, \mathbf{u}_3 \rangle|$, (b) $|\langle c\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle|$,
   (c) $|\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1 \rangle|$, (d) $|c\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle|$.
2. Verify Proposition 3.5 if

   (a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is augmented by $\langle 1, 1, 2 \rangle$,

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ is augmented by $\langle 1, 0, 0 \rangle$.

Proofs

1. Prove that if $A = [a_{ij}]$ is a 3×2 matrix, then $|A|$ is the square root of the sum of squares of determinants of all 2×2 submatrices of A.

2. Given $\mathbf{v} = \langle a, b\rangle \neq \langle 0, 0\rangle$ and $\mathbf{u} = \langle c, d\rangle$,
   (a) prove that the component of $\mathbf{u}$ orthogonal to $\mathrm{Sp}\{\mathbf{v}\}$ is

$$\mathbf{u}^{\perp} = \frac{ad - bc}{|\mathbf{v}|^2}\langle -b, a\rangle,$$

   and

   (b) prove that $|\langle \mathbf{v}, \mathbf{u}\rangle| = |\mathbf{v}|\,|\mathbf{u}^{\perp}|$.

4. *Absolute Determinant as a Volume and a Magnification*

We saw in Chapter V that the size of certain linear figures, such as line segments, parallelograms, and parallelepipeds, could be expressed using the dot and cross product operations. In this section, another method, in terms of the absolute determinant, will be obtained for computing these sizes. The absolute determinant unifies the various formulas for size into a single compact formulation in terms of matrix entries. Furthermore, it provides an extension of the size concept to nongeometric n–parallelotopes, $n > 3$. We begin with a useful formula for computing $|A|$ when m is less than or equal to 3.

Proposition 4.1 If A is an $m \times n$ matrix, $n \leq m$, then $|A|$ is the square root of the sum of the squares of all determinants of $n \times n$ submatrices of A.

We shall not consider the general proof, and our interest in Proposition 4.1 lies only with cases $m \leq 3$. These cases may be individually verified by straightforward methods.

Example 4.1 Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}$, the 2×2 submatrices are

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}; \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix},$$

with respective determinants -6, 1, and 8. Hence,

$$|A| = [(-6)^2 + 1^2 + 8^2]^{1/2} = \sqrt{101}$$

(see Example 3.1).

From Proposition 4.1, some relationships between the absolute determinant and the norm, dot product, and cross product can now be established.

Proposition 4.2 Let **u**, **v**, **w**, be 3-tuples:

(a) If $A = \langle \mathbf{u} \rangle$, then $|A| = |\mathbf{u}|$.
(b) If $A = \langle \mathbf{u}, \mathbf{v} \rangle$, then $|A| = |\mathbf{u} \times \mathbf{v}|$.
(c) If $A = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \rangle$ then $|A| = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$.

For the proof we use the notation

$$\mathbf{u} = \langle a_{11}, a_{21}, a_{31} \rangle, \quad \mathbf{v} = \langle a_{12}, a_{22}, a_{32} \rangle, \quad \text{and} \quad \mathbf{w} = \langle a_{13}, a_{23}, a_{33} \rangle.$$

Then (a) follows from the equality

$$|\mathbf{u}| = (a_{11}{}^2 + a_{21}{}^2 + a_{31}{}^2)^{1/2} = |A|.$$

In the case of (b),

$$|\mathbf{u} \times \mathbf{v}| = \det \begin{bmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \mathbf{k}$$

$$= \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{k}.$$

It may be seen by inspection that $|\mathbf{u} \times \mathbf{v}|$ is the square root of the sum of the squares of determinants of 2×2 submatrices of A, and thus it equals $|A|$. This proves (b). For the proof of (c) see Proofs, exercise 1.

We now recall from Chapter III that $|\mathbf{u}|$, $|\mathbf{u} \times \mathbf{v}|$, and $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$, respectively, represent the size of a line segment, parallelogram, and parallelepiped. Hence, by Proposition 4.2, each is given by the single concept of absolute determinant. The reason for this can be understood directly from Proposition 3.5. Consider in Cartesian space the line segment $\overline{OP}$. Its length is

$$|\overline{OP}| = |\langle \mathbf{OP} \rangle|.$$

If $\overline{OQ}$ is not collinear with $\overline{OP}$, then the parallelogram with sides $\overline{OP}$ and $\overline{OQ}$ has area given by the formula (see Figure 7.1)

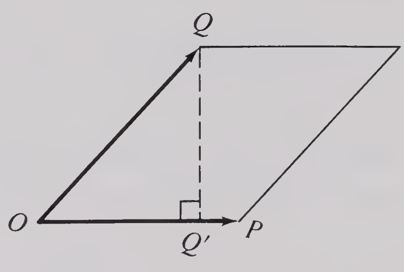

Figure 7.1

$$\text{Area} = \text{Base} \cdot \text{Altitude} = |\mathbf{OP}|\,|\mathbf{Q'Q}|,$$

where Q' is the perpendicular projection of Q onto the base $\overline{OP}$. Since $\mathbf{Q'Q}$ is the component of $\mathbf{OQ}$ perpendicular to the space spanned by $\mathbf{OP}$, it follows from Proposition 3.5 that

$$\text{Area} = |\mathbf{OP}|\,|\mathbf{Q'Q}| = |\langle \mathbf{OP}, \mathbf{OQ} \rangle|.$$

The process may be continued geometrically to one higher dimension. If $\overline{OR}$ is not coplanar with $\overline{OP}$ and $\overline{OQ}$, then the volume of the parallelepiped with sides $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ is (see Figure 7.2)

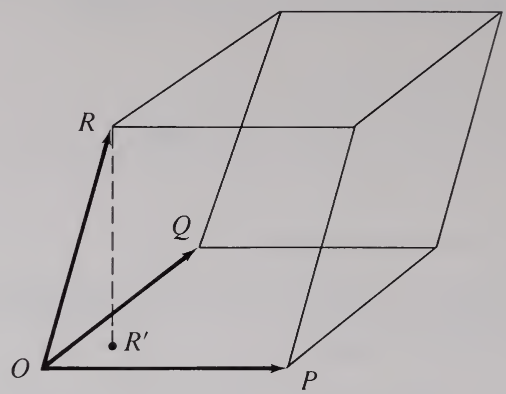

Figure 7.2

$$\begin{aligned}\text{Volume} &= \text{Base Area} \cdot \text{Altitude}\\ &= |\langle \mathbf{OP}, \mathbf{OQ} \rangle|\, |\mathbf{R'R}|,\end{aligned}$$

where R' is the perpendicular projection of R onto the plane containing $\overline{OP}$ and $\overline{OQ}$. Hence, $|\mathbf{R'R}|$ is the length of the component of $\mathbf{OR}$ perpendicular to the space spanned by $\mathbf{OP}$ and $\mathbf{OQ}$. By Proposition 3.5,

$$\text{Volume} = |\langle \mathbf{OP}, \mathbf{OQ} \rangle|\, |\mathbf{R'R}| = |\langle \mathbf{OP}, \mathbf{OQ}, \mathbf{OR} \rangle|.$$

Comparable arguments in the Euclidean plane would show the following facts (see Proofs, exercise 2):

(a) the length of $\overline{OP}$ is $|\langle \mathbf{OP} \rangle|$,
(b) the area of the parallelogram with sides $\overline{OP}$ and $\overline{OQ}$ is $|\langle \mathbf{OP}, \mathbf{OQ} \rangle|$.

Although our analysis cannot proceed to another dimension geometrically, there is no such algebraic limitation. Since translation has no effect on size, we have justified the following definition.

Definition of Volume of an n-Parallelotope

The volume *of the n-parallelotope*

$$\mathbf{T} = \mathbf{u}_o + [r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_n\mathbf{u}_n]$$

in $\mathbf{R}^m$, $n \leq m$, is

$$\text{Vol } \mathbf{T} = |\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \rangle|.$$

Example 4.2 The volume of the 2-parallelotope

$$\langle 1, 0, 3, 2\rangle + [r\langle 1, 1, 0, 0\rangle + s\langle 1, 0, 3, 5\rangle]$$

in $\mathbf{R}^4$ is

$$\left|\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \\ 0 & 5 \end{bmatrix}\right| = [(-1)^2 + 3^2 + 5^2 + 3^2 + 5^2 + 0^2]^{1/2} = \sqrt{69}.$$

In addition to describing size, the absolute determinant can also be used to give the magnification of size of an n–parallelotope in $\mathbf{R}^n$ by an injective affine function. By *magnification* we mean here the ratio of size of the f-image $f(\mathbf{T})$ to the size of $\mathbf{T}$, where $\mathbf{T}$ is an n–parallelotope in $\mathbf{R}^n$. Since translation does not change size, it is sufficient to consider the magnification of the size of

$$\mathbf{T} = [r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_n\mathbf{u}_n]$$

in $\mathbf{R}^n$ by an injective linear function f. From Chapter IV the f-image of $\mathbf{T}$ is the n–parallelotope in $\mathbf{R}^m$

$$f(\mathbf{T}) = [r_1 f(\mathbf{u}_1) + r_2 f(\mathbf{u}_2) + \ldots + r_n f(\mathbf{u}_n)].$$

Application of $|BA| = |B|\,|A|$ gives

$$\begin{aligned} |\mathrm{Mat}\, f|\ \mathrm{Vol}\,\mathbf{T} &= |\mathrm{Mat}\, f|\,|\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\rangle| \\ &= |(\mathrm{Mat}\, f)\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\rangle| \\ &= |\langle f(\mathbf{u}_1), f(\mathbf{u}_2), \ldots, f(\mathbf{u}_n)\rangle| \\ &= \mathrm{Vol}\, f(\mathbf{T}). \end{aligned}$$

We may therefore conclude the following result.

Magnification Theorem

If $\mathbf{u}_o + f$ is an injective, affine function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and $\mathbf{T}$ is an n–parallelotope in $\mathbf{R}^n$, then

$$|\mathrm{Mat}\, f| = \frac{\mathrm{Vol}\, f(\mathbf{T})}{\mathrm{Vol}\,\mathbf{T}}.$$

Example 4.3 Let $\mathbf{T}$ be the parallelogram $\langle 1, 0\rangle + [r\langle 1, 2\rangle + s\langle 0, 3\rangle]$ and $\mathbf{u}_o{}' + f$ the affine function with

$$\mathbf{u}_o{}' = \langle 1, 0, 3\rangle, \qquad \mathrm{Mat}\, f = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Then $f(\mathbf{T})$ is described for $0 \leq r, s \leq 1$ by

$$f(\mathbf{T}) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$+ s \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \langle 2, 0, 5 \rangle + [r\langle 7, 2, 2 \rangle + s\langle 9, 3, 0 \rangle].$$

From $|\text{Mat } f| = \left| \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \right| = \sqrt{41}$, $\text{Vol } \mathbf{T} = \left| \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \right| = 3$,

$$\text{Vol } f(\mathbf{T}) = \left| \begin{bmatrix} 7 & 9 \\ 2 & 3 \\ 2 & 0 \end{bmatrix} \right| = \sqrt{369},$$

the equality $\sqrt{41} = \sqrt{369}/3$ verifies the magnification theorem.

Questions

1. If $\mathbf{T} = \mathbf{u}_o + [r\mathbf{u}_1 + s\mathbf{u}_2]$, then Vol $\mathbf{T}$ is ______.
   (a) $|\langle \mathbf{u}_o, \mathbf{u}_1, \mathbf{u}_2 \rangle|$,
   (b) $|\langle \mathbf{u}_1, \mathbf{u}_2 \rangle|$,
   (c) $|\langle \mathbf{u}_1 - \mathbf{u}_o, \mathbf{u}_2 - \mathbf{u}_o \rangle|$.

2. If f is linear and $\mathbf{T}$ a parallelotope in the domain of f, then f magnifies the volume of $\mathbf{T}$ by ______.
   (a) $|\text{Mat } f|$,
   (b) Vol $f(\mathbf{T})$,
   (c) Vol $\mathbf{T}$.

Problems

1. Do Problem Set C at the end of the Chapter.

Exercises

1. Given Vol $(\mathbf{u}_o + [r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + r_3\mathbf{u}_3]) = a$, find
   (a) $\text{Vol}(\mathbf{u}_1 + [r_1\mathbf{u}_2 + r_2\mathbf{u}_3 + r_3\mathbf{u}_1])$,
   (b) $\text{Vol}(\mathbf{u}_o + [r_1(3\mathbf{u}_1) + r_2(\mathbf{u}_2 - \mathbf{u}_1) + r_3\mathbf{u}_3])$.

2. Verify the equation $|\mathrm{Mat}\, f| = \dfrac{\mathrm{Vol}\, f(\mathbf{T})}{\mathrm{Vol}\, \mathbf{T}}$ if

   (a) $\mathbf{T} = \langle 1, 2\rangle + [r\langle 3, -1\rangle + s\langle 2, 5\rangle], f(\mathbf{i}) = \langle 3, 0, 2\rangle, f(\mathbf{j}) = \langle 1, 1, 0\rangle$,

   (b) $\mathbf{T} = \langle 1, 3, 0\rangle + [r\langle 3, -1, 1\rangle + s\langle 2, 5, 0\rangle + t\langle 1, 0, 1\rangle], f(\mathbf{i}) = \langle 3, 0, 2\rangle, f(\mathbf{j}) = \langle 2, 1, 0\rangle, f(\mathbf{k}) = 1, 0, 0\rangle$.

Proofs

1. Prove that if $A = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle$ is a 3×3 matrix, then $|A| = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$. (*Hint*: Expand $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ and use $|A| = |\det A|$.)

2. Prove that the area of the parallelogram with sides $\overline{OP}$ and $\overline{OQ}$ is $|\langle \mathbf{OP}, \mathbf{OQ}\rangle|$.

Problems

A. *Determinants*

There is associated with each square matrix a number called its *determinant*. The determinant of a 2×2 matrix may be computed by the formula shown in A.1.

A.1 $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc.$

1. Find the determinant of each matrix.

   (a) $\begin{bmatrix} 2 & 6 \\ 3 & -4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 7 & -3 \end{bmatrix}$.

The determinants of larger size matrices are computed more easily by certain procedures other than formulas. A square matrix is *triangular* if either all entries below the diagonal are zero or all entries above the diagonal are zero. A triangular matrix has the following property.

A.2 The determinant of a triangular matrix is the product of its diagonal elements.

2. Find the determinant of each matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 & 0 \\ 5 & -3 & 0 \\ 1 & 7 & 4 \end{bmatrix}$.

If an elementary row or column operation is applied to a matrix, its determinant is affected as set forth in A.3.

A.3 (a) The interchange of two rows or columns of A causes det A to be multiplied by (-1).
(b) Adding a multiple of one row or column of A to another does not change det A.
(c) Multiplication of a row or column of A by c causes det A to be multiplied by c.

Using A.2 and A.3, the determinant of any square matrix can be found by the following process.

A.4 (a) Convert to triangular form by elementary operations, recording the determinant change at each step,
(b) evaluate the determinant of the triangular form, and
(c) adjust the value in (b) by the various changes in (a).

3. Find

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 2 & 2 & 6 \end{bmatrix}$$

using the conversion sequence

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

4. Find the determinant for each of the following matrices,

(a) $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 4 & 6 \\ 3 & 7 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 1 & 0 & 2 \end{bmatrix}$.

An alternative or supplement to the method of A.4 uses the following property.

A.5 If the jth column of a square matrix A has all entries 0 except the entry c in the ith row, then

$$\det A = (-1)^{i+j} c \det A_{ij},$$

where A_{ij} is the submatrix obtained from A by deleting the ith row and jth column.

The statement of A.5 is again true if "row" is replaced by "column" and "column" by "row."

5. Find the determinant in each of the given cases.

$$\text{(a)}\ \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ -3 & 4 & 6 \end{bmatrix}, \qquad \text{(b)}\ \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 5 & 4 \end{bmatrix}, \qquad \text{(c)}\ \begin{bmatrix} 2 & 1 & 3 & 6 \\ 4 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 \\ 8 & 0 & 0 & 0 \end{bmatrix}.$$

6. Find the determinant of each matrix.

$$\text{(a)}\ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 5 \\ 1 & 7 & 6 \end{bmatrix}, \qquad \text{(b)}\ \begin{bmatrix} 1 & 3 & 2 & 7 \\ 2 & 1 & 6 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & 5 & 1 & 5 \end{bmatrix},$$

$$\text{(c)}\ \begin{bmatrix} 4 & 5 & 0 & 4 \\ 2 & -1 & 0 & 3 \\ 3 & 2 & 4 & 2 \\ 6 & 7 & 0 & 11 \end{bmatrix}.$$

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7. Find the determinant for each of the following matrices.

$$\text{(a)}\ \begin{bmatrix} 1 & 7 \\ -2 & -3 \end{bmatrix}, \qquad \text{(b)}\ \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 0 & 6 & 3 \end{bmatrix},$$

$$\text{(c)}\ \begin{bmatrix} 6 & 2 & 4 \\ 3 & 2 & -1 \\ 4 & 0 & 5 \end{bmatrix}, \qquad \text{(d)}\ \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 0 & 3 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \end{bmatrix}.$$

B. *Absolute Determinants*

There is associated with each $m \times n$ matrix A, $m \geq n$, a nonnegative number $|A|$ called its *absolute determinant*. It may be computed by either of two methods.

B.1 $|A| = \sqrt{\det A^*A}$.

B.2 $|A|$ is the square root of the sum of the squares of the determinants of all $n \times n$ submatrices of A.

For $m \leq 3$, the method in B.2 is more suitable for computation. As an easy corollary of B.2, we note the following fact.

B.3 If A is a square matrix, then $|A| = |\det A|$.

1. Using each of B.1 and B.2, compute the absolute determinant for each matrix given.

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 2 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 4 & 5 \\ 6 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

2. Using B.1, compute the absolute determinant in each case below.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$.

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3. Find the absolute determinant for the given matrices.

(a) $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 6 \\ 3 & -5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \\ 2 & 4 \end{bmatrix}$.

C. Parallelotope and Simplex Size

The absolute determinant may be used to compute the size of various types of sets encountered in Chapter IV. The general procedure for finding the size of a parallelotope is shown in the following steps.

C.1 (a) Form the matrix whose column vectors correspond to the sides of the parallelotope.
(b) The size is the absolute determinant of the matrix in (a).

1. Find the size for each figure.
   (a) $(3\mathbf{i} - \mathbf{j}) + [r(2\mathbf{i} + \mathbf{j})]$,
   (b) $(\mathbf{i} - 3\mathbf{j}) + [r(\mathbf{i} - \mathbf{j}) + s(\mathbf{i} + 2\mathbf{j})]$,
   (c) $(\mathbf{i} - \mathbf{j} + \mathbf{k}) + [r(\mathbf{i} + \mathbf{k}) + s(2\mathbf{i} - 3\mathbf{j})]$,
   (d) $(\mathbf{i} + \mathbf{k}) + [r(3\mathbf{i} - \mathbf{j}) + s(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + t(3\mathbf{j} + \mathbf{k})]$.
2. Given that $P = (1, 2)$, $Q = (3, -1)$, and $R = (4, -3)$, find the size of
   (a) the line segment with end points P and Q,
   (b) the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$.
3. Given $P = (1, 0, 4)$, $Q = (2, 1, -5)$, $R = (3, 1, 3)$, and $S = (1, 0, 1)$, find the size of
   (a) the line segment with end points P and Q,
   (b) the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$,
   (c) the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$.

The computation of areas of triangles and volumes of tetrahedrons is described next.

C.2 (a) The area of the triangle with sides $\overline{PQ}$ and $\overline{QR}$ is 1/2 the area of the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$.
(b) The volume of the tetrahedron with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$ is 1/6 the volume of the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$.

4. Find the area of the triangle with vertices as given below.
   (a) $(1, 2), (3, 0), (6, -5)$; (b) $(2, -4), (3, 5), (1,7)$;
   (c) $(1, 0, 3), (6, 1, 2), (7, 0, 4)$; (d) $(1, 1, 5), (2, -1, 3), (1, 7, 0)$.
5. Find the volume of the tetrahedron with vertices given.
   (a) $(1, 0, 0), (1, 1, 0), (2, 3, 5), (6, 0, 4)$;
   (b) $(-1, 3, 2), (1, 6, 0), (5, 0, 4), (3, 8, 0)$.

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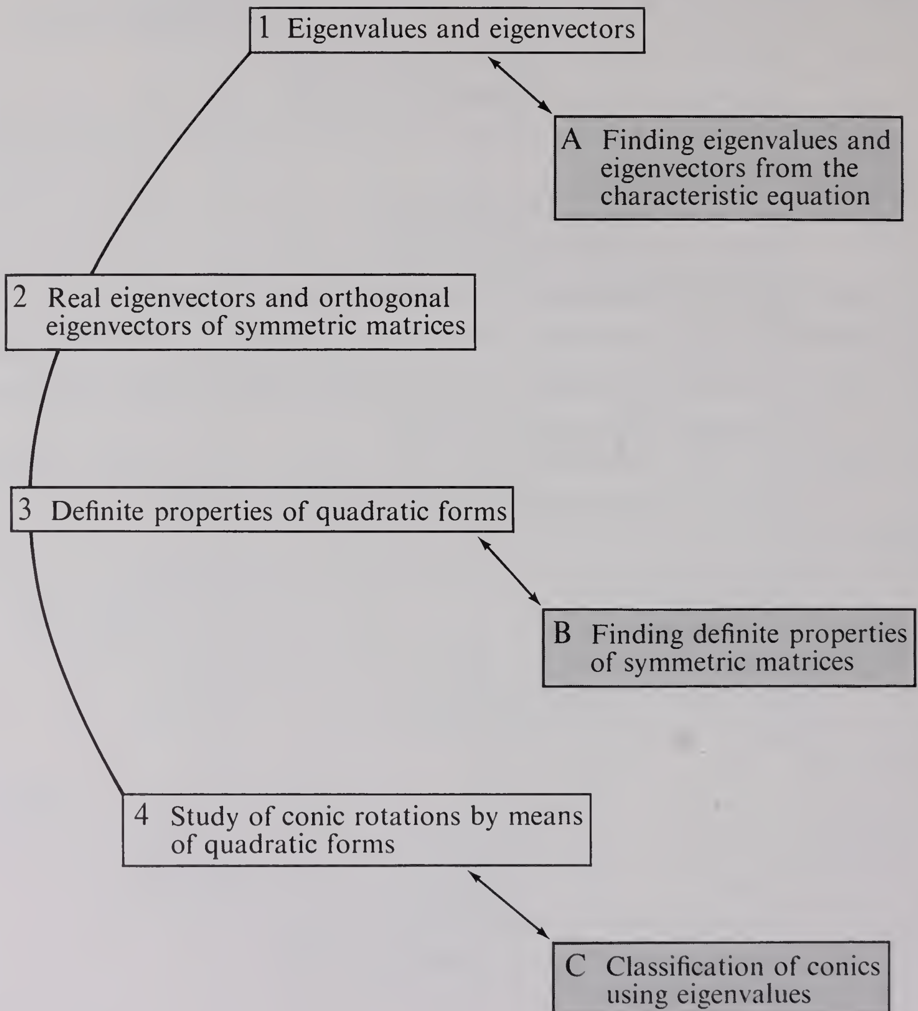
6. Given $P = (1, 0)$, $Q = (3, 2)$, and $R = (4, -7)$, find the size of
   (a) the line segment with end points P and Q,
   (b) the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$,
   (c) the triangle with vertices P, Q, R.
7. Given $P = (1, 3, 0)$, $Q = (2, 1, 5)$, $R = (6, 0, 2)$, and $S = (-1, 2, 4)$, find the size of
   (a) the line segment with end points P and Q,
   (b) the parallelogram with sides $\overline{PQ}$ and $\overline{QR}$,
   (c) the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$,
   (d) the tetrahedron with vertices P, Q, R, and S.


1 Eigenvalues and eigenvectors
A Finding eigenvalues and eigenvectors from the characteristic equation
2 Real eigenvectors and orthogonal eigenvectors of symmetric matrices
3 Definite properties of quadratic forms
B Finding definite properties of symmetric matrices
4 Study of conic rotations by means of quadratic forms
C Classification of conics using eigenvalues

VIII

Eigenvalues and Quadratic Forms

We have already seen that the association of one or more numbers to a mathematical object may produce helpful information about the object. Instances of important numbers are the dimension of a vector space, the rank of a linear function or matrix, the entries in the matrix of a linear function, and the determinant of a matrix. In this chapter other significant numbers, called *eigenvalues*, will be associated with linear functions and matrices. The prefix "eigen" is a German word meaning "self" or "own." Common synonyms for eigenvalues are *proper values* and *characteristic values*. Eigenvalues will be used by us in the study of conics; they also have many other applications to mathematics, physics, and engineering.

1. Eigenvalues and Eigenvectors

In this section f will denote a linear function from $\mathbf{R}^n$ to $\mathbf{R}^n$. Such linear functions, which have the same domain and range, are often called

operators. We now associate with f certain numbers called *eigenvalues,* which satisfy a prescribed condition described in the following definition.

Definition of Eigenvalue

A real number, λ, is an eigenvalue of f if and only if there exists a nonzero vector $\mathbf{u}$, *in* $\mathbf{V}$, *such that* $f(\mathbf{u}) = \lambda\mathbf{u}$.

Thus, an eigenvalue is a number which acts as a scalar multiple of some nonzero vector to give its f-image. If λ is an eigenvalue and $f(\mathbf{u}) = \lambda\mathbf{u}$, then $\mathbf{u}$ is called an *eigenvector* of f belonging to λ. We shall now see that the set of eigenvectors of f which belong to a given eigenvalue λ constitutes a subspace of $\mathbf{R}^n$, called the *eigenspace* of λ.

Proposition 1.1 If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of f which belong to λ and a is a real number, then (a) $\mathbf{u} + \mathbf{v}$ and (b) $a\mathbf{u}$ are also eigenvectors of f which belong to λ.

The proof of (a) follows from the equality chain

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v}).$$

For (b) see Proofs, exercise 1.

Example 1.1 Let P be the graph point of an eigenvector $\mathbf{u}$ which belongs to an eigenvalue λ of an operator f on $\mathbf{R}^2$. If Q is the graph point of $\lambda\mathbf{u}$, then $\overline{OP}$ and $\overline{OQ}$ are collinear. The eigenvalue, λ, signifies an extension, contraction, or reversal of direction of its eigenspace according to whether its value is greater than 1, between 0 and 1, or less than 0 (see Figure 8.1).

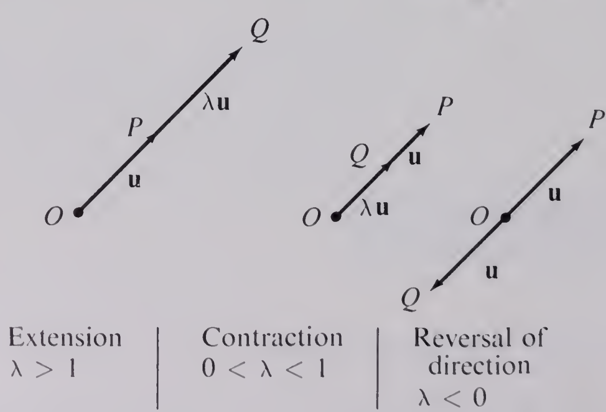


Figure 8.1

Results concerning the existence of eigenvalues can be achieved with the help of matrices. If $A = \text{Mat } f$, then we define λ and $\mathbf{u}$, respectively, to be an eigenvalue and an eigenvector of A belonging to λ according to whether or not it satisfies the same property for f. A prescription for finding eigenvalues is given by the next result.

Proposition 1.2 The real number λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Here I denotes the identity matrix having the same size as A. For the proof let id denote the identity operator on $\mathbf{R}^n$. If λ is an eigenvalue of A and $\text{Mat} f = A$, then λ is also an eigenvalue of f and $f(\mathbf{u}) = \lambda\mathbf{u}$ for some nonzero $\mathbf{u}$. Hence,

$$(f - \lambda \text{ id})(\mathbf{u}) = f(\mathbf{u}) - \lambda\mathbf{u} = \mathbf{0}$$

and $f - \lambda$ id is not injective. Therefore, $\text{Mat}(f - \lambda \text{ id}) = A - \lambda I$ is singular, and $\det(A - \lambda I) = 0$ by Proposition 2.4 in Chapter VII. The converse is obtained by reversing these steps (see Proofs, exercise 2).

It can be seen from the defining sum for a determinant that $\det(A - \lambda I)$ is a polynomial of degree n in λ, called the *characteristic polynomial* of A. This polynomial has at most n roots by the fundamental theorem of algebra, and, therefore, f has at most n eigenvalues. The polynomial may have no real roots, in which case f has no (real) eigenvalues.

Example 1.2 If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\det(A - \lambda I) = \det\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}).$$

Setting $\det(A - \lambda I) = 0$ gives, by the quadratic formula,

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2}$$

as the two eigenvalue candidates. Then A has 2, 1, or 0 eigenvalues according to whether $(a_{11} - a_{22})^2 + 4a_{12}a_{21}$ is positive, zero, or negative.

The eigenvectors belonging to an eigenvalue, λ, may be determined by solving systems of equations. If $\text{Mat} f = A$, then $f(\mathbf{u}) = \lambda\,\mathbf{u}$ is equivalent to $A\mathbf{u} = \lambda\mathbf{u}$. Thus if $A = [a_{ij}]$ and $\mathbf{u} = \langle x_1, x_2, \ldots, x_n \rangle$, then $\mathbf{u}$ is an eigenvalue of A belonging to λ provided

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

Example 1.3 The eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ are found from

$$\det\begin{bmatrix} 2-\lambda & 2 \\ -1 & 5-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12 = 0.$$

The roots are $\lambda_1 = 4$ and $\lambda_2 = 3$, and the eigenspaces belonging to 4 and 3 are, respectively, the solutions sets of

$$\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix}; \qquad \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$

It may be verified these are $\mathrm{Sp}\{\langle 1, 1\rangle\}$ and $\mathrm{Sp}\{\langle 2, 1\rangle\}$.

Questions

1. If $f(\lambda) = \lambda\mathbf{u}$, $\mathbf{u} \neq \mathbf{0}$, then λ is a(n) ______ of f.
2. If A is an $n \times n$ matrix, then the number of eigenvalues of A is ______.
   (a) at least 1,
   (b) at most n,
   (c) n^2.
3. If λ is an eigenvector of f, then the zero vector ______ be an eigenvector belonging to λ.
   (a) must,
   (b) cannot,
   (c) might or might not.
4. Eigenvalues of A are roots of the ______ of A.
5. If λ is an eigenvalue of A, then $A - \lambda I$ is ______.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. Find the (a) eigenvalues of A, (b) eigenspace belonging to each eigenvalue of A if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

Proofs

1. Prove that if $\mathbf{u}$ is an eigenvector of f which belongs to λ, then so also is $a\mathbf{u}$.
2. Prove that if $\det(A - \lambda I) = 0$, then λ is an eigenvalue of A.
3. Prove that A and $B^{-1}AB$ have the same eigenvalues. (*Hint*: $B^{-1}AB - \lambda I = B^{-1}AB - B^{-1}(\lambda I)B$.)
4. Prove that A and A^* have the same eigenvalues. (*Hint*: $(A - \lambda I)^* = A^* - \lambda I$.)

2. Eigentheory of Symmetric Matrices

In the previous section we saw that a square matrix A may have no (real) eigenvalues. Since our eigentheory has value only for matrices which have real eigenvalues, it is natural to ask if any category of matrices necessarily produces real eigenvalues. The answer is affirmative.

Proposition 2.1 If A is a symmetric $n \times n$ matrix, then all roots of the characteristic polynomial of A are real.

A proof of this result is found in the appendix. It will be verified here for the case $n = 2$. If we let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{21} = a_{12}$, then in Example 1.2 the radicand is $(a_{11} - a_{22})^2 + 4a_{12}{}^2$, and this is necessarily non-negative. Thus the roots are real.

We next show that eigenvectors belonging to distinct eigenvalues of a symmetric matrix A are orthogonal.

Proposition 2.2 If λ_1 and λ_2 are distinct eigenvalues of a symmetric matrix A and $\mathbf{u}_1$, $\mathbf{u}_2$ are eigenvectors of A belonging to λ_1, λ_2, respectively, then $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$.

For the proof, let $\operatorname{Mat} f = A$. Then f is self-adjoint and, hence,

$$\begin{aligned} f(\mathbf{u}_1) \cdot \mathbf{u}_2 &= \mathbf{u}_1 \cdot f(\mathbf{u}_2), \\ (\lambda_1 \mathbf{u}_1) \cdot \mathbf{u}_2 &= \mathbf{u}_1 \cdot (\lambda_2 \mathbf{u}_2), \\ (\lambda_1 - \lambda_2)(\mathbf{u}_1 \cdot \mathbf{u}_2) &= 0. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ by hypothesis, it must be concluded that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$.

For the following discussion we shall choose $n = 4$ to simplify notation and ideas. If A is a 4×4 symmetric matrix, then the number of real eigenvalues of A is at least one and at most four. Let us consider the case in which A has three eigenvalues λ_1, λ_2, and λ_3, with respective eigenspaces $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ in $\mathbf{R}^4$. We may then choose for each subspace, $\mathbf{S}_i$, an orthonormal basis, $\mathbf{B}_i$. The union $\mathbf{B}$ of all elements in $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ is an orthonormal set by Proposition 2.2. Hence, $\mathbf{B}$ has at least three and at most four elements, since it is independent. Our next result implies that $\mathbf{B}$ must have four elements, and thus is an orthonormal basis of $\mathbf{R}^4$. A proof is in the appendix.

Proposition 2.3 If A is a symmetric $n \times n$ matrix, then the eigenvectors of A contain an orthonormal basis of $\mathbf{R}^n$.

Example 2.1 Given $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = -\lambda^3 + 5\lambda^2 - 3\lambda - 9$$

has roots $\lambda_1 = -1$ and $\lambda_2 = 3$. The following facts may be verified.

(a) The λ_1 eigenspace of A has for a basis $\left\{\left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right\rangle\right\}$;

(b) the λ_2 eigenspace of A has for a basis $\left\{\langle 0, 0, 1\rangle, \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle\right\}$,

and together these provide an orthonormal basis of $\mathbf{R}^3$.

It is implied by Proposition 2.3 that whenever the roots of the characteristic polynomial of a symmetric matrix are not distinct, then some eigenspace has dimension greater than 1. We now study some of the consequences of Proposition 2.3. If f is a self-adjoint operator on $\mathbf{R}^n$, then the eigenvectors of f contain an orthonormal basis $\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of $\mathbf{R}^n$ such that

$$f(\mathbf{u}_1) = \lambda_1 \mathbf{u}_1, f(\mathbf{u}_2) = \lambda_2 \mathbf{u}_2, \ldots, f(\mathbf{u}_n) = \lambda_n \mathbf{u}_n,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are (not necessarily distinct) eigenvalues of f. Hence,

$$\mathrm{Mat}_{\mathbf{B}} f = \langle \lambda_1 \mathbf{e}_1, \lambda_2 \mathbf{e}_2, \ldots, \lambda_n \mathbf{e}_n \rangle,$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are standard basis vectors of $\mathbf{R}^n$. This proves the following result.

Proposition 2.4 If f is a self-adjoint operator on $\mathbf{R}^n$ and $\mathbf{B}$ is an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of f, then $\mathrm{Mat}_{\mathbf{B}} f$ is a diagonal matrix whose diagonal entries are eigenvalues of f.

Example 2.2 In Example 2.1 we set

$$\mathbf{B} = \left\{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle, \langle 0, 0, 1\rangle, \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle\right\}.$$

Then $\mathbf{B}$ is an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors of A, and

$$\mathrm{Mat}_{\mathbf{B}} f = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Our final result in this section has later application in the study of conic rotations.

Diagonalization Theorem for Symmetric Matrices

Let A be a symmetric matrix and C an orthogonal matrix whose column vectors are an orthonormal basis of eigenvectors of A. Then $C^{-1}AC = C^*AC$ is a diagonal matrix whose diagonal entries are eigenvalues of A.

For the proof we use the equality of Proposition 8.3 in Chapter V with $\mathbf{B}' = \mathbf{B}$.

$$\mathrm{Mat}_{\mathbf{B}} f = C_{\mathbf{B}}{}^{-1}(\mathrm{Mat}\, f)C_{\mathbf{B}},$$

where $\mathrm{Mat}\, f = A$ and $\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of eigenvectors of A. Thus, $C = C_{\mathbf{B}} = \langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\rangle$ is an orthogonal matrix and, hence, $C^{-1} = C^*$. The remainder of the conclusion follows from Proposition 2.4.

Example 2.3 Referring to Examples 2.1 and 2.2, if

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix},$$

then

$$C^{-1}AC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Questions

1. A ________ matrix has real eigenvalues.
2. Eigenvectors belonging to different eigenvalues of a symmetric matrix are ________.
3. If a 3×3 symmetric matrix has two real eigenvalues then some eigenspace ________.
   (a) is all of $\mathbf{R}^3$,
   (b) has dimension >1,
   (c) contains only the zero vector.
4. If $\mathbf{B}$ is a basis of eigenvectors of $A = \operatorname{Mat} f$, then $\operatorname{Mat}_{\mathbf{B}} f$ is ________.
   (a) the identity matrix,
   (b) an orthogonal matrix,
   (c) a diagonal matrix.

Exercises

1. Find an orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix if $A =$

   (a) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

2. Given that $f(\mathbf{i}) = \langle 1, 2\rangle, f(\mathbf{j}) = \langle 2, 1\rangle$, find a basis $\mathbf{B}$ such that $\operatorname{Mat}_{\mathbf{B}} f$ is a diagonal matrix.

3. *Quadratic Forms*

A *quadratic form* is essentially an algebraic expression which is the sum of terms each having degree two. Quadratic forms are nonlinear in character, although, as we shall see, they relate to certain linear algebraic processes. Also, in applications to calculus of several variables, quadratic forms sometimes perform an intermediary role between linear and nonlinear functions.

Quadratic forms of 2 and 3 variables have the respective forms

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$
$$a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2.$$

The reason for the presence of the 2's in the various coefficients will soon be

apparent. These quadratic forms may be written, using matrices, as they are shown below.

$$[x \quad y]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \qquad [x \quad y \quad z]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

as may be readily verified by matrix multiplication. In each case the matrix product gives a 1×1 entry, which is identified with its real-number entry. The general concept of quadratic form will now be formalized.

Definition of Quadratic Form

The quadratic form *of a symmetric* $n \times n$ *matrix* A *is the real-valued function* Q_A *having domain* $\mathbf{R}^n$ *and the rule, for each* $\mathbf{u}$ *in* $\mathbf{R}^n$,

$$Q_A(\mathbf{u}) = \mathbf{u} \cdot A\mathbf{u}.$$

The identification of a column $n \times 1$ matrix and its column vector is used in this definition.

Example 3.1 The quadratic forms of A and B, for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 4 \end{bmatrix},$$

are respectively given by

$$Q_A(\langle x, y\rangle) = [x \quad y]\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4xy + 3y^2$$

and

$$Q_B(\langle x, y, z\rangle) = [x \quad y \quad z]\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= x^2 + 6xy + 10xz + 2y^2 - 2yz + 4z^2.$$

We now investigate certain properties of quadratic forms. By inspection the following properties may be noted.

(a) $x^2 + 0xy + y^2$ *is positive for all* $\langle x, y\rangle \neq \langle 0, 0\rangle$,
(b) $(-1)x^2 + 0xy + (-1)y^2$ *is negative for all* $\langle x, y\rangle \neq \langle 0, 0\rangle$, *and*
(c) $x^2 + 0xy + (-1)y^2$ *is positive if* $\langle x, y\rangle = \langle 1, 0\rangle$ *and negative if* $\langle x, y\rangle = \langle 0, 1\rangle$.

It is not immediately obvious, however, if the forms x^2+xy+y^2 and $x^2+3xy+y^2$ are ever negative. By completing the square we can obtain the equalities shown below.

$$\begin{aligned} x^2+xy+y^2 &= (x^2+xy+1/4\,y^2)+y^2-1/4\,y^2 \\ &= (x+1/2\,y)^2+3/4\,y^2, \\ x^2+3xy+y^2 &= (x^2+3xy+9/4\,y^2)+y^2-9/4\,y^2 \\ &= (x+3/2\,y)^2-5/4\,y^2. \end{aligned}$$

Thus we can see that x^2+xy+y^2 is never negative, whereas $x^2+3xy+y^2$ is negative if $\langle x, y\rangle = \langle 3, -2\rangle$. We formalize this line of investigation by introducing the following:

Definition of Definite Properties

The symmetric matrix A is
(a) *positive-definite if $Q_A(\mathbf{u})>0$ for all $\mathbf{u}\neq\mathbf{0}$,*
(b) *positive-semidefinite if $Q_A(\mathbf{u})\geq 0$ for all $\mathbf{u}$,*
(c) *negative-definite if $Q_A(\mathbf{u})<0$ for all $\mathbf{u}\neq\mathbf{0}$,*
(d) *negative-semidefinite if $Q_A(\mathbf{u})\leq 0$ for all $\mathbf{u}$,*
(e) *indefinite if $Q_A(\mathbf{u})>0$ for some $\mathbf{u}$'s and $Q_A(\mathbf{u})<0$ for some $\mathbf{u}$'s.*

The quadratic form Q_A is said to have a certain definite property according to whether or not A has that property. Since $Q_A(\mathbf{0})=0$ for any A, it is evident that a positive-definite matrix is also positive-semidefinite, and a negative-definite matrix is negative-semidefinite. The definite property of a diagonal matrix can be seen by inspection, as explained by the next result.

Proposition 3.1 If A is a diagonal matrix, then A is
(a) positive-definite if and only if each diagonal entry is positive,
(b) positive-semidefinite if and only if each diagonal entry is nonnegative,
(c) negative-definite if and only if each diagonal entry is negative,
(d) negative-semidefinite if and only if each diagonal entry is nonpositive,
(e) indefinite if there is at least one positive and one negative diagonal entry.

For a proof of (e), if the ith diagonal entry, a_{ii}, is positive, and the jth diagonal entry, a_{jj}, is negative, then

$$Q_A(\mathbf{e}_i)=a_{ii}>0;\ Q_A(\mathbf{e}_j)=a_{jj}<0,$$

where $\mathbf{e}_i$, $\mathbf{e}_j$ are standard basis vectors.

Example 3.2 The following are respectively positive-definite, negative-semidefinite, and indefinite.

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}.$$

The definite property of other symmetric matrices cannot generally be determined by inspection, unless two diagonal elements have different signs, in which case the matrix is indefinite.

Proposition 3.2 If a symmetric matrix A has a positive and a negative diagonal entry, then A is indefinite.

The proof is the same as for Proposition 3.1(e).

Example 3.3 It will be verified from the definition that

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & -7 \end{bmatrix}$$

is indefinite. Letting $\mathbf{u}_1 = \langle 1, 0, 0 \rangle$, $\mathbf{u}_2 = \langle 0, 0, 1 \rangle$, then a simple computation gives $Q_A(\mathbf{u}_1) = 1 > 0$ and $Q_A(\mathbf{u}_2) = -7 < 0$. Therefore, A is indefinite.

It has already been noted that the definite property of certain quadratic forms may be found by completing the square. For the two-variable case, this procedure is seen to be completely general by the equation introduced below.

$$ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{(ac - b^2)}{a}y^2,$$

which is valid when $a \neq 0$. The case $a = 0$ can be handled by a similar equation. Although this process can be extended to the case of more variables, it becomes exceedingly cumbersome, and we shall consider an alternative method. The new approach rests on the fact that the definite property is an invariant with respect to the congruence relation.

Proposition 3.3 If A is symmetric and C is nonsingular, then A and C^*AC have the same definite property.

We consider the proof for A an $n \times n$ matrix. The function property $\mathbf{u} \cdot f(\mathbf{v}) = f^*(\mathbf{u}) \cdot \mathbf{v}$ corresponds to the matrix property $\mathbf{u} \cdot A\mathbf{v} = A^*\mathbf{u} \cdot \mathbf{v}$. Therefore,

$$\begin{aligned} Q_{C^*AC}(\mathbf{u}) &= \mathbf{u} \cdot C^*AC\mathbf{u} \\ &= C\mathbf{u} \cdot AC\mathbf{u} \\ &= Q_A(C\mathbf{u}). \end{aligned}$$

The desired conclusion now comes from the nonsingularity of C, which implies that as $\mathbf{u}$ varies through all of $\mathbf{R}^n$ so does $C\mathbf{u}$, since C corresponds to an injective operator on $\mathbf{R}^n$. Thus, for instance, if C^*AC is positive-definite and $\mathbf{v}$ is a nonzero vector in $\mathbf{R}^n$, then $\mathbf{v} = C\mathbf{u}$ for some nonzero vector $\mathbf{u}$ in $\mathbf{R}^n$ and, hence,

$$Q_A(\mathbf{v}) = Q_A(C\mathbf{u}) = Q_{C^*AC}(\mathbf{u}) > 0$$

shows that Q_A is also positive definite.

The definite property of a matrix can now be stated in terms of its eigenvalues. By the diagonalization theorem for symmetric matrices, every matrix is congruent to a diagonal matrix whose diagonal entries are its eigenvalues. This leads to the following conclusion.

Eigenvalue Characterization of the Definite Property

A symmetric matrix A is

(a) *positive–definite if and only if all the eigenvalues are positive,*
(b) *positive–semidefinite if and only if all the eigenvalues are nonnegative,*
(c) *negative–definite if and only if all the eigenvalues are negative,*
(d) *negative–semidefinite if and only if all the eigenvalues are nonpositive,*
(e) *indefinite if and only if it has a positive and a negative eigenvalue.*

Example 3.4 From Example 2.1, the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

has eigenvalues -1, 3 and is, therefore, indefinite. From Example 1.3 the matrix

$$\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

has eigenvalues 3, 4 and is, therefore, positive–definite.

Since the eigenvalues of an $n \times n$ matrix are computed as roots of the characteristic polynomial of degree n, it is evident that for $n > 2$ the determination of the eigenvalues cannot generally be accomplished easily. A more practical

method for finding the definite property of a matrix relates to the conversion to canonical form under congruence by pairs of elementary operations, a pair consisting of a column operation and the same row operation. After each application of a pair of operations, the resulting matrix is congruent to the original, and the process yields a diagonal matrix whose definite property can be seen by inspection. If at some intermediate stage a pair produces a matrix with a positive and a negative diagonal entry, then the original matrix must be indefinite.

Example 3.5 (a) Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be given, where $a > 0$. Adding $-b/a$ times the first column to the second and then $-b/a$ times the first row to the second gives

$$\begin{bmatrix} a & 0 \\ 0 & (ac - b^2)/a \end{bmatrix}.$$

Therefore, the given matrix is positive–definite if and only if $ac - b^2 > 0$.

(b) The matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ may be successively converted to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

by pairs of elementary operations. It is, therefore, indefinite.

Questions

1. A quadratic form is a ______.
   (a) number,
   (b) vector,
   (c) matrix,
   (d) function.
2. A quadratic form which always assigns nonnegative numbers is called ______ definite.
3. A diagonal matrix which has a zero diagonal entry is either ______, ______, or ______ definite.
4. The definite property is invariant with respect to the ______ relation on matrices.
5. A matrix with eigenvalues 3, 0, -2 is ______ definite.
6. Given $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a > 0 > c$, then $Q_A(\mathbf{u}) > 0$, $Q_A(\mathbf{v}) < 0$ where $\mathbf{u} =$ ______ and $\mathbf{v} =$ ______.

Problems

1. Do Problem Set B at the end of the chapter.

Exercises

1. Determine the definite property of each matrix by (1) completing the square, (2) finding the eigenvalues, and (3) converting to diagonal form by operation pairs.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Proofs

1. Prove in three ways that if $a < 0$, then $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is negative–definite if and only if $ac - b^2 > 0$.

4. Rotation of Conics

In this section we apply the techniques and results of the previous section to a study of rotations of conics in the plane. A conic in the plane is the graph of an equation of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$$

The *linear* and *quadratic* parts are, respectively,

$$dx + ey \quad \text{and} \quad ax^2 + 2bxy + cy^2.$$

The conic is *degenerate* if it is empty or consists of finitely many points and lines. Examples of degenerate conics are the graphs of the equations below.

$$\begin{aligned} x^2 + y^2 &= 0, && \text{graph is origin point;} \\ x^2 + y^2 + 1 &= 0, && \text{graph is empty;} \\ x^2 - y^2 &= 0, && \text{graph consists of lines } y = +x,\ y = -x. \end{aligned}$$

We shall concern ourselves only with nondegenerate conics. Since the linear part is geometrically linked to a shifting or translation of the conic, it will not be explicitly involved in a considerable portion of this section. The quadratic part is the quadratic form of

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

From analytic geometry it is known that if $b=0$, then for nondegenerate cases our conic is

(a) a circle if $a=c$ $(a, c \neq 0)$,
(b) an ellipse if a and c have the same sign and are not equal,
(c) a hyperbola if a and c have different signs,
(d) a parabola if $a=0$ or $c=0$ (not both $=0$).

The case where $b=0$ gives the axes of the conic parallel to the x and y axes. If $b \neq 0$ then the axes are otherwise, and this situation now takes our attention. We seek a change of variable which corresponds to a rotation of the xy axis system into the position of the $x'y'$ axis system of the conic (see Figure 8.2). Letting (x, y) and (x', y') respectively denote the coordinates of an arbitrary point relative to these two systems, by Example 8.6 in Chapter 5,

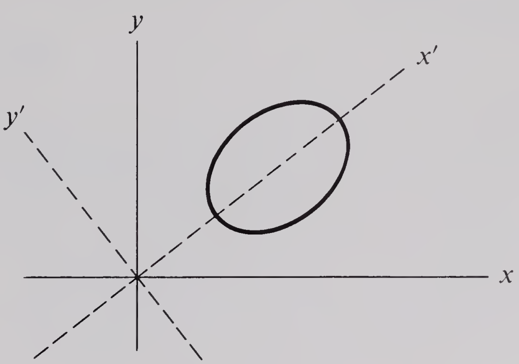


Figure 8.2

$$\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x' \\ y' \end{bmatrix},$$

where $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the matrix of the rotation from the xy to the $x'y'$ system. Then,

$$\begin{aligned} Q_A(\langle x, y\rangle) &= [x \quad y]A\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix}^* A \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \left(B\begin{bmatrix} x' \\ y' \end{bmatrix}\right)^* A \left(B\begin{bmatrix} x' \\ y' \end{bmatrix}\right) \\ &= [x' \quad y']B^*AB\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= Q_{B^*AB}(\langle x', y'\rangle). \end{aligned}$$

Since the quadratic form of the conic has no mixed term relative to the $x'\,y'$ system, and since B corresponds to a rotation, it follows that B satisfies the three properties listed here.

(1) B is orthogonal,
(2) $\det B > 0$,
(3) B^*AB is diagonal.

Conversely, any matrix B which satisfies these three properties rotates the xy axis to an axis system of the given conic, since there is no mixed term. By the diagonalization theorem for symmetric matrices, such a matrix is given by $B = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$, where $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis of eigenvectors of A. Furthermore, the diagonal entries λ_1 and λ_2 of B^*AB are the eigenvalues of A. Since the quadratic part of the conic relative to the $x'\,y'$ system is

$$Q_{B^*AB}(\langle x', y' \rangle) = \lambda_1(x')^2 + \lambda_2(y')^2,$$

we have the following result.

Theorem on Conic Types

If the quadratic part of a nondegenerate conic is the quadratic form of a matrix with eigenvalues λ_1 and λ_2, then the conic is

(a) *a circle if $\lambda_1 = \lambda_2$ $(\lambda_1, \lambda_2 \neq 0)$,*
(b) *an ellipse if λ_1 and λ_2 have the same sign and are not equal,*
(c) *a hyperbola if λ_1 and λ_2 have different signs,*
(d) *a parabola if $\lambda_1 = 0$ or $\lambda_2 = 0$ (not both $= 0$).*

It should be noted that the rotation matrix B is not unique. From Figure 8.3 we can see that the conic axes can be labeled in four ways. Each of these corresponds to a choice of sign and order for the orthonormal basis of eigenvectors of A.

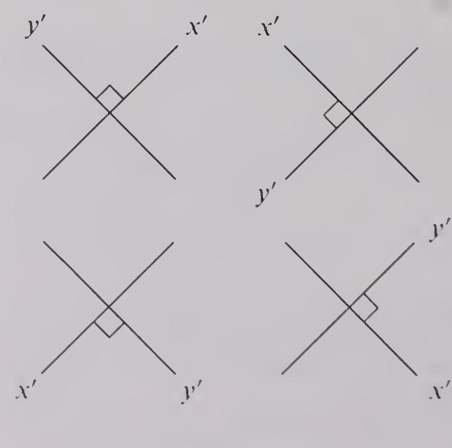


Figure 8.3

Example 4.1 (a) The matrix of the quadratic part of

$$2x^2 - 4xy - y^2 + 2x - 16 = 0$$

is $\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$. It has eigenvalues 3 and -2, and an orthonormal basis of eigenvectors is

$$\mathbf{B} = \left\{ \frac{1}{\sqrt{5}} \langle 2, -1 \rangle, \frac{1}{\sqrt{5}} \langle 1, 2 \rangle \right\}.$$

Hence, if

$$B = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

then

$$B^*AB = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

and the quadratic part of the conic relative to $\mathbf{B}$ is $3x^2 - 2y^2$. This conic is a hyperbola.

(b) The matrix of the quadratic part of

$$9x^2 + 6xy + y^2 + 2x - 4y - 8 = 0$$

is $\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. The eigenvalues are 0 and 10 and, hence, the conic is a parabola.

Three-dimensional analogues of conics are called *quadric surfaces*. The general equation form for these is

$$ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2 + gx + hy + iz + j = 0.$$

The *quadratic part* is the quadratic form of the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

The procedure for identifying the conic type may be extended to quadric surfaces. The problem is considerably more complicated, and a knowledge of both the linear and constant part is necessary for a complete classification. Letting $B = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of eigenvectors of A and $\det B > 0$, then B describes a rotation of $\mathbf{R}^3$ and B^*AB is a diagonal matrix whose diagonal entries, $\lambda_1, \lambda_2, \lambda_3$, are eigenvalues of A. Thus if (x', y', z')

denote coordinates relative to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then the quadratic part of the quadric surface may be written as

$$\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2.$$

The number of positive, negative, and zero eigenvalues of A is then useful in classifying the surface.

Questions

1. The quadratic part of $3x^2 + xy + y^2 - 6x - 40 = 0$ is ________.
2. If $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is indefinite, then $ax^2 + 2bxy + cy^2 + d = 0$ is a ________.
3. The axes of $ax^2 + 2bxy + cy^2 + d = 0$ are the x and y axes provided ________ $= 0$.
4. The axes of $ax^2 + 2bxy + cy^2 + d = 0$ have the direction of ________ of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
5. If the eigenvalues of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ are equal, then $ax^2 + 2bxy + cy^2 + d = 0$ is a ________.

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Assuming nondegeneracy, classify in terms of b the conic $x^2 + 2bxy + y^2 + d = 0$.
2. Sketch the conic $2x^2 + 2\sqrt{10}\, xy + 5y^2 + 4x - 6y = 0$.

Proofs

1. Prove that if $ax^2 + 2bxy + cy^2 + d = 0$ is
   (a) a circle, then $b = 0$;
   (b) a parabola, then $ac - b^2 = 0$;
   (c) an ellipse, then $a > 0$ implies $ac - b^2 > 0$;
   (d) a hyperbola, then $a > 0$ implies $ac - b^2 < 0$.

Problems

A. *Eigenvalues and Eigenvectors*

If A is a square matrix and I is the identity matrix having the same size as A, then

A.1 the *characteristic equation* of A is

$$\det(A - \lambda I) = 0.$$

For A an $n \times n$ matrix, the characteristic equation is a polynomial of degree n in λ.

1. Find the characteristic equation for each matrix.

(a) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

A real number, λ_0, is an *eigenvalue* of A if λ_0 is a root of the characteristic equation of A. By the fundamental theorem of algebra, an $n \times n$ matrix has at most n eigenvalues.

2. Find the eigenvalues for the matrices below.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 \\ -5 & 2 \end{bmatrix}$.

For purposes of simplicity the remainder of this section will be concerned with 2×2 matrices. If λ is an eigenvalue of A, then $\langle x, y \rangle$ is an *eigenvector* of A belonging to λ provided that

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

The eigenvectors may be determined by solving systems of equations.

3. Find the eigenvectors belonging to each eigenvalue in 2.
4. Find the characteristic equation, eigenvalues, and eigenvectors for the two matrices below.

(a) $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$.

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5. Find the eigenvalues and eigenvectors for these matrices.

(a) $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & -3 \\ 4 & -2 \end{bmatrix}$.

B. *Quadratic Forms*

The *quadratic form* of a symmetric matrix A is given by the rule

B.1 (a) $[x \quad y]A\begin{bmatrix} x \\ y \end{bmatrix}$, if A is 2×2;

(b) $[x \quad y \quad z]A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, if A is 3×3.

From B.1 we may verify the following statements.

B.2 (a) $ax^2 + 2bxy + cy^2$ is the quadratic form of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

(b) $ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$ is the quadratic form of

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

1. Find the quadratic form of each matrix.

(a) $\begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & 6 \\ 3 & 1 & -1 \\ 6 & -1 & 4 \end{bmatrix}$.

2. Find the symmetric matrix whose quadratic form is

(a) $4x^2 + 10xy + 6y^2$, (b) $x^2 + 4xy + 6xz + 3y^2 - z^2$.

Certain *definite properties* are associated with each symmetric matrix. By inspection the definite property of a diagonal matrix can be ascertained.

B.3 A diagonal matrix A is

(a) *positive–definite* if all diagonal entries are positive;
(b) *positive–semidefinite* if all diagonal entries are nonnegative;
(c) *negative–definite* if all diagonal entries are negative;
(d) *negative–semidefinite* if all diagonal entries are nonpositive;
(e) *indefinite* if some diagonal entry is positive and some diagonal entry is negative.

The assertion that a matrix is positive-(or negative-) semidefinite usually implies that it is not positive-(or negative-) definite.

3. Determine the definite property for each matrix.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$,

(d) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, (e) $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$, (f) $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$.

A general procedure for finding the definite property of a symmetric matrix A is given in B.4.

B.4 (a) Convert A to diagonal form by applying pairs of elementary operations, a pair consisting of a column operation and the same row operation (applied in either order).
(b) The diagonal matrix in (a) has the same definite property as A; apply B.3.

Any symmetric matrix with a positive and a negative diagonal entry is indefinite. If, after applying some pair of operations in B.4(a), the resulting matrix has a positive and a negative diagonal entry, then A is indefinite.

4. Determine the definite property for each matrix.

(a) $\begin{bmatrix} 2 & 1 \\ 2 & -16 \end{bmatrix}$, (b) $\begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} -1 & 3 \\ 3 & 9 \end{bmatrix}$,

(d) $\begin{bmatrix} 6 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

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5. Find the quadratic form for each matrix.

(a) $\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 6 & -4 \\ 2 & -4 & 7 \end{bmatrix}$.

6. Find a symmetric matrix whose quadratic form is

(a) $2x^2 + 10xy - 6y^2$, (b) $x^2 - 2y^2 + 4yz - 3z^2$.

7. Determine the definite property for each matrix.

(a) $\begin{bmatrix} 4 & -4 \\ -4 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 10 & 3 \\ 3 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$,

(d) $\begin{bmatrix} 5 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

C. *Rotation of Conics*

The general equation of a conic has the form

C.1 $ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$

The *quadratic part* of the conic in C.1 is $ax^2 + 2bxy + cy^2$, which is the quadratic form of

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

1. Find A so that the quadratic part of the following conic is the quadratic form of A.
   (a) $x^2 + 8xy - y^2 + 7x + 3y - 8 = 0$,
   (b) $4x^2 - xy - x + 8 = 0$.

If λ_1 and λ_2 are the (not necessarily distinct) eigenvalues of A, then the graph of the equation in C.1 is determined to be

C.2 (a) a circle, if $\lambda_1 = \lambda_2 \neq 0$;
(b) an ellipse, if λ_1 and λ_2 have the same sign and $\lambda_1 \neq \lambda_2$;
(c) a hyperbola, if λ_1 and λ_2 have different signs; or
(d) a parabola, if $\lambda_1 = 0$ or $\lambda_2 = 0$ (not both $= 0$).

The result in C.2 presumes a certain nondegeneracy which will be assumed throughout this section.

2. Determine if the graph of the following conic is a circle, ellipse, hyperbola, or parabola.
   (a) $2x^2 + 2xy + y^2 - 12 = 0$, (b) $2x^2 + 4xy + y^2 - 12 = 0$,
   (c) $3x^2 + 2\sqrt{3}xy + y^2 + 5x - 7y = 0$,
   (d) $4x^2 + 4y^2 - 3x - 40 = 0$.

The axes of the conic in C.1 have the direction of the eigenvectors of A. The equation of the conic relative to its x' and y' axes has no mixed $x'y'$ term. This equation can be obtained in the following manner.

C.3 (a) Find eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of A, each having norm 1 and such that $\mathbf{u}_2$ can be obtained from $\mathbf{u}_1$ by a counterclockwise rotation of $\pi/2$.

(b) Form the matrix B whose column vectors are $\mathbf{u}_1, \mathbf{u}_2$. (By (a), B is orthonormal and $\det B = +1$.)

(c) Substitute from $\begin{bmatrix} x \\ y \end{bmatrix} = B \begin{bmatrix} x' \\ y' \end{bmatrix}$ into C.1 to obtain an equation in x' and y'.

It can be shown that the substitution of C.3(c) gives λ_1 and λ_2 as respective coefficients of $(x')^2$, $(y')^2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of A which respectively belong to λ_1 and λ_2. The solution is not unique, since there are four choices for the $\mathbf{u}_1$, $\mathbf{u}_2$ combination.

3. For the following conics, find unit vectors in the direction of the axes of the conic, and write the equation of the conic relative to these axes.

(a) $6x^2 + 6xy - 2y^2 - 21 = 0$, (b) $5x^2 + 4xy + 2y^2 - 6 = 0$,
(c) $4x^2 + 4xy + y^2 + \sqrt{5}x - \sqrt{5}y - 9 = 0$.

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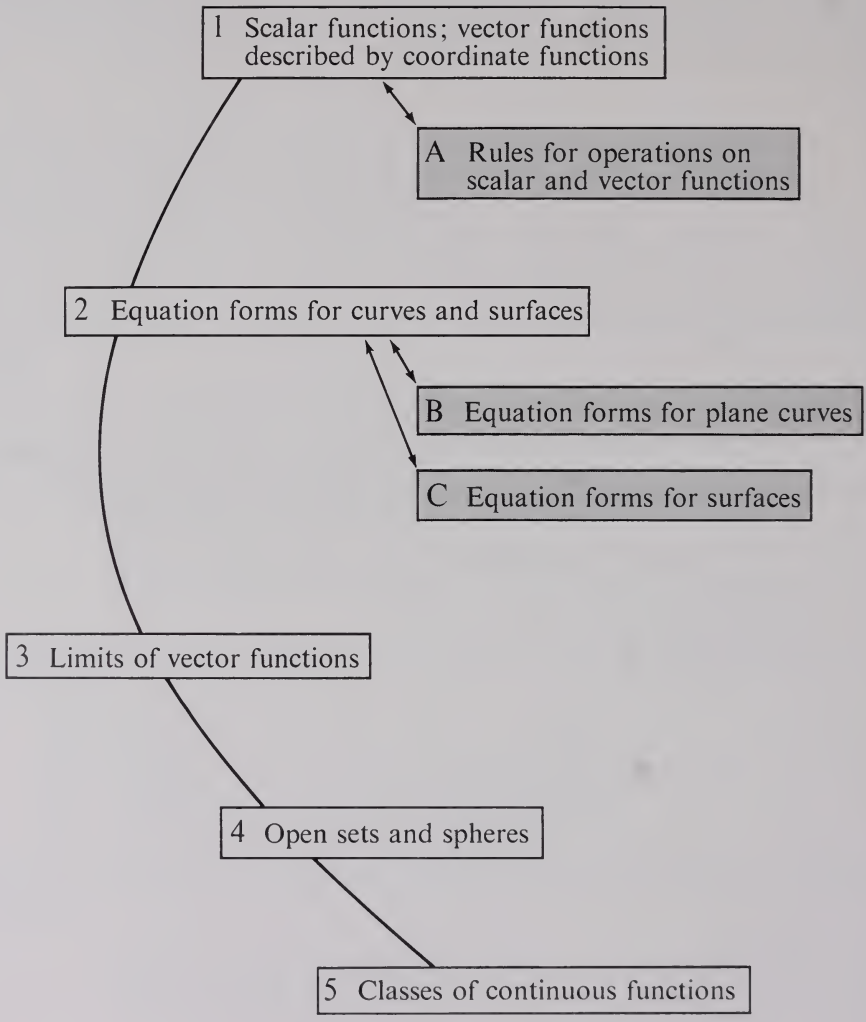
4. Classify each of the following conics as a circle, ellipse, hyperbola, or parabola. Also find unit vectors in the directions of the axes of the conic, and write the equation of the conic relative to these axes.

(a) $2x^2 + 2\sqrt{2}xy + 3y^2 - 8 = 0$, (b) $2xy - 1 = 0$,
(c) $3x^2 + 4xy - \sqrt{5}x - 3 = 0$.


1 Scalar functions; vector functions described by coordinate functions
A Rules for operations on scalar and vector functions
2 Equation forms for curves and surfaces
B Equation forms for plane curves
C Equation forms for surfaces
3 Limits of vector functions
4 Open sets and spheres
5 Classes of continuous functions

IX

Scalar and Vector Functions; Limits

Many of the functions used to describe relations in geometry, science, and engineering are not linear. In first-year calculus it was observed that real-valued nonlinear functions of one variable can be studied by means of the limit concept, provided they satisfy certain desirable properties such as continuity or differentiability. Thus such problems as finding velocities and areas, which for linear situations can be solved by algebraic methods, are attacked in calculus by means of derivatives and integrals.

Although we shall now depart abruptly from linear mathematics, the linear techniques and results which we have studied will have widespread applications and implications in the development that follows. For one thing, the spaces of our study will continue to be vector spaces or subsets of vector spaces. Furthermore, we shall see that all sufficiently smooth physical change can be approximated locally by linear change. Also, sufficiently smooth physical or geometric configurations can be approximated locally by parallelotopes.

We shall begin with a study of functions with unspecified characteristics.

Significant properties of functions will come later. Our functions will fall into two classes,

(1) *scalar* (or *real*) *functions*, and
(2) *vector functions*.

In physical applications these are often called *scalar fields* and *vector fields*. The terms scalar and vector, as used here, denote the nature of the range elements of the function. Some physical examples will illustrate (see Figure 9.1).

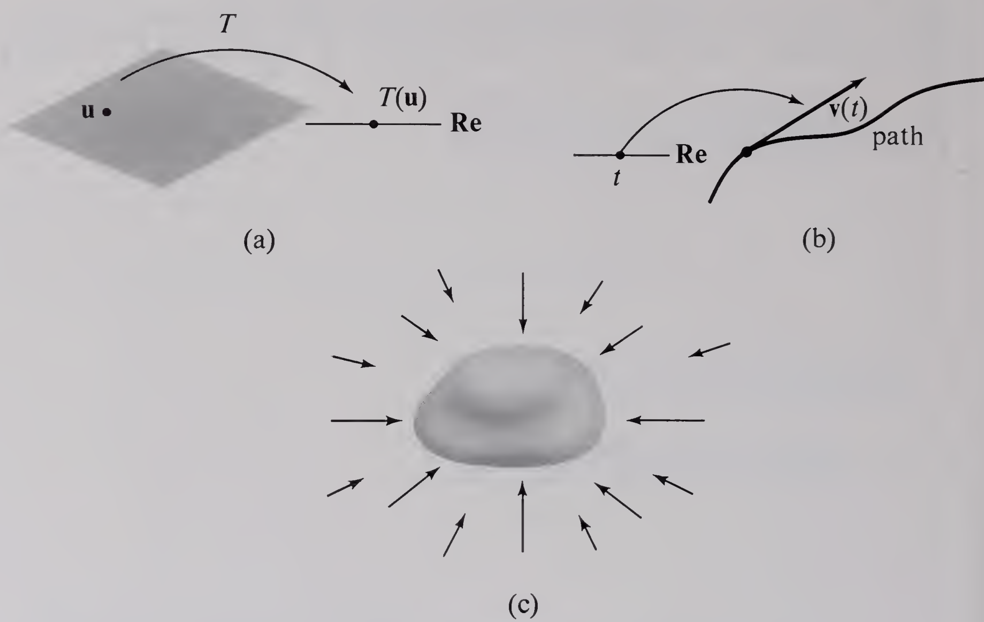


Figure 9.1

(a) The temperature T of a flat plate is a scalar function. If we symbolize a point on the plate by **u**, regarded as a tuple vector relative to some coordinate system, then T assigns to each **u** a real number, $T(\mathbf{u})$.
(b) The velocity, $\mathbf{v}(t)$, of a moving particle in space is a vector function of the (scalar) time t.
(c) The gravitational force field, **F**, produced by a physical object in space is a vector function with domain elements the position vectors in space.

1. Algebra of Scalar and Vector Functions

In this section we consider functions having as their domain a subset U of $\mathbf{R}^n$ and having range $\mathbf{R}^m$ (see Figure 9.2).

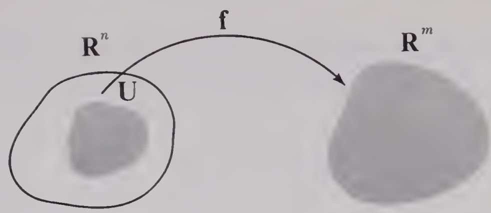


Figure 9.2

These are called *functions of n variables.* If $m = 1$, they are called *scalar* or *real functions*; if $m > 1$, they are called *vector functions.* Boldface symbols, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, and so forth denote vector functions. Italic symbols f, g, h, are used for scalar functions. At times it will be convenient to let boldface symbols denote a function which may be either a vector or a scalar function.

The rule of a scalar function can often be described by a formula. For example, $f(\langle r, s\rangle) = rs^2$ might be the rule of a scalar function of 2 variables. This function rule is usually abbreviated to $f(r, s) = rs^2$. A vector function $\mathbf{f}$ may be described by certain associated scalar functions called the *coordinate functions of* $\mathbf{f}$. For instance, let $\mathbf{f}$ be a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ and $\mathbf{u}$ a variable domain element of $\mathbf{f}$ (see Figure 9.3). If x and y denote coordinates of the range of $\mathbf{f}$, then $\mathbf{f}(\mathbf{u}) = \langle x, y\rangle$ is a variable range element of $\mathbf{f}$; thus x and y each vary with $\mathbf{u}$ and may be regarded as (scalar) functions of $\mathbf{u}$. We write $x = \bar{x}(\mathbf{u})$, $y = \bar{y}(\mathbf{u})$; then $\bar{x}(\mathbf{u})$ and $\bar{y}(\mathbf{u})$ are the *coordinate functions* of $\mathbf{f}$ and for each $\mathbf{u}$ we have

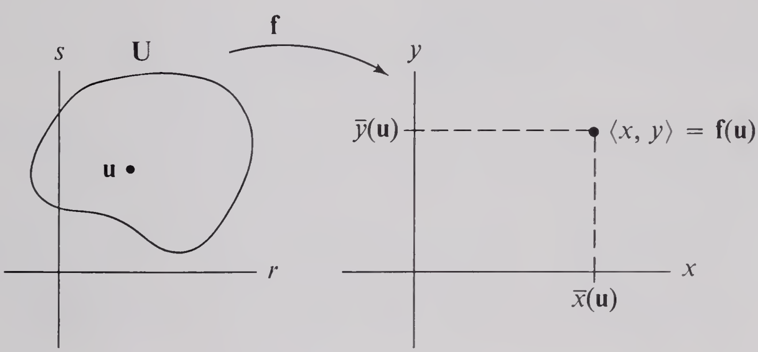


Figure 9.3

$$\mathbf{f}(\mathbf{u}) = \langle \bar{x}(\mathbf{u}), \bar{y}(\mathbf{u})\rangle.$$

Thus, a vector function $\mathbf{f}$ from $\mathbf{R}^2$ to $\mathbf{R}^2$ might be given by a rule

$$\mathbf{f}(r, s) = \langle rs^2, r - 2s\rangle,$$

where $\bar{x}(r, s) = rs^2$, $\bar{y}(r, s) = r - 2s$ are the coordinate functions of $\mathbf{f}$. Using standard basis notation, this may be written

$$\mathbf{f}(r, s) = rs^2\mathbf{i} + (r - 2s)\mathbf{j}.$$

Operations on vector functions will be considered next. Let $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and $\mathbf{g} = \bar{x}'\mathbf{i} + \bar{y}'\mathbf{j}$ have the same domain and same range space. From the equality chain,

$$\begin{aligned}(\mathbf{f} + \mathbf{g})(\mathbf{u}) &= (\bar{x}(\mathbf{u})\mathbf{i} + \bar{y}(\mathbf{u})\mathbf{j}) + (\bar{x}'(\mathbf{u})\mathbf{i} + \bar{y}'(\mathbf{u})\mathbf{j})\\ &= (\bar{x}(\mathbf{u}) + \bar{x}'(\mathbf{u}))\mathbf{i} + (\bar{y}(\mathbf{u}) + \bar{y}'(\mathbf{u}))\mathbf{j}\\ &= (\bar{x} + \bar{x}')(\mathbf{u})\mathbf{i} + (\bar{y} + \bar{y}')(\mathbf{u})\mathbf{j},\end{aligned}$$

we see that $\mathbf{f} + \mathbf{g}$ is obtained by adding corresponding coordinate functions. In a similar way, it can be seen that $c\mathbf{f}(\mathbf{u}) = c\bar{x}(\mathbf{u})\mathbf{i} + c\bar{y}(\mathbf{u})\mathbf{j}$, and, hence, the product of c and $\mathbf{f}$ is obtained by multiplying each coordinate function of $\mathbf{f}$ by c (see Proofs, exercise 1).

Example 1.1 If $\mathbf{f} = r^2\mathbf{i} + rs\mathbf{j}$ and $\mathbf{g} = s^2\mathbf{i} + r\mathbf{j}$, then

$$\mathbf{f} + \mathbf{g} = (r^2 + s^2)\mathbf{i} + (rs + r)\mathbf{j}$$

and

$$3\mathbf{f} = 3r^2\mathbf{i} + 3rs\mathbf{j}.$$

If the domain of $\mathbf{g}$ contains the range of $\mathbf{f}$, then the composition $\mathbf{g} \circ \mathbf{f}$ is defined. Given that

$$\mathbf{f} = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} \quad \text{and} \quad \mathbf{g}(x, y) = \bar{x}'(x, y)\mathbf{i} + \bar{y}'(x, y)\mathbf{j},$$

we have

$$\begin{aligned}\mathbf{g} \circ \mathbf{f}(\mathbf{u}) &= \mathbf{g}(\mathbf{f}(\mathbf{u}))\\ &= \bar{x}'(\mathbf{f}(\mathbf{u}))\mathbf{i} + \bar{y}'(\mathbf{f}(\mathbf{u}))\mathbf{j}\\ &= \bar{x}'(\bar{x}(\mathbf{u}), \bar{y}(\mathbf{u}))\mathbf{i} + \bar{y}'(\bar{x}(\mathbf{u}), \bar{y}(\mathbf{u}))\mathbf{j}.\end{aligned}$$

Therefore, the coordinate functions of $\mathbf{g} \circ \mathbf{f}$ are obtained by substituting each coordinate function of $\mathbf{f}$ for the corresponding domain variable of $\mathbf{g}$.

Example 1.2 Given $\mathbf{f} = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} = rs\mathbf{i} + (r - s)\mathbf{j}$ and $\mathbf{g} = x^2\mathbf{i} + xy\mathbf{j}$, then substitution of rs for x and $r - s$ for y in the formula for $\mathbf{g}$ gives $\mathbf{g} \circ \mathbf{f} = (rs)^2\mathbf{i} + rs(r - s)\mathbf{j}$.

If the range of $\mathbf{f}$ is $\mathbf{R}^3$, then the coordinate functions of $\mathbf{f}$ may be denoted by $\bar{x}$, $\bar{y}$, and $\bar{z}$; hence, we may write $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$. If the range of $\mathbf{f}$ is $\mathbf{R}^m$, $m \geq 3$, then a different form of notation must be adopted. In this more general case we let $\bar{f}_i$ denote the ith coordinate function of $\mathbf{f}$, and write

$$\mathbf{f} = \langle \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_m \rangle.$$

Thus, $\bar{f}_i$ is a scalar function with domain equal to the domain of $\mathbf{f}$, and $\bar{f}_i(\mathbf{u})$ is the ith coordinate of $\mathbf{f}(\mathbf{u})$. The properties of our various operations may then be formally stated.

Proposition 1.1 If $\mathbf{f} = \langle \bar{f}_1, \ldots, \bar{f}_m \rangle$ and $\mathbf{g} = \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$, then
(a) $\mathbf{f} + \mathbf{g} = \langle \bar{f}_1 + \bar{g}_1, \ldots, \bar{f}_m + \bar{g}_m \rangle$,
(b) $c\mathbf{f} = \langle c\bar{f}_1, \ldots, c\bar{f}_m \rangle$.

Proposition 1.2 If $\mathbf{g} = \langle \bar{g}_1, \ldots, \bar{g}_l \rangle$ and $\mathbf{g} \circ \mathbf{f}$ is defined, then

$$\mathbf{g} \circ \mathbf{f} = \langle \bar{g}_1 \circ \mathbf{f}, \ldots, \bar{g}_l \circ \mathbf{f} \rangle.$$

Some new operations for vector and scalar functions will now be introduced. Each of these plays a significant role in the subject area of vector calculus. The rules are defined as follows whenever domain and range spaces make them meaningful.

(a) $fg(\mathbf{u}) = f(\mathbf{u})g(\mathbf{u})$,	*Product of scalar functions*,
(b) $\dfrac{f}{g}(\mathbf{u}) = \dfrac{f(\mathbf{u})}{g(\mathbf{u})}$,	*Quotient of scalar functions*,
(c) $f\mathbf{g}(\mathbf{u}) = f(\mathbf{u})\mathbf{g}(\mathbf{u})$,	*Product of scalar function and vector function*,
(d) $\dfrac{\mathbf{f}}{g}(\mathbf{u}) = \dfrac{\mathbf{f}(\mathbf{u})}{g(\mathbf{u})}$,	*Quotient of vector function and scalar function*,
(e) $\mathbf{f} \cdot \mathbf{g}(\mathbf{u}) = \mathbf{f}(\mathbf{u}) \cdot \mathbf{g}(\mathbf{u})$,	*Dot product of vector functions*,
(f) $\mathbf{f} \times \mathbf{g}(\mathbf{u}) = \mathbf{f}(\mathbf{u}) \times \mathbf{g}(\mathbf{u})$,	*Cross product of vector functions.*

The products (c), (d), and (f) each yield vector functions. Their coordinate functions are easily seen to be given by the next result.

Proposition 1.3

(a) $f\langle \bar{g}_1, \ldots, \bar{g}_m \rangle = \langle f\bar{g}_1, \ldots, f\bar{g}_m \rangle$,

(b) $\dfrac{\langle \bar{f}_1, \ldots, \bar{f}_m \rangle}{g} = \left\langle \dfrac{\bar{f}_1}{g}, \ldots, \dfrac{\bar{f}_m}{g} \right\rangle$,

(c) $(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) \times (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \\ \bar{g}_1 & \bar{g}_2 & \bar{g}_3 \end{bmatrix}$.

It is assumed that the right side of (c) is evaluated in the same manner as the cross product of vectors. Many properties may be established for the various

operations using corresponding properties for vectors. For instance, from $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ we have, for an arbitrary $\mathbf{u}$,

$$\begin{aligned}(\mathbf{f} \times \mathbf{g}) \cdot \mathbf{h}(\mathbf{u}) &= (\mathbf{f} \times \mathbf{g})(\mathbf{u}) \cdot \mathbf{h}(\mathbf{u}) \\ &= [\mathbf{f}(\mathbf{u}) \times \mathbf{g}(\mathbf{u})] \cdot \mathbf{h}(\mathbf{u}) \\ &= \mathbf{f}(\mathbf{u}) \cdot [\mathbf{g}(\mathbf{u}) \times \mathbf{h}(\mathbf{u})] \\ &= \mathbf{f}(\mathbf{u}) \cdot (\mathbf{g} \times \mathbf{h})(\mathbf{u}) \\ &= \mathbf{f} \cdot (\mathbf{g} \times \mathbf{h})(\mathbf{u}).\end{aligned}$$

Therefore, $(\mathbf{f} \times \mathbf{g}) \cdot \mathbf{h} = \mathbf{f} \cdot (\mathbf{g} \times \mathbf{h})$.

Questions

1. A function from $\mathbf{R}^2$ to $\mathbf{R}^3$ is a function of ______ variables.
2. A function from $\mathbf{R}^2$ to $\mathbf{R}^3$ has ______ coordinate functions.
3. If $\mathbf{g} = \langle \bar{g}_1, \bar{g}_2 \rangle$ and $\mathbf{f} = \langle \bar{f}_1, \bar{f}_2 \rangle$, then the coordinate functions of $\mathbf{g} \circ \mathbf{f}$ are ______.
4. The two product operations on vector functions are the ______ and ______.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. Prove that if $\mathbf{f} = \langle \bar{x}, \bar{y} \rangle$, then $c\mathbf{f} = \langle c\bar{x}, c\bar{y} \rangle$.
2. Prove, using a corresponding property for vectors, each function property below:
   (a) $\mathbf{f} \times (\mathbf{g} + \mathbf{h}) = (\mathbf{f} \times \mathbf{g}) + (\mathbf{f} \times \mathbf{h})$, (b) $\mathbf{f} \times \mathbf{g} = -\mathbf{g} \times \mathbf{f}$,
   (c) $\mathbf{f} \times (c\mathbf{g}) = c(\mathbf{f} \times \mathbf{g})$.

2. Curves and Surfaces

Geometric configurations give rise to various scalar and vector functions which may be used to describe and study them. From analytic geometry, we are already familiar with functions related to straight lines and

conics in the plane. There are generally many ways to describe a given set by a function, some ways having special advantages.

The term "curve" has intuitive meaning to the extent that we can classify certain simple sets as being curves or noncurves. On the other hand, a precise definition of curve is not easily attained and must be postponed until a later place in the text. For our present purposes it is immaterial whether we regard a plane curve as a set of vectors in $\mathbf{R}^2$, or as the set of points which constitute its graph. In fact, it will be convenient to identify these corresponding algebraic and geometric entities and use them interchangeably. Furthermore, extent of curves, determined by endpoints, will usually be ignored in this section. We shall, instead, be primarily interested in the rules, expressed by equations, of various types of functions which are used to describe a curve.

Let $\mathscr{C}$ be the curve in the Cartesian plane which is the graph of $y=f(x)$ (see Figure 9.4(a)). Thus, (x_0, y_0) is on $\mathscr{C}$ if and only if $y_0=f(x_0)$. The equation $y=f(x)$ is called the *explicit equation* of $\mathscr{C}$. The form $y-f(x)=0$ is called an *implicit equation* of $\mathscr{C}$. If $g(x, y)=y-f(x)$, then $\mathscr{C}$ is the set of all points whose g-image is 0 (see Figure 9.4(b)). More generally, an implicit equation has the form $g(x, y)=a$, where a is a constant. A third description of $\mathscr{C}$, called a *parametric form*, is obtained by regarding the x and y coordinates of points on $\mathscr{C}$ as functions, $x=\bar{x}(r)$ and $y=\bar{y}(r)$, of a single variable r. Simple parametric equations for $\mathscr{C}$ are $x=r$ and $y=f(r)$. Corresponding to these parametric equations is a *vector equation*, $\mathbf{f}(r)=r\mathbf{i}+f(r)\mathbf{j}$, of $\mathscr{C}$. The curve $\mathscr{C}$ is called the *image graph* of $\mathbf{f}$ (Figure 9.4(c)).

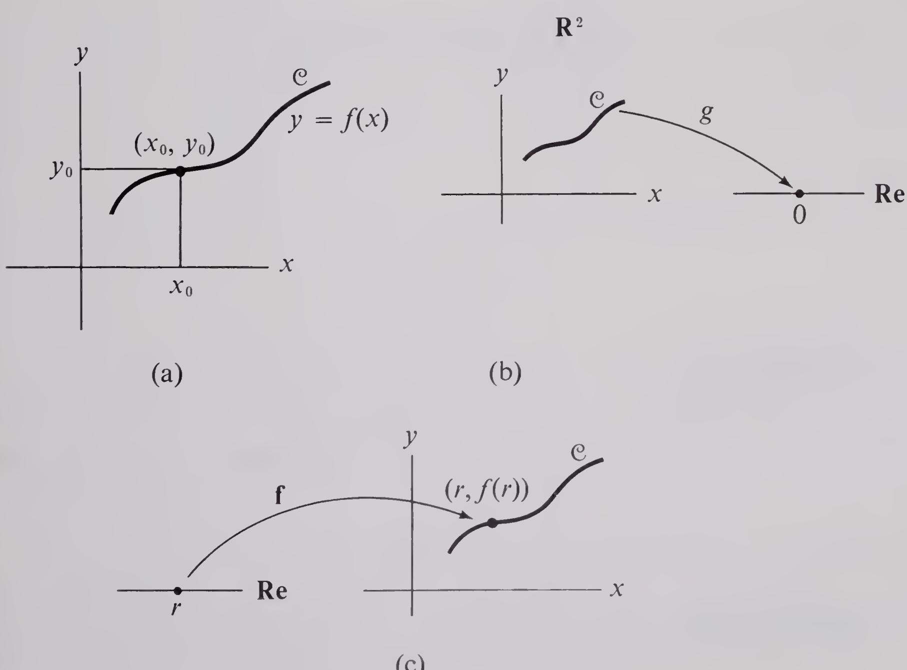


Figure 9.4

Summarizing, we have seen that a curve $\mathscr{C}$ in the Cartesian plane may be described by any of the following forms:

(a) an explicit equation $y = f(x)$; f is a scalar function with domain in **Re**,
(b) an implicit equation $g(x, y) = a$; g is a scalar function with domain in $\mathbf{R}^2$,
(c) a vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)j$; $\mathbf{f}$ is a vector function with domain in **Re**.

Example 2.1 Let $\mathscr{C}$ be the semicircle of radius 1 centered at the origin and lying in the upper half of the xy plane. Then $\mathscr{C}$ is described by the following equations.

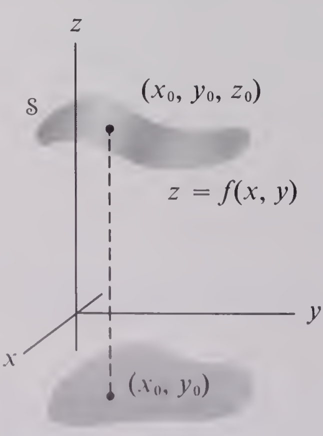
(a) $y = \sqrt{1 - x^2}$, explicit equation,
(b) $x^2 + y^2 = 1$, implicit equation,
(c) $\mathbf{f}(r) = r\mathbf{i} + \sqrt{1 - r^2}\,\mathbf{j}$, vector equation.

In this case another vector equation,

$$\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad 0 \le \theta \le \pi,$$

is more often used to describe $\mathscr{C}$.

Many basic curves cannot be completely described by an explicit equation. For instance, the circle of radius 1 centered at the origin can be described by the implicit equation $x^2 + y^2 = 1$ and the vector equation $\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $0 \le \theta \le 2\pi$. Various portions of this circle can be described by the explicit equations $y = \sqrt{1 - x^2}$, $y = -\sqrt{1 - x^2}$, $x = \sqrt{1 - y^2}$, and $x = -\sqrt{1 - y^2}$, but no explicit equation encompasses the entire curve. This suggests that there is a disadvantage in the explicit form. A familiarity with all three equation forms, however, is necessary for a basic working knowledge of vector calculus.



We next study functions associated with surfaces. A surface may be pictured as a 2-dimensional set embedded in space. There will be no attempt here at precise definitions. The same type of equation forms hold for surfaces as for curves. Given the real function $z = f(x, y)$ of two variables, there is the surface graph $\mathscr{S}$ consisting of points (x_0, y_0, z_0) which satisfy the requirement, $z_0 = f(x_0, y_0)$. Analogous to the case for curves, there are the following equation forms associated with $\mathscr{S}$ (see Figure 9.5).

(a) an explicit equation $z = f(x, y)$; f is a scalar function with domain in $\mathbf{R}^2$,
(b) an implicit equation, $g(x, y, z) = a$; g is a scalar function with domain in $\mathbf{R}^3$,
(c) a vector equation $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}$; $\mathbf{f}$ is a vector function with domain in $\mathbf{R}^2$.

Example 2.2 Let $\mathscr{S}$ be the surface of revolution obtained by revolving the curve $y = x^2$ in the xy plane about the x axis. If an arbitrary point $(x_0, x_0^2, 0)$ on the curve is revolved about the x axis through an angle ϕ, then the point $(x_0, x_0^2 \cos\phi, x_0^2 \sin\phi)$ is obtained (see Figure 9.6). Therefore, a vector equation of $\mathscr{S}$ is

$$\mathbf{f}(x, \phi) = x\mathbf{i} + x^2 \cos\phi\,\mathbf{j} + x^2 \sin\phi\,\mathbf{k}.$$

From the parametric equations $y = x^2 \cos\phi$, $z = x^2 \sin\phi$ and the identity $(x^2 \cos\phi)^2 + (x^2 \sin\phi)^2 = x^4$, we easily obtain $x^4 - y^2 - z^2 = 0$ as an implicit equation of $\mathscr{S}$. An explicit equation of the positive z portion of $\mathscr{S}$ is $z = \sqrt{x^4 - y^2}$.

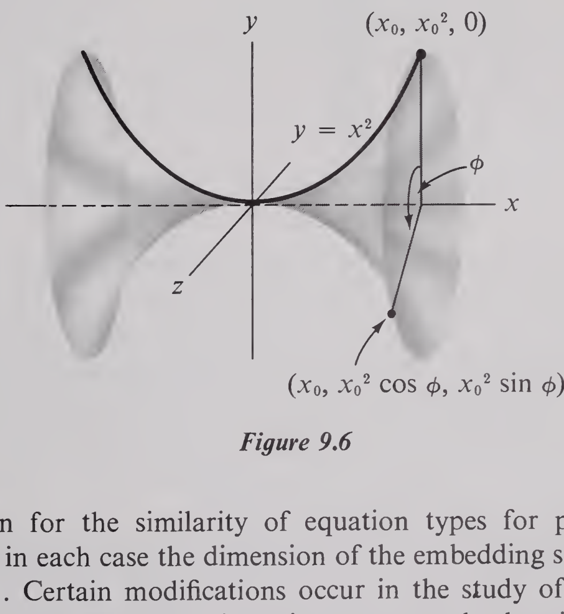


Figure 9.6

The reason for the similarity of equation types for plane curves and surfaces is that in each case the dimension of the embedding space exceeds that of the set by 1. Certain modifications occur in the study of curves in space, where this is not true. A curve in Cartesian space may be described by any of the equation forms below:

(a) an explicit equation $z = f(x, y)$, where the domain of f is a curve in the xy plane,
(b) two implicit equations $g_1(x, y, z) = a_1$ and $g_2(x, y, z) = a_2$; the curve is then the intersection of the surface graphs of g_1 and g_2, or
(c) a vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$.

Example 2.3 Let $\mathscr{C}$ be the curve which is implicitly described as the intersection of the plane, $2x + 3y - z = 1$, and the cylinder, $x^2 + y^2 = 1$ (see Figure 9.7). Then, $z = 2x + 3y - 1$ is an explicit equation of $\mathscr{C}$, where the values of (x, y) are restricted to the circle $x^2 + y^2 = 1$ in the xy plane. Using the vector equation $\mathbf{h}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ of the circle $x^2 + y^2 = 1$, we obtain for $\mathscr{C}$ the following vector equation:

$$\mathbf{f}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + (2 \cos \theta + 3 \sin \theta - 1)\mathbf{k}.$$

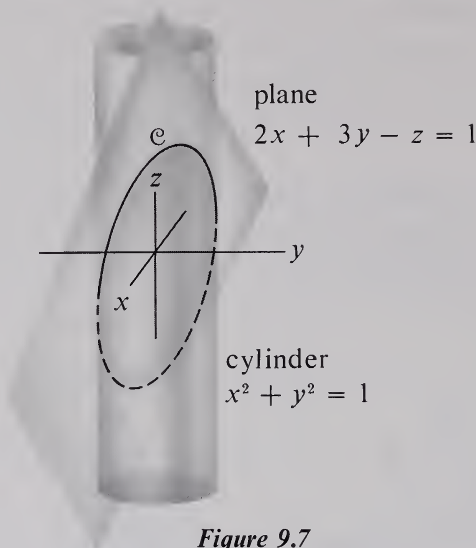


Figure 9.7

The vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ describes a curve $\mathscr{C}$ as the $\mathbf{f}$-image of a set of real numbers. However, as was stated in the introduction to Chapter VI, a *rectangle* may be the $\mathbf{f}$-image of a continuous function whose domain is an interval of real numbers. On the other hand, the $\mathbf{f}$-image of any constant function is a *point*, which could hardly be recognized as a curve. We therefore observe a need for finding properties of such functions whose image sets could properly be called curves; these properties will be formulated later in the text. A similar discussion applies to curves and surfaces in $\mathbf{R}^3$.

The explicit equations for a surface $\mathscr{S}$ and curve $\mathscr{C}$ in space have the same form, the distinction being the nature of the domain set. For the curve $\mathscr{C}$ the domain was specified as a curve (1-dimensional) in the xy plane. The domain set for $\mathscr{S}$ was implicitly assumed to be 2-dimensional. A precise formulation of dimension for lines and planes was obtained without great difficulty in the study of vector algebra. A careful study of dimension for nonlinear sets, however, involves subtleties which are not appropriate at the level of this text. Therefore, an intuitive grasp of the meaning of the important concept of dimension must suffice for our understanding of certain descriptions of sets by means of functions.

Questions

1. The equation $y = f(x)$ is a(n) ______ equation; $g(x, y) = a$ is a(n) ______ equation; $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ is a(n) ______ equation.

2. An ellipse cannot be completely described by ______.
   (a) an implicit equation,
   (b) parametric equations,
   (c) an explicit equation.

3. A curve is ______ -dimensional; a surface is ______ -dimensional.

4. A curve in $\mathbf{R}^3$ can be described by a single ______ equation.
   (a) implicit,
   (b) parametric,
   (c) vector.

5. A surface may be described as the image of a ______ in the Cartesian plane.
   (a) point,
   (b) curve,
   (c) region.

Problems

1. Do Problem Sets B and C at the end of the chapter.

Exercises

1. Find a vector equation of the surface obtained by revolving the circle $y = 3 + 2 \sin \theta$, $z = 2 + 2 \cos \theta$ in the yz plane about the z axis. (*Hint*: Revolving the point $(0, y_o, z_o)$ through an angle ϕ gives

$$(y_o \sin \phi, y_o \cos \phi, z_o).$$

2. Find a vector equation of the skew cylindrical surface (see Figure 9.8) generated by a line with direction vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, moving along the circle $x^2 + y^2 = 1$ in the xy plane. (*Hint*: An arbitrary vector in the set has the form $\mathbf{OP} + r(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, where $P = (\cos \theta, \sin \theta, 0)$ is on the circle.)

3. Find a vector equation of the skew conical surface consisting of all lines passing through $(1, 2, 3)$ and the circle $x^2 + y^2 = 1$ in the xy plane (see Figure 9.9). (*Hint*: An arbitrary vector in the conical surface has the form $\mathbf{OP} + r\mathbf{PQ}$, where $P = (1, 2, 3)$ and $Q = (\cos \theta, \sin \theta, 0)$ are points on the circle.)

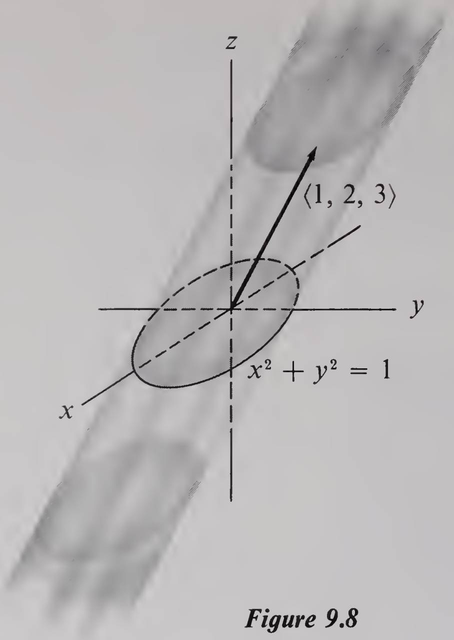


Figure 9.8

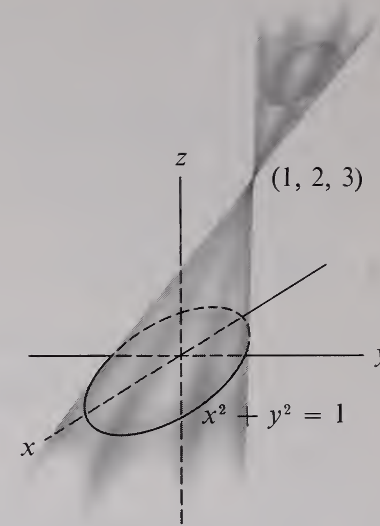


Figure 9.9

4. Give a vector equation of the curve which is the portion of the surface described.
   (a) $z = x^2 + 4y^2$ lying above the parabola $y = 3x^2$,
   (b) $z = 3x^2 + 2y^2$ lying above the ellipse $x^2/4 + y^2/9 = 1$.

Proofs

1. Prove that the surface obtained by revolving about the z axis the curve $x = 0$, $y = \bar{y}(r)$, $z = \bar{z}(r)$, where $\bar{y}(r) \geq 0$ for all r, has a vector equation

$$\mathbf{f}(r, \phi) = \bar{y}(r)\sin\phi\,\mathbf{i} + \bar{y}(r)\cos\phi\,\mathbf{j} + \bar{z}(r)\mathbf{k}.$$

2. Prove that the skew cylindrical surface generated by a line with direction vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ moving along the curve

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$$

has a vector equation

$$\mathbf{g}(r, s) = (as + \bar{x}(r))\mathbf{i} + (bs + \bar{y}(r))\mathbf{j} + (cs + \bar{z}(r))\mathbf{k}.$$

3. Prove that the skew conical surface consisting of lines passing through the point (a, b, c) and the curve $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$ has a vector equation

$$\mathbf{g}(r, s) = (a(1 - s) + s\bar{x}(r))\mathbf{i} + (b(1 - s) + s\bar{y}(r))\mathbf{j} + (c(1 - s) + s\bar{z}(r))\mathbf{k}.$$

3. Limits

The entire theory of calculus rests on the idea of limit. Although the beginning of the calculus, generally credited independently to Newton and Leibniz, goes back to the 17th century, it was not until the 19th century that the limit concept was made reasonably sound by the Frenchman Cauchy [1789–1857] and later the German mathematician Weierstrass [1815–1897].

If $\mathbf{f}$ denotes a vector or scalar function from a set $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$, and if $\mathbf{u}_o$ and $\mathbf{u}_o'$ are, respectively, elements in $\mathbf{R}^n$ and $\mathbf{R}^m$, then the expression,

$$\lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f} = \mathbf{u}_o',$$

may be read "limit of $\mathbf{f}$ as $\mathbf{u}$ approaches $\mathbf{u}_o$ is $\mathbf{u}_o'$." In this expression $\mathbf{u}$ is regarded as a variable element in $\mathbf{U}$. Loosely interpreted, the limit statement means

"if $\mathbf{u}$ is near $\mathbf{u}_o$, then $\mathbf{f}(\mathbf{u})$ is near $\mathbf{u}_o'$."

Since the vector algebra concept of norm corresponds to the geometric notion of distance, the expression "$\mathbf{u}$ is near $\mathbf{u}_o$" is conveyed by the statement "$|\mathbf{u} - \mathbf{u}_o|$ is small." Thus, a more precise meaning for $\lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$ is

"if $|\mathbf{u} - \mathbf{u}_o|$ is small, then $|\mathbf{f}(\mathbf{u}) - \mathbf{u}_o'|$ is small."

A precise replacement for the word "small" gives the following extension of the limit definition usually found in first-year calculus.

Definition of Limit

$\lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$ if and only if for each number $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta$ and $\mathbf{u} \neq \mathbf{u}_o$, then $|\mathbf{f}(\mathbf{u}_o) - \mathbf{u}_o'| < \varepsilon$.

It should be noted that the assignment which $\mathbf{f}$ makes to $\mathbf{u}_o$ is immaterial in determining $\lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f}$; in fact, $\mathbf{u}_o$ may not be in $\mathbf{U}$. The definition is nontrivial, however, only if there are vectors in $\mathbf{U}$ arbitrarily near $\mathbf{u}_o$. This will always be assumed when limits are employed.

Example 3.1 Let $\mathbf{f}$ be a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ (see Figure 9.10). The condition that $|\mathbf{u} - \mathbf{u}_o| < \delta$, $\mathbf{u} \neq \mathbf{u}_o$, says that $\mathbf{u}$ is in the circular disk $\mathbf{D}$, which has radius δ and is centered at $\mathbf{u}_o$, but does not include either the center point $\mathbf{u}_o$ or the points on the boundary circle. The

inequality $|\mathbf{f}(\mathbf{u}) - \mathbf{u}_0'| < \varepsilon$ says that $\mathbf{f}(\mathbf{u})$ is in the circular disk $\mathbf{D}'$, without boundary, centered at $\mathbf{u}_0'$, and having radius ε. Thus $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} = \mathbf{u}_0'$ asserts that for each $\mathbf{D}'$ about $\mathbf{u}_0'$ there is a $\mathbf{D}$ about $\mathbf{u}_0$ such that the $\mathbf{f}$-image of $\mathbf{D}$ is contained in $\mathbf{D}'$.

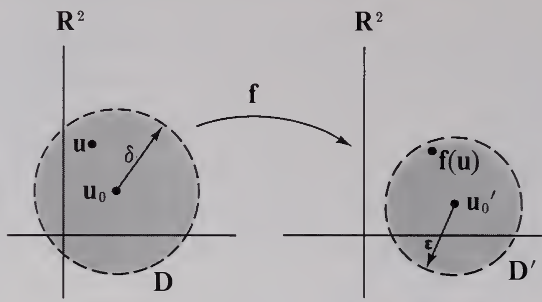


Figure 9.10

Many fundamental limit properties of first-year calculus remain true with the extended definition. In fact the same proofs remain valid with only slight modifications. Our first result gives a linearity property for limits.

Proposition 3.1 (a) $\lim_{\mathbf{u}\to\mathbf{u}_0} (\mathbf{f}+\mathbf{g}) = \lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} + \lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{g}$,
(b) $\lim_{\mathbf{u}\to\mathbf{u}_0} (c\mathbf{f}) = c \lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}$.

It is assumed here that the limits on the right side of the equalities exist. For a proof of (b) when $c \neq 0$, let $\varepsilon > 0$ and $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} = \mathbf{u}_0'$. Then there exists $\delta > 0$ such that $\mathbf{u} \neq \mathbf{u}_0$ and $|\mathbf{u} - \mathbf{u}_0| < \delta$ implies $|\mathbf{f}(\mathbf{u}) - \mathbf{u}_0'| < \varepsilon/|c|$. The latter inequality is equivalent to $|c\mathbf{f}(\mathbf{u}) - c\mathbf{u}_0'| < \varepsilon$, and this proves (for $c \neq 0$) the desired equality:

$$\lim_{\mathbf{u}\to\mathbf{u}_0} c\mathbf{f} = c\mathbf{u}_0' = c \lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}.$$

The limit of a vector function can be obtained from the limits of its coordinate functions. For instance, if $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, then

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} = \lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x}\mathbf{i} + \lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y}\mathbf{j}.$$

This equality is valid provided that either (a) $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}$ exists or (b) $\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x}$ and $\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y}$ each exist. For the proof, assuming (b), we set

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x} = a \quad \text{and} \quad \lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y} = b.$$

If $\varepsilon > 0$, then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

(a) $|\mathbf{u} - \mathbf{u}_0| < \delta_1(\mathbf{u} \neq \mathbf{u}_0)$ implies $|\bar{x}(\mathbf{u}) - a| < \varepsilon/2$,
(b) $|\mathbf{u} - \mathbf{u}_0| < \delta_2(\mathbf{u} \neq \mathbf{u}_0)$ implies $|\bar{y}(\mathbf{u}) - b| < \varepsilon/2$.

If δ is the smaller of δ_1, δ_2, then $|\mathbf{u}-\mathbf{u}_\circ|<\delta$ ($\mathbf{u}\neq\mathbf{u}_\circ$) implies that

$$\begin{aligned}|f(\mathbf{u})-(a\mathbf{i}+b\mathbf{j})| &= |(\bar{x}(\mathbf{u})\mathbf{i}+\bar{y}(\mathbf{u})\mathbf{j})-(a\mathbf{i}+b\mathbf{j})|\\ &= |(\bar{x}(\mathbf{u})-a)\mathbf{i}+(\bar{y}(\mathbf{u})-b)\mathbf{j}|\\ &\leq |(\bar{x}(\mathbf{u})-a)|+|(\bar{y}(\mathbf{u})-b)| \quad \text{(triangle inequality)}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &= \varepsilon.\end{aligned}$$

This completes the proof. The result may be extended to the following theorem, which is valid in the sense that if either limit exists, then the other does also and they are equal expressions.

Coordinate Theorem for Limits

If $\mathbf{f}=\langle\bar{f}_1,\ldots,\bar{f}_m\rangle$, *then*

$$\lim_{\mathbf{u}\to\mathbf{u}_\circ}\mathbf{f}=\left\langle\lim_{\mathbf{u}\to\mathbf{u}_\circ}\bar{f}_1,\ldots,\lim_{\mathbf{u}\to\mathbf{u}_\circ}\bar{f}_m\right\rangle.$$

Example 3.2 Let $\mathbf{f}(x)=(\sin x/x)\mathbf{i}+x\mathbf{j}$, $x>0$. Then

$$\lim_{x\to 0}\mathbf{f}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\mathbf{i}+\left(\lim_{x\to 0}x\right)\mathbf{j}=\mathbf{i}.$$

Limits behave well with the other function operations. For scalar functions we have additional properties, as shown below.

Proposition 3.2

(a) $\displaystyle\lim_{\mathbf{u}\to\mathbf{u}_\circ}fg=\left(\lim_{\mathbf{u}\to\mathbf{u}_\circ}f\right)\left(\lim_{\mathbf{u}\to\mathbf{u}_\circ}g\right)$,

(b) $\displaystyle\lim_{\mathbf{u}\to\mathbf{u}_\circ}\frac{f}{g}=\frac{\lim_{\mathbf{u}\to\mathbf{u}_\circ}f}{\lim_{\mathbf{u}\to\mathbf{u}_\circ}g}$.

It is again assumed that the limits on the right side of the equality exist. Furthermore, in (b) it must be assumed that $g(\mathbf{u})\neq 0$ for every $\mathbf{u}$ and that $\lim_{\mathbf{u}\to\mathbf{u}_\circ}g\neq 0$. The proofs of (a) and (b) are similar to those of the corresponding properties of functions of one variable found in elementary calculus.

Using Proposition 3.2(a) and the coordinate theorem for limits, the following properties for vector functions can be established.

Proposition 3.3

(a) $\lim_{\mathbf{u}\to\mathbf{u}_0} f\mathbf{g} = (\lim_{\mathbf{u}\to\mathbf{u}_0} f)\ (\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{g})$,

(b) $\lim_{\mathbf{u}\to\mathbf{u}_0} \dfrac{\mathbf{f}}{g} = \dfrac{\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}}{\lim_{\mathbf{u}\to\mathbf{u}_0} g}$,

(c) $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}\cdot\mathbf{g} = (\lim_{\mathbf{u}\to u_0} \mathbf{f})\cdot(\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{g})$,

(d) $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}\times\mathbf{g} = (\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f})\times(\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{g})$.

It is again assumed that the right-side limits exist. A proof of (c) will be made for the case where

$$\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}, \qquad \mathbf{g} = \bar{x}'\mathbf{i} + \bar{y}'\mathbf{j},$$

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x} = a,$$

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y} = b,$$

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x}' = a',$$

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y}' = b'.$$

Then,

$$\begin{aligned}
\lim_{\mathbf{u}\to\mathbf{u}_0} (\mathbf{f}\cdot\mathbf{g}) &= \lim_{\mathbf{u}\to\mathbf{u}_0} (\bar{x}\bar{x}' + \bar{y}\bar{y}') \\
&= \left(\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x}\right)\left(\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{x}'\right) + \left(\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y}\right)\left(\lim_{\mathbf{u}\to\mathbf{u}_0} \bar{y}'\right) \\
&= aa' + bb' \\
&= (a\mathbf{i} + b\mathbf{j})\cdot(a'\mathbf{i} + b'\mathbf{j}) \\
&= \left(\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f}\right)\cdot\left(\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{g}\right).
\end{aligned}$$

The behavior of limits with the composition operation is not so simple. Let it be assumed, that

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} = \mathbf{u}_0' \quad \text{and} \quad \lim_{\mathbf{u}'\to\mathbf{u}_0'} \mathbf{g} = \mathbf{u}_0'',$$

where $\mathbf{g}\circ\mathbf{f}$ is defined (see Figure 9.11). If $\mathbf{u}_1$ is near $\mathbf{u}_0$, then $\lim_{\mathbf{u}\to\mathbf{u}_0} \mathbf{f} = \mathbf{u}_0'$ implies that $\mathbf{f}(\mathbf{u}_1)$ is near $\mathbf{u}_0'$, and hence, $\mathbf{g}\circ\mathbf{f}(\mathbf{u}_1) = \mathbf{g}(\mathbf{f}(\mathbf{u}_1))$ is near $\mathbf{u}_0''$, since

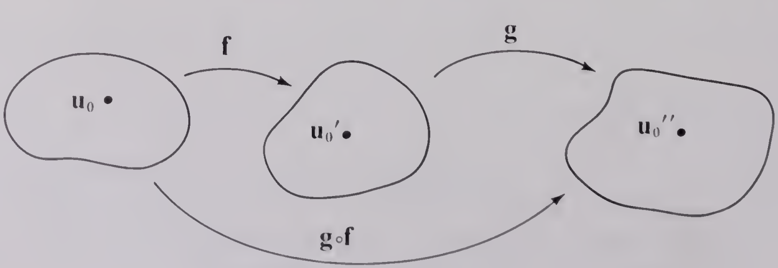


Figure 9.11

$\lim_{\mathbf{u}' \to \mathbf{u}_o'} \mathbf{g} = \mathbf{u}_o''$. Therefore, it would seem plausible that $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{g} \circ \mathbf{f} = \mathbf{u}_o''$. The following example shows this conclusion is erroneous, however, and the reason lies in the inequality condition $\mathbf{u}' \neq \mathbf{u}_o'$ occurring in the definition of $\lim_{\mathbf{u}' \to \mathbf{u}_o'} \mathbf{g}$.

Example 3.3 Let $f(x) = 0$ for all x in **Re** and let $g(x) = 1$ for $x \neq 0$, $g(0) = 0$. Then

$$\lim_{x \to 0} f = 0 \quad \text{and} \quad \lim_{x' \to 0} g = 1;$$

but $\lim_{x \to 0} g \circ f = 0$, since $g \circ f(x) = 0$ for all x.

The foregoing difficulty does not occur if $\lim_{\mathbf{u}' \to \mathbf{u}_o'} \mathbf{g} = \mathbf{g}(\mathbf{u}_o')$, which is later seen to be the condition that $\mathbf{g}$ is *continuous* at $\mathbf{u}_o'$.

Proposition 3.4 If $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$ and $\lim_{\mathbf{u} \to \mathbf{u}_o'} \mathbf{g} = \mathbf{g}(\mathbf{u}_o')$, then $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{g} \circ \mathbf{f} = \mathbf{g}(\mathbf{u}_o')$.

For the proof let $\varepsilon > 0$ and choose $\delta_1 > 0$ and $\delta > 0$ to satisfy the following conditions:

(a) If $|\mathbf{u} - \mathbf{u}_o'| < \delta_1$, then $|\mathbf{g}(\mathbf{u}') - \mathbf{g}(\mathbf{u}_o')| < \varepsilon$.

(b) If $|\mathbf{u} - \mathbf{u}_o| < \delta$ and $\mathbf{u} \neq \mathbf{u}_o$, then $|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_o)| < \delta_1$.

Combining (a) and (b) gives the desired implication.

(c) If $|\mathbf{u} - \mathbf{u}_o| < \delta$ and $\mathbf{u} \neq \mathbf{u}_o$, then $|\mathbf{g} \circ \mathbf{f}(\mathbf{u}) - \mathbf{g} \circ \mathbf{f}(\mathbf{u}_o)| < \varepsilon$.

Questions

1. If $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if ________, then ________.

2. If $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$, then $\mathbf{f}(\mathbf{u}_o)$ ________.
   (a) is always defined,
   (b) $= \mathbf{u}_o'$,
   (c) is undefined,
   (d) might be any range vector.

3. The limit property is not necessarily preserved by the operation of ________.
   (a) addition,
   (b) composition,
   (c) cross product.

4. If $\mathbf{f} = <\bar{x}, \bar{y}>$, $\lim_{\mathbf{u} \to \mathbf{u}_o} \bar{x} = 4$, $\lim_{\mathbf{u} \to \mathbf{u}_o} \bar{y} = 3$, then $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} =$ ________.

Exercises

1. Find, using the properties of limits, the vector given by each expression.
   (a) $\lim_{x \to 2} \langle x, x^2 - 1, 3x + 4 \rangle$,

   (b) $\lim_{\langle x, y \rangle \to \langle 1, 0 \rangle} xe^y\mathbf{i} + \cos y\mathbf{j}$,

   (c) $\lim_{x \to 0} \left\langle \dfrac{\tan x}{x}, x - 3 \right\rangle$ (use L'hopital's rule),

   (d) $\lim_{x \to 0} \left(\left\langle \dfrac{\cos x}{1 - \sin x}, \dfrac{1 - e^x}{x}, 1 \right\rangle \times \left\langle \dfrac{x^2 - x}{x}, e^x, x + 1 \right\rangle \right)$.

Proofs

1. Prove that if $\mathbf{f}(\mathbf{u}) = \mathbf{u}_o'$ for all $\mathbf{u}$, then $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$.

2. Prove that if $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} = \mathbf{u}_o'$ and $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{g} = \mathbf{u}_1'$, then $\lim_{\mathbf{u} \to \mathbf{u}_o} (\mathbf{f} + \mathbf{g}) = \mathbf{u}_o' + \mathbf{u}_1'$. (*Hint*: For $\varepsilon > 0$ choose δ_1 and δ_2 so that
   (a) if $|\mathbf{u} - \mathbf{u}_o| < \delta_1$ $(\mathbf{u} \neq \mathbf{u}_o)$, then $|\mathbf{f}(\mathbf{u}) - \mathbf{u}_o'| < \varepsilon/2$,
   (b) if $|\mathbf{u} - \mathbf{u}_o| < \delta_2$ $(\mathbf{u} \neq \mathbf{u}_o)$, then $|\mathbf{g}(\mathbf{u}) - \mathbf{u}_1'| < \varepsilon/2$.
   Show $\delta =$ the smaller of δ_1, δ_2 satisfies the definition of $\lim_{\mathbf{u} \to \mathbf{u}_o} (\mathbf{f} + \mathbf{g}) = \mathbf{u}_o' + \mathbf{u}_1'$.)

3. Given $\lim_{\mathbf{u} \to \mathbf{u}_o} f = a$ and $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{g} = \langle b_1, b_2 \rangle$ where $\mathbf{g} = \langle g_1, g_2 \rangle$, prove $\lim_{\mathbf{u} \to \mathbf{u}_o} f\mathbf{g} = a\langle b_1, b_2 \rangle$ using the coordinate theorem for limits.

4. Open Sets

In first-year calculus the function domains most often used are intervals. Open intervals are more convenient for the study of derivatives than the intervals which include one or both end points. The next few chapters will deal with extensions of the notion of the derivative to functions of several variables. In these extensions the simplest and most meaningful analysis results when the domains are restricted to *open sets*. Loosely stated, a set $\mathbf{U}$ is open in $\mathbf{R}^n$ if, for each $\mathbf{u}_o$ in $\mathbf{U}$, all vectors in $\mathbf{R}^n$ sufficiently near $\mathbf{u}_o$ are also in $\mathbf{U}$. Otherwise stated, each vector in $\mathbf{U}$ can be moved a short distance without leaving $\mathbf{U}$. Open sets may be geometrically interpreted as those which do not include any boundary points. They possess the property that everywhere in the set their dimension is locally the same as that of the embedding space.

Definition of Open Set

A set $\mathbf{U}$ *in* $\mathbf{R}^n$ *is* open *if and only if for each* $\mathbf{u}$ *in* $\mathbf{U}$ *there exists a number* $\delta > 0$ *such that if* $\mathbf{v}$ *is in* $\mathbf{R}^n$ *and* $|\mathbf{v} - \mathbf{u}| < \delta$, *then* $\mathbf{v}$ *is in* $\mathbf{U}$.

The following example shows that this definition is consistent with the prior usage of "open" when describing intervals.

Example 4.1 (a) The interval (a, b) is an open set in **Re**. If $a < c < b$, then choosing δ as the smaller of the two numbers, $c - a$ and $b - c$, gives a value which satisfies the definition of open.
(b) The half-closed interval [a, b) is not open in **Re**, since no $\delta > 0$ can be selected for a to satisfy the definition of open. This may be seen by noting that if $x = a - \delta/2$, then $|x - a| < \delta$ and x is not in $[a, b)$.

It should be recognized that the open property of a set depends not only on the character of the set, but also on the embedding space. If a set $\mathbf{U}$ is open in $\mathbf{R}^n$, then near each $\mathbf{u}$ in $\mathbf{U}$ the points of $\mathbf{U}$ form a set having dimension n.

Example 4.2 If $\mathbf{U} = \{\langle x, 0\rangle : 0 < x < 1\}$, then the graph of $\mathbf{U}$ is a segment, without end points, in the Cartesian plane (see Figure 9.12). However in the plane $\mathbf{U}$ is not open, for given $\mathbf{u} = \langle c, 0\rangle$ in $\mathbf{U}$ and $\delta > 0$, then $\mathbf{v} = \langle c, \delta/2\rangle$ satisfies $|\mathbf{v} - \mathbf{u}| = \delta/2 < \delta$, whereas $\mathbf{v}$ is not in $\mathbf{U}$.

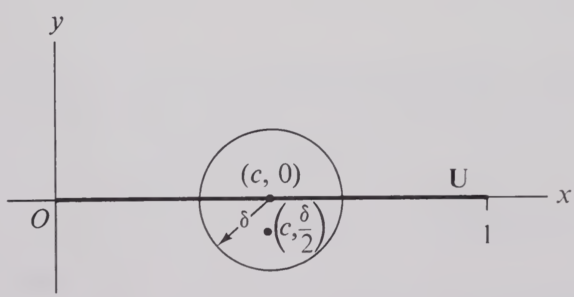


Figure 9.12

The open property may be described in terms of spheres. If $\mathbf{v}_o$ is in $\mathbf{R}^n$, then the *open sphere* of radius $\delta > 0$ centered at $\mathbf{v}_o$ is defined by the equation

$$\sigma[\mathbf{v}_o : \delta] = \{\mathbf{v} : |\mathbf{v} - \mathbf{v}_o| < \delta\}.$$

The set **U** is open provided each vector in **U** is the center of some open sphere lying entirely within **U**.

> **Example 4.3** (a) In **Re**, identified with $\mathbf{R}^1$, the inequality $|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$. Hence $\sigma[a:\delta]$ is the open interval $(a-\delta, a+\delta)$.
> (b) In $\mathbf{R}^2$, $|\mathbf{v}-\mathbf{v}_0|<\delta$ provided $\mathbf{v}$ is inside the circle of radius δ centered at $\mathbf{v}_0$. Thus, the graph of an open sphere in $\mathbf{R}^2$ is a circular disc which does not include its boundary.
> (c) In $\mathbf{R}^3$ the graph of an open sphere is a solid sphere which does not include its outer shell.

We can associate with each set in $\mathbf{R}^n$ the (possibly empty) subset which comprises the *open part* of the set. The open part of **U** means the set of vectors in **U** which are the center of some open sphere contained within **U**. Formally, $\mathbf{u}_0$ is an *interior vector* of **U** if there exists $\delta>0$ such that $\sigma[\mathbf{u}_0:\delta]$ is a subset of **U**. The *interior* of **U** consists of all interior vectors of **U**. It is an immediate consequence that **U** is open if and only if each vector in **U** is an interior vector of **U**. Equivalently, **U** is open if it equals its interior.

> **Example 4.4** (a) Let **U** be the closed interval $[a, b]$; then each number in **U**, except a and b, is in the interior of **U**.
> (b) Let $\mathbf{U}=\{\langle x,y\rangle: x^2+y^2\leq 1\}$. The graph of **U** is the circle of radius 1 centered at the origin, together with all points inside the circle. A vector $\langle x, y\rangle$ is interior to **U** if $x^2+y^2<1$.

Questions

1. **U** is open in $\mathbf{R}^n$ provided that ______.
   (a) **U** is not closed,
   (b) each vector in **U** is an interior vector,
   (c) if **u**, **v** are in **U**, then $|\mathbf{v}-\mathbf{u}|<\delta$ for some $\delta>0$.
2. A segment without end points ______ an open set.
   (a) is,
   (b) is not,
   (c) is in some cases and is not in other cases.
3. The graph of an open sphere in $\mathbf{R}^2$ is ______.
   (a) an open interval,
   (b) a circle,
   (c) a disc.
4. The open part of a set is called its ______.

Exercises

1. Sketch each of the following subsets of $\mathbf{R}^2$, specify its interior, and determine whether it is open.
   (a) $\{\langle x, y\rangle : x^2 + y^2 < 1\}$,
   (b) $\{\langle x, y\rangle : x + y \leq 1\}$,
   (c) $\{\langle x, y\rangle : x = 0 \text{ or } y = 0\}$,
   (d) $\{\langle x, y\rangle : x > 0\}$,
   (e) $\{\langle x, y\rangle : y \leq 0\}$,
   (f) $\{\langle x, y\rangle : x < y\}$.

2. Given $\mathbf{U} = \{\langle x, y\rangle : x^2 + y^2 < 1\}$, find for each of the following vectors, $\mathbf{u}_o$, those numbers $\delta > 0$ such that $\sigma[\mathbf{u}_o : \delta]$ is contained within $\mathbf{U}$.
   (a) $\mathbf{u}_o = \langle 0, 1/2\rangle$, (b) $\mathbf{u}_o = \langle 1/3, 5/6\rangle$,
   (c) $\mathbf{u}_o = \langle r\cos\theta, r\sin\theta\rangle$, $0 < r < 1$.

Proofs

1. Prove that a 1-plane in $\mathbf{R}^2$ cannot be an open set.

2. Prove in $\mathbf{R}^2$ that $\sigma[\langle 0, 0\rangle : \delta]$ is an open set, where $\delta > 0$ is arbitrary. (*Hint*: For $\mathbf{u}_o$ in $\sigma[\langle 0, 0\rangle : \delta]$ let $\delta_1 = \delta - |\mathbf{u}_o|$, and show by the triangle inequality that $\sigma[\mathbf{u}_o : \delta_1]$ is a subset of $\sigma[\langle 0, 0\rangle : \delta]$.)

5. Continuity

A fundamental function property in calculus is *continuity*. The property of continuity means, for a function which has a graph, that the graph has no jumps.

Definition of Continuity

Let $\mathbf{f}$ *be a vector function with domain* $\mathbf{U}$ *in* $\mathbf{R}^n$. *If* $\mathbf{u}_o$ *is in* $\mathbf{U}$, *then* $\mathbf{f}$ *is* continuous *at* $\mathbf{u}_o$ *provided* $\lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f} = \mathbf{f}(\mathbf{u}_o)$.

From the definitions of limit and sphere, there may be obtained alternate formulations of the continuity property.

Proposition 5.1 The following statements are equivalent.

(a) $\mathbf{f}$ is continuous at $\mathbf{u}_o$.
(b) For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta$, then $|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_o)| < \varepsilon$.
(c) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbf{f}(\sigma[\mathbf{u}_o : \delta])$ is a subset of $\sigma[\mathbf{f}(\mathbf{u}_o) : \varepsilon]$.

From the properties of limit, it can be seen that continuity is preserved by the various operations on scalar and vector functions. For example, if $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{u}_o$, then $\mathbf{f} + \mathbf{g}$ is shown to be continuous at $\mathbf{u}_o$ by the following equality chain.

$$\begin{aligned}\lim_{\mathbf{u}\to\mathbf{u}_o} (\mathbf{f} + \mathbf{g}) &= \lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{f} + \lim_{\mathbf{u}\to\mathbf{u}_o} \mathbf{g} \\ &= \mathbf{f}(\mathbf{u}_o) + \mathbf{g}(\mathbf{u}_o) \\ &= (\mathbf{f} + \mathbf{g})(\mathbf{u}_o).\end{aligned}$$

Although continuity has been defined at a point, we shall usually be more interested in the global property of continuity. A function $\mathbf{f}$ is *continuous* if it is continuous at each element of its domain. The preservation of continuity by the various operations will now be formalized.

Proposition 5.2 If f, g, $\mathbf{f}$, and $\mathbf{g}$ are continuous, then so are

(a) $\mathbf{f} + \mathbf{g}$, (b) $c\mathbf{f}$, (c) fg,
(d) f/g, (e) $f\mathbf{g}$, (f) $\mathbf{f}/g$,
(g) $\mathbf{f} \cdot \mathbf{g}$, (h) $\mathbf{f} \times \mathbf{g}$, (i) $\mathbf{g} \circ \mathbf{f}$.

It is of course assumed that the denominators in (d) and (f) are never zero.

The continuity of various particular functions will now be investigated. From first-year calculus the following functions are known to be continuous on their domains:

(a) *Rational functions* (of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials),
(b) *Trigonometric functions* ($\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, $\cot x$),
(c) *Logarithmic functions* ($\log_a x$, where $a > 1$),
(d) *Exponential functions* (a^x where $a > 0$).

Corresponding conclusions for functions of several variables will now be obtained. We first prove the continuity of the scalar function f with domain $\mathbf{R}^2$ and rule $f(x, y) = x$. This function is called the *first coordinate function on* $\mathbf{R}^2$.

In order to prove the continuity of $\mathbf{f}$ at $\mathbf{u}_o = \langle x_o, y_o \rangle$, we let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $\mathbf{u} = \langle x, y \rangle$ satisfies the condition $|\mathbf{u} - \mathbf{u}_o| < \delta = \varepsilon$, then

$$\begin{aligned} |f(\mathbf{u}) - f(\mathbf{u}_o)| &= |f(x, y) - f(x_o, y_o)| \\ &= |x - x_o| \\ &\leq \sqrt{(x - x_o)^2 + (y - y_o)^2} \\ &= |\mathbf{u} - \mathbf{u}_o| \\ &< \varepsilon. \end{aligned}$$

Similarly, the *second coordinate function* $f(x, y) = y$ on $\mathbf{R}^2$ is also continuous. The following example shows the manner in which various particular functions of several variables can be shown to be continuous.

Example 5.1 Given $f(r, s) = \sin rs$, then $f = h \circ g$ where $h(x) = \sin x$ and g is the product of the coordinate functions $g_1(r, s) = r$ and $g_2(r, s) = s$ on $\mathbf{R}^2$. Since the product and composition operations preserve continuity, it follows that f is continuous.

More generally, the ith *coordinate function* on $\mathbf{R}^m$ is the scalar function with domain $\mathbf{R}^m$ and rule

$$f(x_1, \ldots, x_m) = x_i.$$

Each of these coordinate functions is continuous, and, hence, all scalar functions constructed from these and other continuous functions of one variable, using the operations of addition, multiplication, subtraction, division, and composition, are also continuous (since these operations preserve continuity).

Questions

1. If $\mathbf{f}$ is continuous at $\mathbf{u}_o$, then $\lim_{\mathbf{u} \to \mathbf{u}_o} \mathbf{f} =$ ________.
2. A function is continuous provided that it is continuous at ________ element of its domain.
   (a) some,
   (b) every,
   (c) every interior.
3. If $\mathbf{f} = \langle \bar{x}, \bar{y} \rangle$ and $\bar{x}$ is continuous, then $\mathbf{f}$ ________ be continuous.
   (a) must,
   (b) cannot,
   (c) may or may not.

Exercises

1. Construct the following functions from real functions of one variable and coordinate functions.
(a) $f(r, s) = \cos(rs^2)$, (b) $f(r, s) = re^s$,

(c) $f(r, s, t) = \dfrac{r}{s \ln t}$.

Proofs

1. Prove that if $\mathbf{f} = \langle \bar{f}_1, \bar{f}_2 \rangle$ and $\mathbf{g} = \langle \bar{g}_1, \bar{g}_2 \rangle$ are continuous, then so are (a) $\mathbf{f} \cdot \mathbf{g}$ and (b) $\mathbf{g} \circ \mathbf{f}$.

Problems

A. *Operations on Functions*

A *scalar function* is a function from a subset of $\mathbf{R}^n$ to the reals, where n is a positive integer. Scalar functions may be symbolized as shown below.

$$\begin{aligned} &f(x), && n = 1,\\ &f(x, y), && n = 2,\\ &f(x, y, z), && n = 3. \end{aligned}$$

A *vector function* is a function from a subset of $\mathbf{R}^n$ to $\mathbf{R}^m$, where m and n are positive integers. These may be symbolized in the following way.

$$\begin{aligned} \mathbf{f}(x) &= \bar{f}_1(x)\mathbf{i} + \bar{f}_2(x)\mathbf{j}, && n = 1, m = 2;\\ \mathbf{f}(x) &= \bar{f}_1(x)\mathbf{i} + \bar{f}_2(x)\mathbf{j} + \bar{f}_3(x)\mathbf{k}, && n = 1, m = 3;\\ \mathbf{f}(x, y) &= \bar{f}_1(x, y)\mathbf{i} + \bar{f}_2(x, y)\mathbf{j}, && n = 2, m = 2;\\ \mathbf{f}(x, y) &= \bar{f}_1(x, y)\mathbf{i} + \bar{f}_2(x, y)\mathbf{j} + \bar{f}_3(x, y)\mathbf{k}, && n = 2, m = 3;\\ \mathbf{f}(x, y, z) &= \bar{f}_1(x, y, z)\mathbf{i} + \bar{f}_2(x, y, z)\mathbf{j}, && n = 3, m = 2;\\ \mathbf{f}(x, y, z) &= \bar{f}_1(x, y, z)\mathbf{i} + \bar{f}_2(x, y, z)\mathbf{j} + \bar{f}_3(x, y, z)\mathbf{k}, && n = 3, m = 3. \end{aligned}$$

The scalar functions $\bar{f}_1, \bar{f}_2$, and $\bar{f}_3$ are called the *coordinate* functions of **f**. Various operations on these functions will now be considered. It will be assumed that domains and ranges are suitable to make the defined rules meaningful.

A.1 Sum of Vector Functions

(a) $(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j}) + (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j}) = (\bar{f}_1 + \bar{g}_1)\mathbf{i} + (\bar{f}_2 + \bar{g}_2)\mathbf{j}$,
(b) $(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) + (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}) =$
$(\bar{f}_1 + \bar{g}_1)\mathbf{i} + (\bar{f}_2 + \bar{g}_2)\mathbf{j} + (\bar{f}_3 + \bar{g}_3)\mathbf{k}$.

A.2 Scalar Multiplication of Vector Functions

(a) $c(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j}) = (c\bar{f}_1)\mathbf{i} + (c\bar{f}_2)\mathbf{j}$,
(b) $c(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) = (c\bar{f}_1)\mathbf{i} + (c\bar{f}_2)\mathbf{j} + (c\bar{f}_3)\mathbf{k}$.

1. Given that $\mathbf{f} = xy\mathbf{i} + xz\mathbf{j}$, $\mathbf{g} = xz\mathbf{i} - yz\mathbf{j}$, perform the indicated operations.

   (a) $\mathbf{f} + \mathbf{g}$, (b) $3\mathbf{f}$,
   (c) $\mathbf{f} + 2\mathbf{g}$, (d) $4\mathbf{f} - \mathbf{g}\ (= 4\mathbf{f} + (-1)\mathbf{g})$.

2. Given $\mathbf{f} = xy\mathbf{i} - yz\mathbf{j} + x^2\mathbf{k}$, $\mathbf{g} = x\mathbf{i} + yz\mathbf{j} - y^2\mathbf{k}$, perform the indicated operations.

   (a) $\mathbf{f} + \mathbf{g}$, (b) $3\mathbf{f} + 2\mathbf{g}$, (c) $2\mathbf{f} - \mathbf{g}$.

The defining rules for three additional operations are given next.

A.3 Product of Scalar Function and Vector Function

(a) $f(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j}) = f\bar{f}_1\mathbf{i} + f\bar{f}_2\mathbf{j}$,
(b) $f(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) = (f\bar{f}_1)\mathbf{i} + (f\bar{f}_2)\mathbf{j} + (f\bar{f}_3)\mathbf{k}$.

A.4 Dot Product of Vector Functions

(a) $(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j}) \cdot (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j}) = \bar{f}_1\bar{g}_1 + \bar{f}_2\bar{g}_2$,
(b) $(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) \cdot (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}) = \bar{f}_1\bar{g}_1 + \bar{f}_2\bar{g}_2 + \bar{f}_3\bar{g}_3$.

A.5 Cross Product of Vector Functions

$$(\bar{f}_1\mathbf{i} + \bar{f}_2\mathbf{j} + \bar{f}_3\mathbf{k}) \times (\bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}) =$$
$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \\ \bar{g}_1 & \bar{g}_2 & \bar{g}_3 \end{bmatrix}.$$

The determinant in A.5 is evaluated in the same manner as in finding the cross product of 3-tuples.

3. Given that $f = xy$, $\mathbf{f} = x^2\mathbf{i} + xy\mathbf{j}$, and $\mathbf{g} = y^2\mathbf{i} - xy\mathbf{j}$, find

   (a) $f\mathbf{f}$, (b) $(3f)\mathbf{g}$, (c) $\mathbf{f} \cdot \mathbf{g}$, (d) $(f\mathbf{f}) \cdot \mathbf{g}$.

4. Given $f = xy$, $\mathbf{f} = x\mathbf{i} - y\mathbf{j} + xz\mathbf{k}$, $\mathbf{g} = y\mathbf{i} - x\mathbf{j} + yz\mathbf{k}$, find

   (a) $f\mathbf{f}$, (b) $\mathbf{f} \cdot \mathbf{g}$, (c) $\mathbf{f} \times \mathbf{g}$, (d) $\mathbf{f} \cdot (f\mathbf{g})$,
   (e) $(f\mathbf{g}) \times \mathbf{f}$, (f) $\mathbf{f} \times \mathbf{f}$.

The *composition* operation is achieved by substitution. For $\mathbf{g} \circ \mathbf{f}$, we let r, s, and t denote the domain coordinate variables of $\mathbf{f}$ and x, y, z the domain coordinate variables of $\mathbf{g}$. The definition of $\mathbf{g} \circ \mathbf{f}$ implies that the number of coordinate functions of $\mathbf{f}$ equals the number of domain coordinate variables of $\mathbf{g}$. It is convenient to let $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the coordinate functions of $\mathbf{f}$.

A.6 The composition $\mathbf{g} \circ \mathbf{f}$ is achieved by substitution of each coordinate function of $\mathbf{f}$ for the corresponding domain coordinate variable of $\mathbf{g}$.

5. Let

$$\mathbf{f} = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} = (r - s)\mathbf{i} + rs\mathbf{j},$$

and

$$\mathbf{g} = x^2y^2\mathbf{i} + x\mathbf{j} - y\mathbf{k}.$$

Find $\mathbf{g} \circ \mathbf{f}$ by substituting $\bar{x}(r, s)$ for x and $\bar{y}(r, s)$ for y in the formula for $\mathbf{g}$.

6. Find $\mathbf{g} \circ \mathbf{f}$ (or $g \circ \mathbf{f}$).

(a) $\mathbf{f} = r^2s\mathbf{i} + (r - s)\mathbf{j}$; $\mathbf{g} = (x - y)\mathbf{i} + xy\mathbf{j}$,
(b) $\mathbf{f} = r\mathbf{i} + r^2\mathbf{j}$; $\mathbf{g} = (x - y)\mathbf{i} + x \ln y\, \mathbf{j}$,
(c) $\mathbf{f} = rs\mathbf{i} + (s - r)\mathbf{j} + r^2\mathbf{k}$; $g = xy \cos z$,
(d) $\mathbf{f} = rst\mathbf{i} - r^2\mathbf{j}$; $\mathbf{g} = x^2y\mathbf{i} - e^x\mathbf{j}$.

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7. Given $f = xy^2$, $\mathbf{f} = x\mathbf{i} - y\mathbf{j}$, $\mathbf{g} = x^2\mathbf{i} + y^2\mathbf{j}$, find

(a) $f(\mathbf{f} + \mathbf{g})$, (b) $\mathbf{f} \cdot (f\mathbf{g})$, (c) $3\mathbf{f} \cdot (2\mathbf{g})$, (d) $\mathbf{f} \cdot (\mathbf{f} - \mathbf{g})$.

8. Given $\mathbf{f} = x\mathbf{i} + x^2\mathbf{j} + y\mathbf{k}$ and $\mathbf{g} = y\mathbf{i} - y^2\mathbf{j} + x^2\mathbf{k}$, find

(a) $\mathbf{f} \cdot (3\mathbf{g})$, (b) $(2\mathbf{f}) \times \mathbf{g}$.

9. Find $\mathbf{g} \circ \mathbf{f}$ from the following information.

(a) $\mathbf{f}(r, s) = r^2\mathbf{i} - rs\mathbf{j}$; $\mathbf{g}(x, y) = x\mathbf{i} + y^2\mathbf{j}$,
(b) $\mathbf{f}(r, s) = rs\mathbf{i} + e^s\mathbf{j} - s\mathbf{k}$; $\mathbf{g}(x, y, z) = x^2z\mathbf{i} - yz\mathbf{j}$.

B. *Curves in the Cartesian Plane*

A plane curve may be described in the following four ways.

B.1 (a) *Explicit equation*: $y = f(x)$ or $x = f(y)$.
(b) *Implicit equation*: $g(x, y) = a$.
(c) *Parametric equations*: $x = \bar{x}(r)$, $y = \bar{y}(r)$.
(d) *Vector equation*: $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$.

The explicit-equation form is essentially unique whereas the other forms are not, although for some curves the other forms have unique standard choices. The explicit form is restricted to a smaller class of curves than the others; when it exists, the other forms can be obtained in the following way.

B.2 (a) Explicit equation: $y = f(x)$.
(b) Implicit equation: $y - f(x) = 0$.
(c) (Standard) parametric equations: $x = r$, $y = f(r)$.
(d) (Standard) vector equation: $\mathbf{f}(r) = r\mathbf{i} + f(r)\mathbf{j}$.

Other similar forms can be obtained for the explicitly defined curve $x = f(y)$.

1. Given $y = x^3 - 3x + 2$, find
   (a) an implicit equation,
   (b) the standard parametric equations, and
   (c) the standard vector equation.
2. Repeat Problem 1 for the curve $x = 4y^2 - 2y + 7$.

Equations of the conics centered at the origin and with axes parallel to the coordinate axes will now be considered.

B.3 The circle of radius a, centered at the origin, has equation forms as shown below.

(a) $y = \sqrt{a^2 - x^2}$, $y = -\sqrt{a^2 - x^2}$, $x = \sqrt{a^2 - y^2}$, $x = -\sqrt{a^2 - y^2}$;
(b) $x^2 + y^2 = a^2$;
(c) $x = a \cos \theta$, $y = a \sin \theta$;
(d) $\mathbf{f}(\theta) = a \cos \theta\mathbf{i} + a \sin \theta\mathbf{j}$.

The particular explicit equation is chosen, when possible, to describe a suitable portion of the circle.

3. Given the circle $x^2 + y^2 = 4$, write
   (a) parametric equations,
   (b) a vector equation,
   (c) an explicit equation for the portion of the circle above the x axis, and
   (d) an explicit equation for the portion of the circle to the left of the y axis.

B.4 The ellipse centered at the origin and having radius a in the x direction and b in the y direction has the following equation forms:

(a) $y = b/a\sqrt{a^2 - x^2}$, $y = -b/a\sqrt{a^2 - x^2}$, $x = a/b\sqrt{b^2 - y^2}$,
$x = -a/b\sqrt{b^2 - y^2}$;
(b) $x^2/a^2 + y^2/b^2 = 1$;
(c) $x = a \cos \theta$, $y = b \sin \theta$;
(d) $\mathbf{f}(\theta) = a \cos \theta\, \mathbf{i} + b \sin \theta\, \mathbf{j}$.

4. Given the ellipse $x^2/4 + y^2/9 = 1$, find
   (a) parametric equations,
   (b) a vector equation, and
   (c) an explicit equation for the portion of the ellipse to the right of the y axis.

For the study of the hyperbola, it is convenient to introduce the *hyperbolic functions*, which are defined by

$$\sinh r = \frac{e^r - e^{-r}}{2}$$

and

$$\cosh r = \frac{e^r + e^{-r}}{2}.$$

It may be verified by direct computation that, for all numbers r;

$$\cosh^2 r - \sinh^2 r = 1.$$

This equality implies that $x = a \cosh r$ and $y = b \sinh r$ satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where it is assumed that $a > 0$, $b > 0$. These parametric equations give only the portion of the hyperbola to the right of the y axis, since $\cosh r > 0$ for all r. The portion to the left of the y axis is given by $x = -a \cosh r$, $y = b \sinh r$.

B.5 The hyperbola in Figure 9.13 has the equation forms given below.

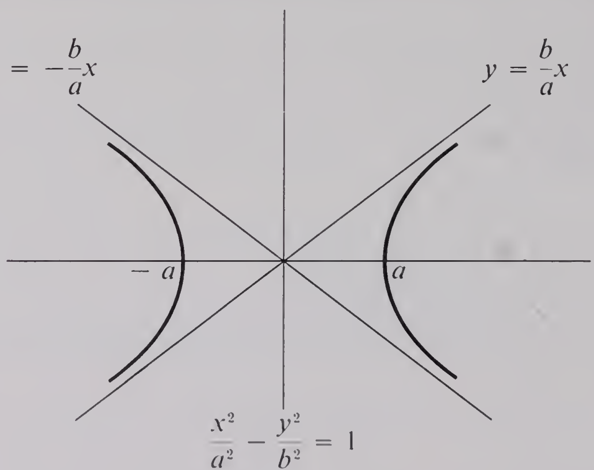

Figure 9.13

(a) $y = b/a\sqrt{x^2 - a^2}$, $y = -b/a\sqrt{x^2 - a^2}$, $x = a/b\sqrt{y^2 + b^2}$, $x = -a/b\sqrt{y^2 + b^2}$;
(b) $x^2/a^2 - y^2/b^2 = 1$;
(c) $x = a \cosh r$, $y = b \sinh r$; $x = -a \cosh r$, $y = b \sinh r$;
(d) $\mathbf{f}(r) = a \cosh r\, \mathbf{i} + b \sinh r\, \mathbf{j}$; $\mathbf{f}(r) = -a \cosh r\, \mathbf{i} + b \sinh r\, \mathbf{j}$.

Similar equations hold for the hyperbola $y^2/b^2 - x^2/a^2 = 1$.

5. For the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

give parametric equations, a vector equation, and an explicit equation of the portion lying in (a) the first quadrant, (b) the second quadrant, (c) the third quadrant.

6. Repeat Problem 5 for $y^2/16 - x^2/9 = 1$.

Parabolas have explicit equations of the form $y = ax^2$, $x = ay^2$, from which the other forms can be obtained in a standard way using B.2.

7. Given the parabola $y = 3x^2$, find

   (a) an implicit equation,
   (b) parametric equations, and
   (c) a vector equation.

8. Repeat Problem 7 for the parabola $x = -5y^2$.

If the explicitly described curve $y = f(x)$ is shifted c units to the right and d units up, then a point (x_o, y_o) on the curve is shifted to the point $(x_o', y_o') = (x_o + c, y_o + d)$. From the equality chain,

$$y_o' - d = y_o = f(x_o) = f(x_o' - c),$$

it is seen that an equation of the translated curve can be obtained from $y = f(x)$ by replacing x by $x - c$ and y by $y - d$. A similar analysis can be made for other equation forms of curves. Shifting to the left (or down) corresponds to a negative value of c (or d). Shifting a curve c units to the right and d units up produces the following equation changes.

B.6 (a) The explicit equation $y = f(x)$ is changed to $y = d + f(x - c)$; the explicit equation $x = f(y)$ is changed to $x = c + f(y - d)$.
(b) The implicit equation $h(x, y) = a$ is changed to $h(x - c, y - d) = a$.
(c) The parametric equations $x = \bar{x}(r)$ and $y = \bar{y}(r)$ are changed to $x = \bar{x}(r) + c$, $y = \bar{y}(r) + d$.
(d) The vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ is changed to

$$\mathbf{f}(r) = (\bar{x}(r) + c)\mathbf{i} + (\bar{y}(r) + d)\mathbf{j}.$$

9. Give an explicit equation of the parabola

   (a) $y = x^2 + 3x - 2$, shifted 3 units to the right and 2 units down;
   (b) $x = 4y^2$, shifted 2 units to the left and 1 unit up.

10. Give an implicit equation of

   (a) the circle, $x^2 + y^2 = 4$, shifted 2 units to the right and 3 units up;
   (b) the ellipse, $x^2/4 + y^2/9 = 1$, shifted 3 units to the left; and
   (c) the hyperbola, $x^2/4 - y^2/9 = 1$, shifted 1 unit to the right and 2 units down.

11. Give a vector equation for the curves in Problem 10. For (c) use the right half of the hyperbola.

12. Find a corresponding vector equation for each of the following curves.
    (a) $(y-3)=4(x-2)^2$,
    (b) $(x-4)^2+(y-1)^2=16$,
    (c) $(x+1)^2/4+(y-3)^2/9=1$,
    (d) $x^2/1-(y-5)^2/4=1$ (use the left half of the hyperbola).

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13. Give a vector equation for each of the following forms.
    (a) $y=11x^3-4x$, (b) $x=e^{3y}$,
    (c) $x^2+(y-2)^2=9$, (d) $(x-1)^2/4+y^2/1=1$,
    (e) $y=2+3(x-1)^2$, (f) $x=4(y+1)^2$,
    (g) $(x-1)^2/4-y^2/9=1$ (use the right half of the hyperbola),
    (h) $(y+2)^2/9-x^2/16=1$ (use the lower half of the hyperbola).
14. Write an explicit equation for each of the forms below.
    (a) $3\cos\theta\,\mathbf{i}+(2\sin\theta-1)\mathbf{j}$, $0\le\theta\le\pi$;
    (b) $(-3+r^2)\mathbf{i}+(-1+r)\mathbf{j}$; (c) $(r+3)\mathbf{i}+\ln r\,\mathbf{j}$.
15. Give an implicit equation.
    (a) $(\cos\theta-3)\mathbf{i}+(2\sin\theta+1)\mathbf{j}$,
    (b) $(2\cosh r+1)\mathbf{i}+(3\sinh r-1)\mathbf{j}$.

C. *Surfaces in Cartesian Space*

Surfaces are described in much the same manner as are plane curves; the main difference is the involvement of an additional variable. There are the following forms.

C.1 (a) *Explicit equation*: $z=f(x,y)$, $y=f(x,z)$, $x=f(y,z)$.
(b) *Implicit equation*: $h(x,y,z)=a$.
(c) *Parametric equations*: $x=\bar{x}(r,s)$, $y=\bar{y}(r,s)$, $z=\bar{z}(r,s)$.
(d) *Vector equation*: $\mathbf{f}(r,s)=\bar{x}(r,s)\mathbf{i}+\bar{y}(r,s)\mathbf{j}+\bar{z}(r,s)\mathbf{k}$.

The surface defined explicitly by $z=f(x,y)$ has the following equation forms.

C.2 (a) Explicit equation $z=f(x,y)$.
(b) Implicit equation: $z-f(x,y)=0$.
(c) (Standard) parametric equations: $x=r$, $y=s$, $z=f(r,s)$.
(d) (Standard) vector equation: $\mathbf{f}(r,s)=r\mathbf{i}+s\mathbf{j}+f(r,s)\mathbf{k}$.

Similar descriptions hold for $y=f(x,z)$ and $x=f(y,z)$.

1. Find the standard vector equation of the surfaces,
   (a) $z=x^2e^y$, (b) $x=y^2\cos yz$.
2. Find an explicit equation of the portion of the surface $3x^2+x-y^2+z^2=0$, lying above the xy plane.

We shall now study two important classes of surfaces,

(a) surfaces of revolution, and
(b) quadric surfaces.

A *surface of revolution* is obtained by revolving a curve about a line. We shall consider only the case in which the curve is in a coordinate plane and the line is a coordinate axis. First, let $z = f(y)$ be a curve in the yz plane, where $f(y) \geq 0$ for each y. If the point $(0, y_o, f(y_o))$ is rotated about the y axis through an angle ϕ (see Figure 9.14), then the resulting point is $(x, y, z) = (f(y_o) \sin \phi, y_o, f(y_o) \cos \phi)$. From the expression,

$$x^2 + z^2 = [f(y_o) \sin \phi]^2 + [f(y_o) \cos \phi]^2 = f(y_o)^2,$$

we can conclude the following result.

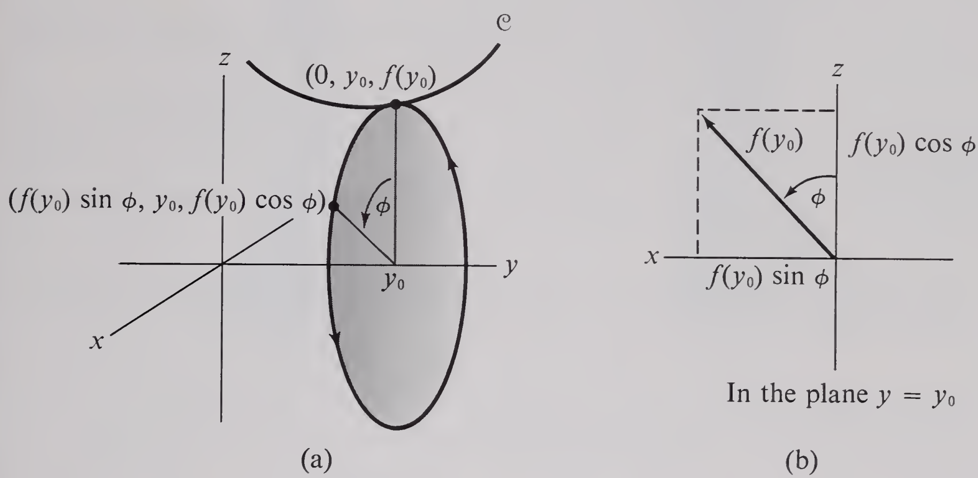


Figure 9.14

C.3 The surface obtained by revolving $z = f(y)$, $f(y) \geq 0$, about the y axis has equation forms as shown below.

(a) Implicit: $x^2 + z^2 - f(y)^2 = 0$, and
(b) Vector: $\mathbf{f}(y, \phi) = f(y) \sin \phi\, \mathbf{i} + y\mathbf{j} + f(y) \cos \phi\, \mathbf{k}$.

Interchanging $\sin \phi$ and $\cos \phi$ in (b) would give another vector equation of the surface. We next consider the same curve, $z = f(y)$, this time with the condition $y \geq 0$, revolved about the z axis. Rotating the point $(0, y_o, f(y_o))$ through an angle ϕ about the z axis results in the point $(x, y, z) = (y_o \sin \phi, y_o \cos \phi, f(y_o))$. From $z = f(y_o)$ and $\sqrt{x^2 + y^2} = y_o$, we can conclude the following facts.

C.4 The surface obtained by revolving $z = f(y)$, $y \geq 0$, about the z axis has equation forms as shown.

(a) Explicit: $z = f(\sqrt{x^2 + y^2})$;
(b) Vector: $\mathbf{f}(y, \phi) = y \sin \phi\, \mathbf{i} + y \cos \phi\, \mathbf{j} + f(y)\mathbf{k}$.

If the curves in other coordinate planes are rotated about a coordinate axis, then similar equations to those in C.3 and C.4 can be found. It is recommended that the

procedure used to derive the formulas, rather than the formulas themselves, be learned.

3. For the surface obtained by revolving $z = y^2$ about the y axis, find (a) a vector and (b) an implicit equation.
4. For the surface obtained by revolving $z = y^2$ about the z axis, find (a) a vector and (b) an explicit equation.
5. (a) Find the point obtained by revolving $(x_o, x_o^2, 0)$, $x_o > 0$, through an angle ϕ about the y axis.
   (b) From (a) find a vector equation of the surface obtained by revolving $y = x^2$ about the y axis.
   (c) Find an explicit equation of the surface described in (b).
6. (a) Find the point obtained by revolving $(x_o, x_o^2, 0)$ through an angle ϕ about the x axis.
   (b) From (a) find a vector equation of the surface obtained by revolving $y = x^2$ about the x axis.
   (c) Find an implicit equation of the surface described in (b).
7. Find a vector equation of the conical surface obtained by revolving the line $z = 3y$, $y > 0$ about the z axis.
8. Find a vector equation of the cylinder obtained by revolving the line $x = 4$ in the xz plane about the z axis.

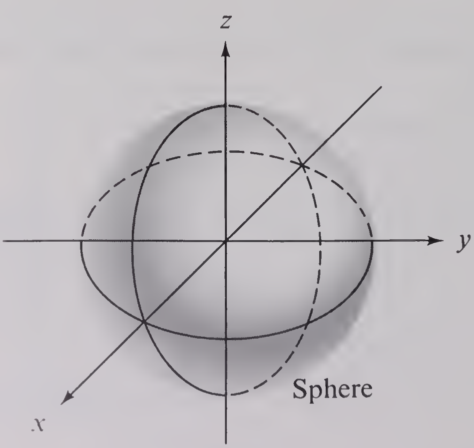
The analogue of the conic in the Cartesian plane is the *quadric surface* in space. A general implicit equation for axes parallel to the coordinate axes is

$$ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0.$$

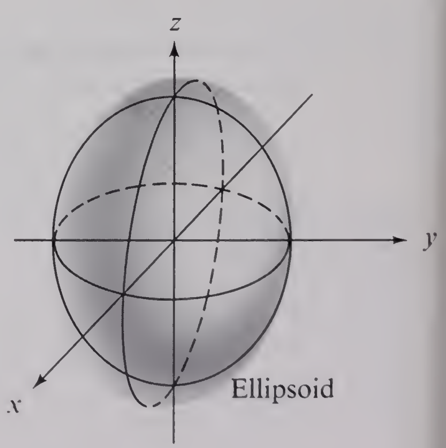
Special cases of quadric surfaces centered at the origin are given in C.5.

C.5 (a) *Sphere* with implicit equation $x^2 + y^2 + z^2 = a^2$;

(b) *Ellipsoid* with implicit equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where $a, b, c > 0$;



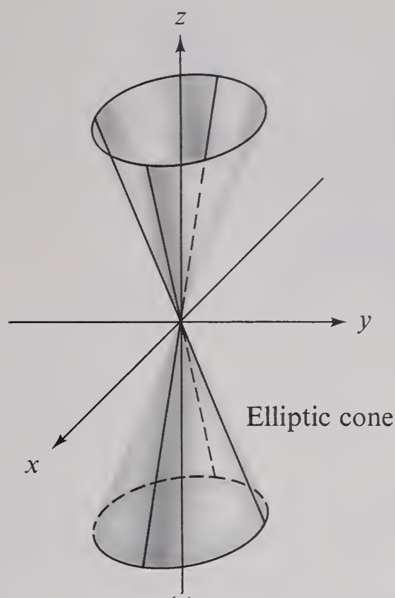
(a)



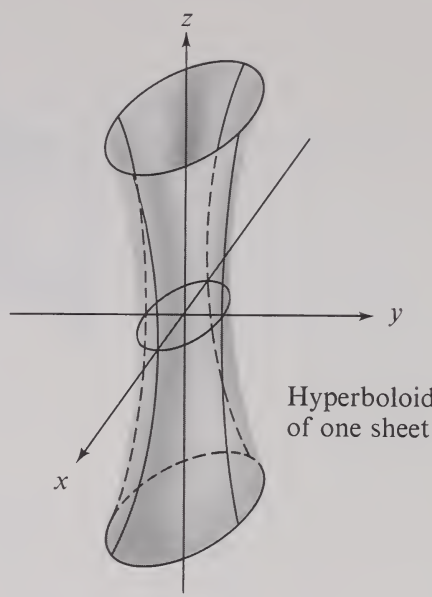
(b)

(c) *Elliptic cone* with implicit equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$, where $a, b, c > 0$;

(d) *Hyperboloid of one sheet* with implicit equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, where $a, b, c > 0$;



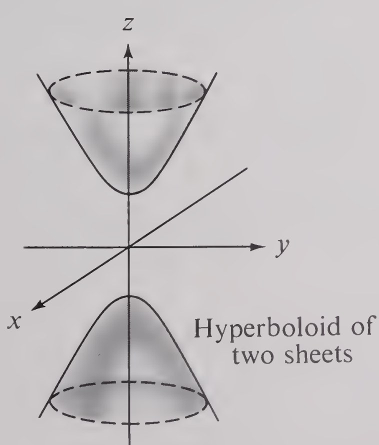
(c)



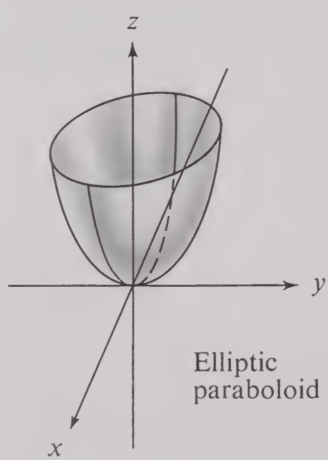
(d)

(e) *Hyperboloid of two sheets* with implicit equation $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a, b, c > 0$;

(f) *Elliptic paraboloid* with explicit equation $z = ax^2 + by^2$, where $a > 0$, $b > 0$;

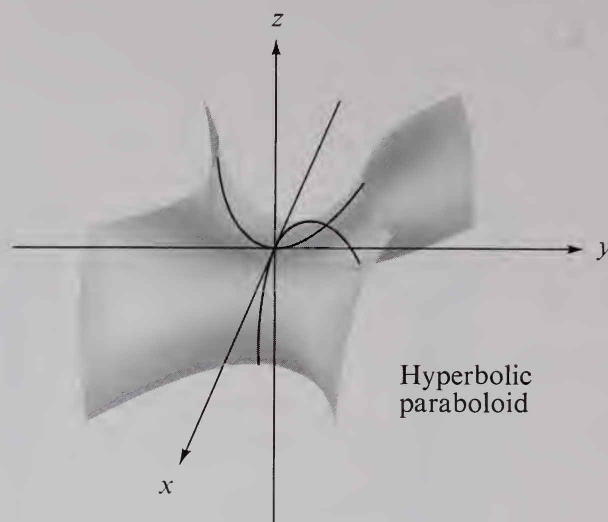


(e)



(f)

(g) *Hyperbolic paraboloid* with explicit equation $z = ax^2 - by^2$, where $a > 0$, $b > 0$.



(g)

9. Find parametric equations $x = \bar{x}(\theta, \phi)$, $y = \bar{y}(\theta, \phi)$, $z = \bar{z}(\theta, \phi)$, for the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ using the following steps.
   (a) Set $x^2/a^2 + y^2/b^2 = \sin^2\phi$ and $z^2/c^2 = \cos^2 \phi$.
   (b) Solve the second equation in (a) for z.
   (c) Convert the first equation in (a) to

   $$\frac{x^2}{(a \sin \phi)^2} + \frac{y^2}{(b \sin \phi)^2} = 1$$

   and find parametric equations for $\bar{x}$ and $\bar{y}$ in terms of $\cos \theta$ and $\sin \theta$.

10. Find parametric equations of the first-octant ($x \geq 0, y \geq 0,\ z \geq 0$) portion of the surface,

    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

    (*Hint*: Let $x^2/a^2 + y^2/b^2 = \cosh^2\phi$ and $z^2/c^2 = \sinh^2\phi$, then proceed as in Problem 9.)

11. Find parametric equations for the following surfaces.
    (a) $x^2/4 + y^2/9 - z^2 = 0$,
    (b) $z = 2x^2 + 3y^2$.

Review

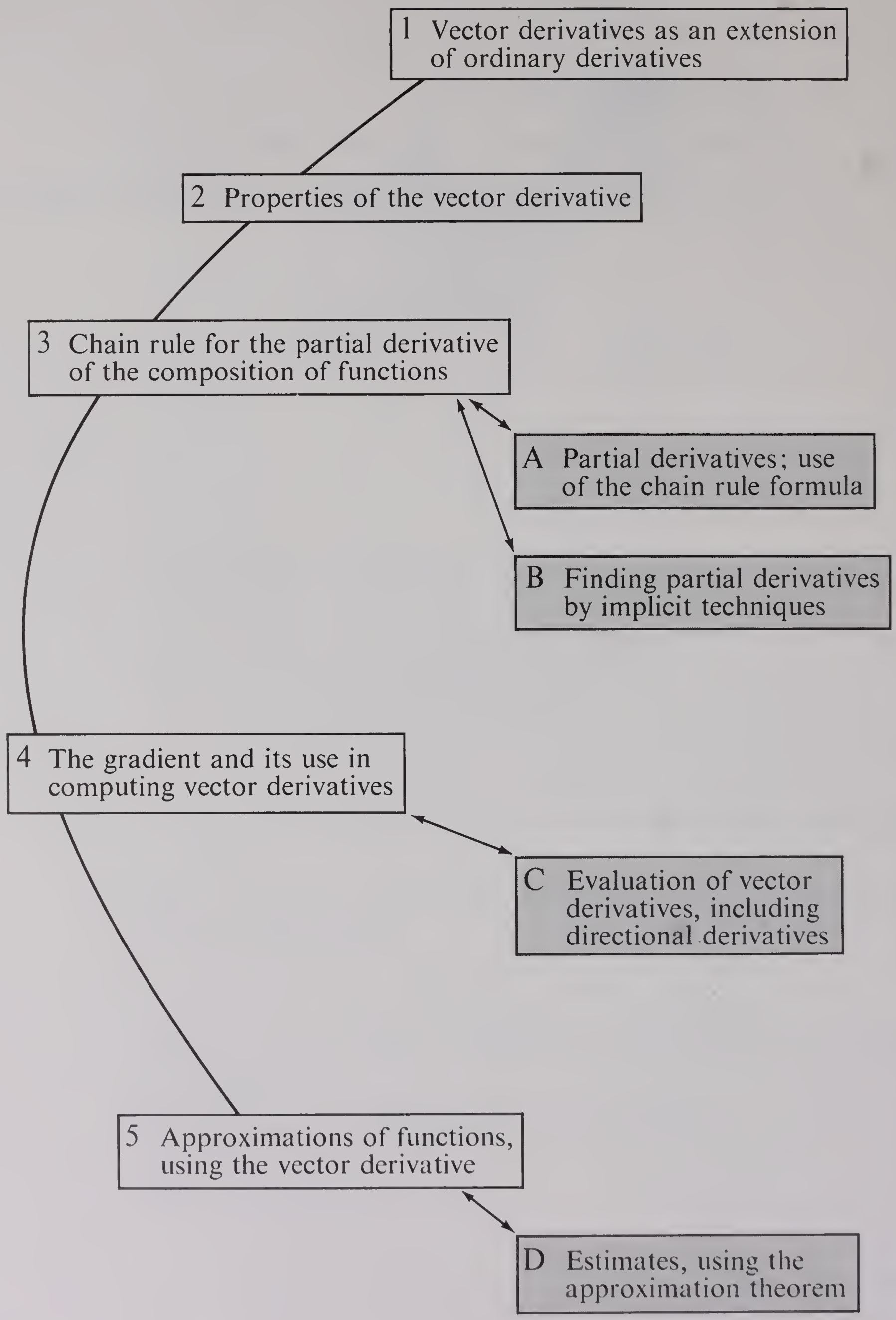
12. Give the standard vector equation of the surfaces,
    (a) $y = x^3z + 4$, (b) $z = \cos^2 xy$.

13. Find a vector equation of the surface obtained by revolving the curve of

    (a) $z = y^3$ about the y axis ($y \geq 0$),
    (b) $z = y^3$ about the z axis,
    (c) $y = 3x$ about the x axis ($x \geq 0$),
    (d) $x = e^y$ about the y axis.

14. Find a vector equation for each surface below.

    (a) $-(x+1)^2/4 - (y+3)^2/16 + z^2/9 = 1$ (use the first-octant portion),
    (b) $z - 3 = 2(x-1)^2 - 4(y+3)^2$.


1 Vector derivatives as an extension of ordinary derivatives
2 Properties of the vector derivative
3 Chain rule for the partial derivative of the composition of functions
A Partial derivatives; use of the chain rule formula
B Finding partial derivatives by implicit techniques
4 The gradient and its use in computing vector derivatives
C Evaluation of vector derivatives, including directional derivatives
5 Approximations of functions, using the vector derivative
D Estimates, using the approximation theorem

X

Vector Derivatives of Scalar Functions

The derivative concept arises whenever a rate-of-change problem occurs in mathematics or physics. Velocities and slopes of curves are examples of quantities which may be described by derivatives. In the case of functions of two or more variables, rates of change are always associated with a direction.

Example 0.1 A container of water is placed on a hot stove (see Figure 10.1(a)). Let $T(\mathbf{u})$ denote the temperature at a variable position vector, $\mathbf{u}$, in the water, the time being assumed fixed. Movement from a fixed point $\mathbf{u}_0$ to nearby points will produce a change in T, and the rate of change will depend on the direction of movement.

Example 0.2 For simplicity, let it be assumed that locally the earth is flat, and the height above sea level is a function $f(x, y)$ of the east and north coordinates, (x, y), of a point (see Figure 10.1(b)). Movement from a fixed point (x_0, y_0) to a nearby point (x, y) produces a change in elevation. The rate of change of height, which in this case

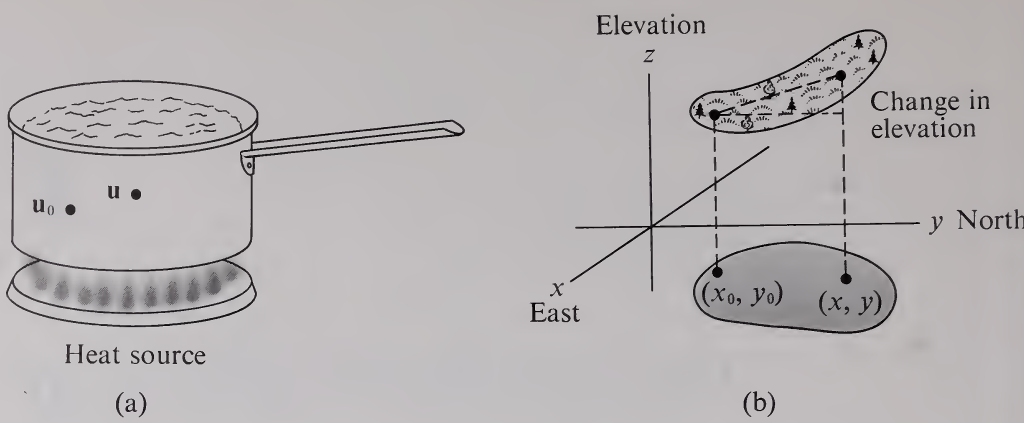


Figure 10.1

is the slope of the terrain, depends on the direction of (x, y) from (x_o, y_o).

If $f(\mathbf{u})$ is a given function, then between $\mathbf{u}_o$ and $\mathbf{u}$ there is an *average rate of change*, loosely described by the difference quotient

$$\frac{\text{change in } f \text{ between } \mathbf{u}_o \text{ and } \mathbf{u}}{\text{change in position between } \mathbf{u}_o \text{ and } \mathbf{u}}.$$

For differentiable functions $f(x)$ of one variable, the average rate of change,

$$\frac{f(x) - f(x_o)}{x - x_o},$$

is approximately the same for all values of x chosen sufficiently near x_o. For well-behaved functions of several variables, however, a comparable difference quotient can vary greatly for values of $\mathbf{u}$ chosen near $\mathbf{u}_o$, but in different directions from $\mathbf{u}_o$. Thus, we shall consider only the average rates of change in specified directions.

The *instantaneous rate of change* is given by the limit of an average rate of change, and for a function $f(x)$ of one variable this is the derivative,

$$\frac{df}{dx}(x_o) = \lim_{x \to x_0} \frac{f(x) - f(x_o)}{x - x_o}.$$

An alternate form of the derivative, using a parameter t, is

$$\frac{df}{dx}(x_o) = \lim_{t \to 0} \frac{f(x_o + t) - f(x_o)}{t}.$$

It is this form which is used in extending the derivative to functions of several variables.

1. Vector Derivatives

In this section it will be assumed that f is a scalar function having as its domain an open set $\mathbf{U}$ in $\mathbf{R}^n$; also $\mathbf{u}_o$ will denote a vector in $\mathbf{U}$ and $\mathbf{v}_o$ a vector in $\mathbf{R}^n$.

Definition of Vector Derivative (Scalar Functions)

The vector derivative *of f at $\mathbf{u}_o$ with respect to $\mathbf{v}_o$ is*

$$f_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t \to 0} \frac{f(\mathbf{u}_o + t\mathbf{v}_o) - f(\mathbf{u}_o)}{t}.$$

This definition assumes that the limit of the difference quotient exists. The following example shows that this is a generalization of the ordinary derivative.

Example 1.1 Let $\mathbf{U}$ be an open interval in $\mathbf{R}^1$, $\mathbf{u}_o = \langle x_o \rangle$, and $\mathbf{v}_o = \langle 1 \rangle$. Then

$$f_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t \to 0} \frac{f(\langle x_o \rangle + t\langle 1 \rangle) - f(\langle x_o \rangle)}{t}.$$

Identifying a 1-tuple with its entry converts the right side of this equality to

$$\lim_{t \to 0} \frac{f(x_o + t) - f(x_o)}{t} = \frac{df}{dx}(x_o).$$

The next example gives a geometric interpretation of the vector derivative.

Example 1.2 Let $n = 2$ and $|\mathbf{v}_o| = 1$. The graph of f is a surface $\mathscr{S}$ (see Figure 10.2). Let $\mathbf{P}$ be the plane which is perpendicular to the xy plane and passes through the line in the xy plane which contains $\mathbf{u}_o$ and has direction determined by $\mathbf{v}_o$. Then $\mathbf{P}$ intersects $\mathscr{S}$ in a curve, $\mathscr{C}$, and intersects the xy plane in the line $\mathbf{L} = \mathbf{u}_o + t\mathbf{v}_o$. Let $\mathbf{P}$ be geometrically identified with the Cartesian plane, the line $\mathbf{L}$ corresponding to the horizontal axis and a perpendicular line through $\mathbf{u}_o$ corresponding

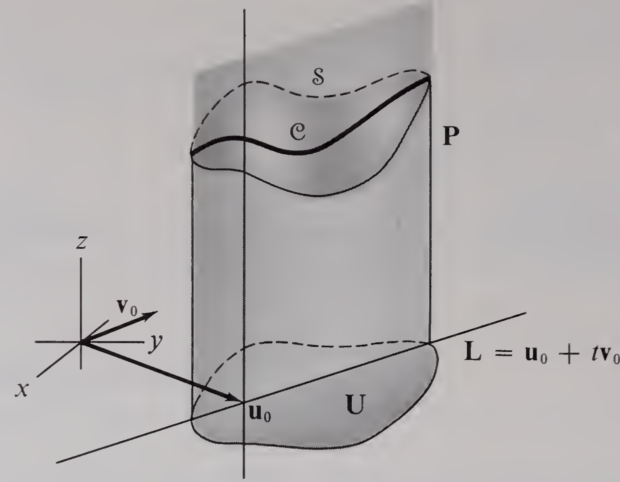


Figure 10.2

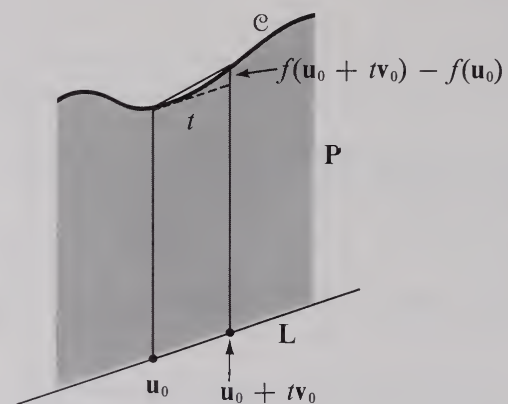


Figure 10.3

to the vertical axis. The tangent line to $\mathscr{C}$ at $(\mathbf{u}_o, f(\mathbf{u}_o))$ has the slope (see Figure 10.3) given by

$$\lim_{t \to 0} \frac{f(\mathbf{u}_o + t\mathbf{v}_o) - f(\mathbf{u}_o)}{t} = f_{\mathbf{v}_o}(\mathbf{u}_o),$$

provided this limit exists. The condition $|\mathbf{v}_o| = 1$ is necessary in order that the scale along **L** be prescribed by a unit vector. The parameter t necessarily assumes both positive and negative values since the domain of f is open. Thus the tangent is two-sided.

The procedure for computing a vector derivative from its definition will now be shown.

Example 1.3 If $f(x, y) = xy^2$, $\mathbf{u}_o = \langle 2, 1 \rangle$, and $\mathbf{v}_o = \langle -3, 1 \rangle$, then

$$\begin{aligned} f_{\mathbf{v}_o}(\mathbf{u}_o) &= \lim_{t \to 0} \frac{f(\langle 2, 1 \rangle + t\langle -3, 1 \rangle) - f(\langle 2, 1 \rangle)}{t} \\ &= \lim_{t \to 0} \frac{(2 - 3t)(1 + t)^2 - 2}{t} \\ &= \lim_{t \to 0} \frac{-3t^3 - 4t^2 + t}{t} \\ &= 1. \end{aligned}$$

As with the ordinary derivative, it is desirable to find formulas for computing vector derivatives. Our first formula method comes in relating vector derivatives to ordinary derivatives. This is accomplished by the introduction of a function, which we shall call an *identification function*, which converts f to a real function of one variable. The identification function "identifies" the line $\mathbf{u}_o + t\mathbf{v}_o$ with $\mathbf{Re}$, in the same manner as $\mathbf{L}$ is identified with the horizontal axis in Example 1.2. Since the definition of $f_{\mathbf{v}_o}(\mathbf{u}_o)$ involves only those values of f on the line $\mathbf{u}_o + t\mathbf{v}_o$, it follows that the procedure merely extracts from the vector derivative situation its one-dimensional essence. Formally, the *identification function* λ is defined to be the function from $\mathbf{Re}$ to $\mathbf{R}^n$ having the rule

$$\lambda(t) = \mathbf{u}_o + t\mathbf{v}_o .$$

It is evident from this definition that

(a) λ is injective, and

(b) the λ-image of $\mathbf{Re}$ is the line $\mathbf{u}_o + t\mathbf{v}_o$.

The basic relation between the vector derivative and an ordinary derivative will now be established.

Proposition 1.1 For each number c,

$$f_{\mathbf{v}_o}(\mathbf{u}_o + c\mathbf{v}_o) = \frac{d(f \circ \lambda)}{dt}(c) .$$

This equality is valid whenever either side exists. An important special case is obtained by setting $c = 0$.

Proposition 1.2

$$f_{\mathbf{v}_o}(\mathbf{u}_o) = \frac{d(f \circ \lambda)}{dt}(0) .$$

A proof of the equality in Proposition 1.1, assuming that the left side exists, is

$$\begin{aligned} f_{\mathbf{v}_o}(\mathbf{u}_o + c\mathbf{v}_o) &= \lim_{t\to 0} \frac{f((\mathbf{u}_o + c\mathbf{v}_o) + t\mathbf{v}_o) - f(\mathbf{u}_o + c\mathbf{v}_o)}{t} \\ &= \lim_{t\to 0} \frac{f(\mathbf{u}_o + (c+t)\mathbf{v}_o) - f(\mathbf{u}_o + c\mathbf{v}_o)}{t} \\ &= \lim_{t\to 0} \frac{f \circ \lambda(c+t) - f \circ \lambda(c)}{t} \\ &= \frac{d(f \circ \lambda)}{dt}(c). \end{aligned}$$

If the right side is assumed to exist, then a proof is made by reversing these steps.
An example will illustrate the use of λ in computing vector derivatives.

Example 1.4 Given $f(x, y) = xe^y$, $\mathbf{u}_o = \langle 1, 3\rangle$, $\mathbf{v}_o = \langle 2, 1\rangle$, then

$$\lambda(t) = \langle 1, 3\rangle + t\langle 2, 1\rangle = \langle 1 + 2t, 3 + t\rangle,$$
$$f \circ \lambda(t) = (1 + 2t)e^{3+t} = e^3(1 + 2t)e^t,$$
$$f_{\mathbf{v}_o}(\mathbf{u}_o) = \frac{d(f \circ \lambda)}{dt}(0) = e^3(3 + 2t)e^t\big|_{t=0} = 3e^3.$$

Questions

1. $f_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t \to 0}$ ________.
2. If the domain of f is $\mathbf{R}^2$, then $f_{\mathbf{v}_o}(\mathbf{u}_o)$ is the slope of the graph of f at $\mathbf{u}_o$ in the direction of $\mathbf{v}_o$, provided ________.
3. The equality $f_{\mathbf{v}_o}(\mathbf{u}_o) = (d(f \circ \lambda)/dt)(0)$ holds true where λ has the rule $\lambda(t) =$ ________.
4. The definition of $f_{\mathbf{v}_o}(\mathbf{u}_o)$ uses only those values of f on the line ________.

Exercises

1. Find $f_{\mathbf{v}_o}(\mathbf{u}_o)$ if $f(x, y) = x^3y$, $\mathbf{u}_o = \langle 1, 2\rangle$, and $\mathbf{v}_o = \langle -1, 0\rangle$; by
   (a) the definition of $f_{\mathbf{v}_o}(\mathbf{u}_o)$,
   (b) using the identification function,
   (c) using L'hopital's rule.
2. Find $f_{\mathbf{v}_o}(\mathbf{u}_o)$ if
   (a) $f(x, y) = x^2e^y$, $\mathbf{u}_o = \langle 1, 0\rangle$, $\mathbf{v}_o = \langle 2, 1\rangle$,
   (b) $f(x, y, z) = xy \sin z$, $\mathbf{u}_o = \langle 1, 2, \pi\rangle$, $\mathbf{v}_o = \langle 0, 1, 1\rangle$.
3. Find the slope of the graph of $f(x, y) = x^2y$ at $(2, 1, 4)$ in the direction of $\langle 3, 4\rangle$.

Proofs

1. Prove that if $\mathbf{v}_o = \mathbf{0}$, then $f_{\mathbf{v}_o}(\mathbf{u}_o) = 0$.

2. *Properties of the Vector Derivative*

The identification function also serves a useful theoretic purpose. It can be used in conjunction with known properties of the ordinary derivative to prove corresponding properties of the vector derivative.

We again let f denote a scalar function having as its domain an open set $\mathbf{U}$ in $\mathbf{R}^n$. If $\mathbf{v}_o$ is in $\mathbf{R}^n$, then the *vector derivative function* $f_{\mathbf{v}_o}$ is the scalar function with domain $\mathbf{U}$ and rule given by $f_{\mathbf{v}_o}(\mathbf{u}_o)$, the existence of the vector derivative being assumed for each $\mathbf{u}_o$ in $\mathbf{U}$. The following properties are valid.

Proposition 2.1

(a) $(cf)_{\mathbf{v}_o} = cf_{\mathbf{v}_o}$,
(b) $(f+g)_{\mathbf{v}_o} = f_{\mathbf{v}_o} + g_{\mathbf{v}_o}$,
(c) $(fg)_{\mathbf{v}_o} = fg_{\mathbf{v}_o} + f_{\mathbf{v}_o}g$,
(d) $\left(\dfrac{f}{g}\right)_{\mathbf{v}_o} = \dfrac{f_{\mathbf{v}_o}g - fg_{\mathbf{v}_o}}{g^2}$.

For (a) and (b) see Proofs, exercises 1 and 2. For a proof of (c), if $\mathbf{u}_o$ is arbitrary in $\mathbf{U}$, then

$$\begin{aligned}(fg)_{\mathbf{v}_o} &= \frac{d(fg\circ\lambda)}{dt}(0)\\ &= \frac{d((f\circ\lambda)(g\circ\lambda))}{dt}(0)\\ &= \left[(f\circ\lambda)\frac{d(g\circ\lambda)}{dt} + \frac{d(f\circ\lambda)}{dt}(g\circ\lambda)\right](0)\\ &= (fg_{\mathbf{v}_o} + f_{\mathbf{v}_o}g)(\mathbf{u}_o).\end{aligned}$$

It should be noted that the equality $fg\circ\lambda = (f\circ\lambda)(g\circ\lambda)$ follows from the definitions of scalar product and composition operation on functions.

Further properties of the vector derivative will now be observed.

Proposition 2.2

(a) If f is linear, then $f_{\mathbf{v}_o}(\mathbf{u}_o) = f(\mathbf{v}_o)$,
(b) $f_{c\mathbf{v}_o} = cf_{\mathbf{v}_o}$.

A proof of (a) follows from the equalities

$$\begin{aligned}\frac{f(\mathbf{u}_o + t\mathbf{v}_o) - f(\mathbf{u}_o)}{t} &= \frac{f(\mathbf{u}_o) + tf(\mathbf{v}_o) - f(\mathbf{u}_o)}{t}\\ &= f(\mathbf{v}_o).\end{aligned}$$

For the proof of (b) when $c \neq 0$,

$$\begin{aligned}f_{c\mathbf{v}_o}(\mathbf{u}_o) &= \lim_{t\to 0}\frac{f(\mathbf{u}_o + t(c\mathbf{v}_o)) - f(\mathbf{u}_o)}{t}\\ &= c\lim_{t\to 0}\frac{f(\mathbf{u}_o + (ct)\mathbf{v}_o) - f(\mathbf{u}_o)}{ct}.\end{aligned}$$

Substitution of $s = ct$ and application of Proposition 3.5 in Chapter IX gives

$$\begin{aligned} f_{c\mathbf{v}_o}(\mathbf{u}_o) &= c \lim_{s\to 0} \frac{f(\mathbf{u}_o + s\mathbf{v}_o) - f(\mathbf{u}_o)}{s} \\ &= cf_{\mathbf{v}_o}(\mathbf{u}_o). \end{aligned}$$

One of the most useful theorems in first-year calculus is the one-variable mean value theorem:

If $f(x)$ is continuous on $[a, b]$ and df/dx exists on (a, b), then there exists c, $a < c < b$, such that

$$f(b) - f(a) = (b - a)\frac{df}{dx}(c).$$

It says geometrically that at some point on the graph of f the tangent line is parallel to the line through the end points of the graph (see Figure 10.4). We shall now extend this result to functions of several variables. The following geometric example indicates the desired extension.

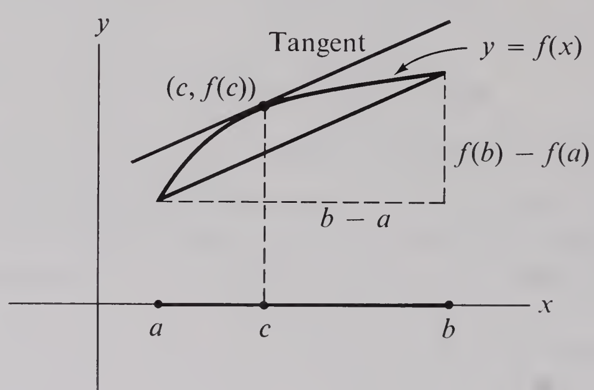


Figure 10.4

Example 2.1 Let $n = 2$ and assume that $\mathbf{U}$ contains the line segment $\mathbf{L}$ whose end points are $\mathbf{u}_o$, $\mathbf{u}_1$. Then

$$\mathbf{v}_o = \frac{\mathbf{u}_1 - \mathbf{u}_o}{|\mathbf{u}_1 - \mathbf{u}_o|}$$

is a vector which has unit norm and is in the direction from $\mathbf{u}_o$ to $\mathbf{u}_1$. From Example 1.2 and the one-variable mean value theorem, for some $\mathbf{u}_o{}^*$ on $\mathbf{L}$ (see Figure 10.5),

$$f(\mathbf{u}_1) - f(\mathbf{u}_o) = |\mathbf{u}_1 - \mathbf{u}_o| f_{\mathbf{v}_o}(\mathbf{u}_o{}^*).$$

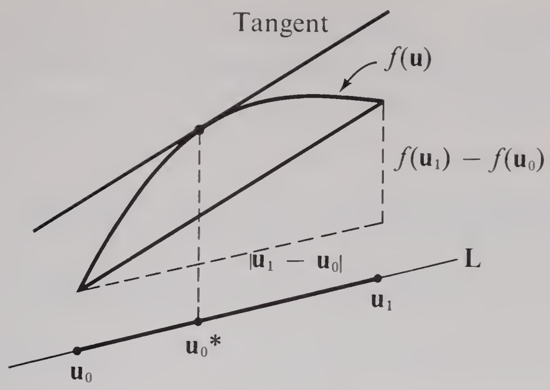


Figure 10.5

Since

$$f_{\mathbf{v}_o}(\mathbf{u}_o{}^*) = \frac{1}{|\mathbf{u}_1 - \mathbf{u}_o|} f_{(\mathbf{u}_1 - \mathbf{u}_o)}(\mathbf{u}_o{}^*)$$

by Proposition 2.2 (b), it follows that

$$f(\mathbf{u}_1) - f(\mathbf{u}_o) = f_{(\mathbf{u}_1 - \mathbf{u}_o)}(\mathbf{u}_o{}^*).$$

From vector algebra the (closed) line segment in $\mathbf{R}^n$ with end points $\mathbf{u}_o$, $\mathbf{u}_1$ is

$$[\mathbf{u}_o, \mathbf{u}_1] = \{\mathbf{u}_o + r(\mathbf{u}_1 - \mathbf{u}_o) : 0 \le r \le 1\}.$$

With this definition we now state our extension:

n-Variable Mean Value Theorem

If the segment $[\mathbf{u}_o, \mathbf{u}_1]$ *is contained in* $\mathbf{U}$ *and* $f_{\mathbf{u}_1 - \mathbf{u}_o}$ *exists at each vector in* $[\mathbf{u}_o, \mathbf{u}_1]$, *then there exists* $\mathbf{u}_o{}^*$ *in* $[\mathbf{u}_o, \mathbf{u}_1]$ *such that*

$$f(\mathbf{u}_1) - f(\mathbf{u}_o) = f_{\mathbf{u}_1 - \mathbf{u}_o}(\mathbf{u}_o{}^*).$$

For the proof we use the identification function $\lambda(t) = \mathbf{u}_o + t(\mathbf{u}_1 - \mathbf{u}_o)$. Then

$$f \circ \lambda(0) = f(\mathbf{u}_o),$$
$$f \circ \lambda(1) = f(\mathbf{u}_1),$$
$$\frac{d(f \circ \lambda)}{dt}(r) = f_{\mathbf{u}_1 - \mathbf{u}_o}(\mathbf{u}_o + r(\mathbf{u}_1 - \mathbf{u}_o)),\ 0 \le r \le 1.$$

The last equality follows from Proposition 1.1. We now apply the one-variable mean value theorem to f to obtain c, $0 \le c \le 1$, such that

$$f \circ \lambda(1) - f \circ \lambda(0) = \frac{d(f \circ \lambda)}{dt}(c).$$

Letting $\mathbf{u}_o{}^* = \mathbf{u}_o + c(\mathbf{u}_1 - \mathbf{u}_o)$, substitution gives the desired equality.

Example 2.2 Given $f(x, y) = xy^2$, $\mathbf{u}_o = \langle 1, 2 \rangle$, $\mathbf{u}_1 = \langle 1, -3 \rangle$, we shall find a suitable $\mathbf{u}_o^*$ by following the proof of the n-variable mean value theorem. Letting

$$\lambda(t) = \langle 1, 2 \rangle + t(\langle 1, -3 \rangle - \langle 1, 2 \rangle) = \langle 1, 2 - 5t \rangle$$

and forming the function $f \circ \lambda(t) = 1(2 - 5t)^2$, then

$$-10(2 - 5t) = \frac{d(f \circ \lambda)}{dt} = f(\mathbf{u}_1) - f(\mathbf{u}_o) = 5$$

gives $t = 1/2$. Hence, $\mathbf{u}_o^* = \lambda(1/2) = \langle 1, -1/2 \rangle$ is a solution.

Questions

1. If $f_{\mathbf{v}}(\mathbf{u}_o) = 4$, then $f_{c\mathbf{v}}(\mathbf{u}_o) =$ ______.
2. The n-variable mean value theorem gives a $\mathbf{u}_o^*$ in $[\mathbf{u}_o, \mathbf{u}_1]$ such that $f_{\mathbf{u}_1 - \mathbf{u}_o}(\mathbf{u}_o^*) =$ ______.
3. The vector $\mathbf{u}_o + r(\mathbf{u}_1 - \mathbf{u}_o)$ is in $[\mathbf{u}_o, \mathbf{u}_1]$ provided that r is ______.

Exercises

1. Find a vector $\mathbf{u}_o^*$ which satisfies the n-variable mean value theorem from the following information.

   (a) $f = 2x + y^2$, $\mathbf{u}_o = \langle 1, 1 \rangle$, $\mathbf{u}_1 = \langle 0, 2 \rangle$;
   (b) $f = x^2 y$, $\mathbf{u}_o = \langle 2, 0 \rangle$, $\mathbf{u}_1 = \langle 3, 2 \rangle$.

Proofs

1. Justify each step in the following proof of $(f + g)_{\mathbf{v}_o} = f_{\mathbf{v}_o} + g_{\mathbf{v}_o}$:

$$(f + g)_{\mathbf{v}_o}(\mathbf{u}_o) = \frac{d((f + g) \circ \lambda)}{dt}(0)$$

$$= \frac{d(f \circ \lambda + g \circ \lambda)}{dt}(0)$$

$$= \frac{d(f \circ \lambda)}{dt}(0) + \frac{d(g \circ \lambda)}{dt}(0)$$

$$= f_{\mathbf{v}_o}(\mathbf{u}_o) + g_{\mathbf{v}_o}(\mathbf{u}_o).$$

2. Prove that $(cf)_{\mathbf{v}_o} = cf_{\mathbf{v}_o}$.

3. Partial Derivatives; Chain Rule

In this section we consider vector derivatives with respect to standard basis vectors and obtain a composition operation formula for these. If $f(x, y)$ has as its domain an open set in $\mathbf{R}^2$, then the *partial derivative* functions of f are $f_{\mathbf{i}}$, $f_{\mathbf{j}}$, where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. It is conventional to denote these using coordinate rather than vector symbols. Thus, these partial derivatives may be written as

$$f_x(=f_{\mathbf{i}}); \qquad f_y(=f_{\mathbf{j}}).$$

Common symbols used elsewhere for these partial derivative functions are

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}.$$

The three partial derivatives of $f(x, y, z)$ are f_x, f_y, and f_z, where

$$f_x = f_{\mathbf{i}}, f_y = f_{\mathbf{j}}, \text{ and } f_z = f_{\mathbf{k}}.$$

In general, the partial derivative functions of $f(x_1, \ldots, x_n)$ are $f_{x_1}, \ldots, f_{x_n}$, where $f_{x_i} = f_{\mathbf{e}_i}$, $\mathbf{e}_i$ representing the standard basis vector.

Partial derivatives may be evaluated by ordinary differentiation with respect to one variable, the other variables being treated as constants. Justification for this will be demonstrated for the case where $n = 2$.

Example 3.1 Given $f(x, y)$ and $\mathbf{u}_o = \langle a, b\rangle$, then

$$\begin{aligned} f_x(\langle a, b\rangle) &= \lim_{t\to 0} \frac{f(\langle a, b\rangle + t\langle 1, 0\rangle) - f(\langle a, b\rangle)}{t} \\ &= \lim_{t\to 0} \frac{f(\langle a+t, b\rangle) - f(\langle a, b\rangle)}{t}. \end{aligned}$$

Letting g be the one-variable function with rule $g(x) = f(\langle x, b\rangle)$, then, from the equality above, we obtain the result

$$\begin{aligned} f_x(\langle a, b\rangle) &= \lim_{t\to 0} \frac{g(a+t) - g(a)}{t} \\ &= \frac{dg}{dx}(a). \end{aligned}$$

The conclusion in Example 3.1 suggests the procedure in the next example.

Example 3.2 (a) Given that $f = x^2y^3$ and $\mathbf{u}_o = \langle 2, 3\rangle$, if we let $g(x) = f(\langle x, 3\rangle) = 27x^2$, we obtain

$$f_x(\mathbf{u}_o) = \frac{dg}{dx}(2) = 108.$$

(b) Given that $f = x^2yz^2$, $\mathbf{u}_o = \langle 3, 1, 2\rangle$, if we let $g(x) = f(\langle x, 1, 2\rangle) = 4x^2$, we obtain

$$f_x(\mathbf{u}_o) = \frac{dg}{dx}(3) = 24.$$

We next consider a formula for the partial derivative of the composition of two functions. This will generalize the chain rule from first-year calculus, which says that for given $f(r)$ and $g(x)$;

$$\frac{d(g\circ f)}{dr} = \left(\frac{dg}{dx}\circ f\right)\frac{df}{dr}.$$

The extension for $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ and $g(x, y)$ will first be investigated. Letting $\mathbf{f}(\mathbf{u}_o) = \langle x_o, y_o\rangle$, $\mathbf{f}(\mathbf{u}_o + t\mathbf{i}) = \langle x, y\rangle$, where it is observed that x and y vary with t, and setting

$$\Delta(g\circ\mathbf{f}) = g\circ\mathbf{f}(\mathbf{u}_o + t\mathbf{i}) - g\circ\mathbf{f}(\mathbf{u}_o),$$

then, (see Figure 10.6)

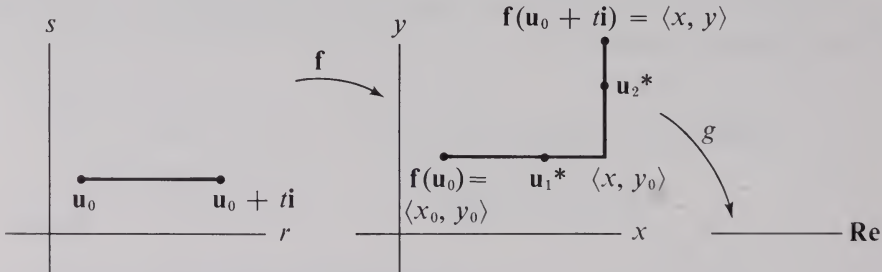


Figure 10.6

$$\begin{aligned}\Delta(g\circ\mathbf{f}) &= g(x, y) - g(x_o, y_o)\\ &= [g(x, y_o) - g(x_o, y_o)] + [g(x, y) - g(x, y_o)].\end{aligned}$$

By the n-variable mean value theorem and Proposition 2.2(b) there exist $\mathbf{u}_1{}^*$ and $\mathbf{u}_2{}^*$, also varying with t and lying respectively on the segments

$$[\langle x_o, y_o\rangle, \langle x, y_o\rangle], [\langle x, y_o\rangle, \langle x, y\rangle],$$

such that

$$g(x, y_o) - g(x_o, y_o) = g_{\langle x - x_o, 0\rangle}(\mathbf{u}_1{}^*) = g_x(\mathbf{u}_1{}^*)(x - x_o)$$

and

$$g(x, y) - g(x, y_\circ) = g_{\langle 0,\, y-y_\circ \rangle}(\mathbf{u}_2^*) = g_y(\mathbf{u}_2^*)(y - y_\circ).$$

Hence,

$$(g\circ\mathbf{f})_r = \lim_{t\to 0} \frac{\Delta(g\circ\mathbf{f})}{t},$$

$$= \lim_{t\to 0} \left[g_x(\mathbf{u}_1^*)\, \frac{x - x_\circ}{t} + g_y(\mathbf{u}_2^*)\, \frac{y - y_\circ}{t} \right].$$

If it is assumed that $\bar{x}_r$, $\bar{y}_r$, g_x, and g_y are continuous, then

$$\lim_{t\to 0} \mathbf{u}_1^* = \lim_{t\to 0} \mathbf{u}_2^* = \lim_{t\to 0} \langle x, y\rangle = \langle x_\circ, y_\circ\rangle = \mathbf{f}(\mathbf{u}_\circ);$$

and also

$$\lim_{t\to 0} g_x(\mathbf{u}_1^*) = g_x(\mathbf{f}(\mathbf{u}_\circ)),$$

$$\lim_{t\to 0} g_y(\mathbf{u}_2^*) = g_y(\mathbf{f}(\mathbf{u}_\circ)),$$

$$\lim_{t\to 0} \frac{x - x_\circ}{t} = \lim_{t\to 0} \frac{\bar{x}(\mathbf{u}_\circ + t\mathbf{i}) - \bar{x}(\mathbf{u}_\circ)}{t} = \bar{x}_r(\mathbf{u}_\circ),$$

$$\lim_{t\to 0} \frac{y - y_\circ}{t} = \lim_{t\to 0} \frac{\bar{y}(\mathbf{u}_\circ + t\mathbf{i}) - \bar{y}(\mathbf{u}_\circ)}{t} = \bar{y}_r(\mathbf{u}_\circ).$$

Substitution gives the equality

$$(g\circ\mathbf{f})_r(\mathbf{u}_\circ) = g_x(\mathbf{f}(\mathbf{u}_\circ))\bar{x}_r(\mathbf{u}_\circ) + g_y(\mathbf{f}(\mathbf{u}_\circ))\bar{y}_r(\mathbf{u}_\circ).$$

Thus, for the function $(g\circ\mathbf{f})_r$, we have

$$(g\circ\mathbf{f})_r = (g_x\circ\mathbf{f})\bar{x}_r + (g_y\circ\mathbf{f})\bar{y}_r .$$

Frequently the composition symbols are omitted and this equation is written

$$g_r = g_x \bar{x}_r + g_y \bar{y}_r .$$

The analogous formula for the partial derivative with respect to s is

$$(g\circ\mathbf{f})_s = (g_x\circ\mathbf{f})\bar{x}_s + (g_y\circ\mathbf{f})\bar{y}_s ,$$

or in abbreviated form,

$$g_s = g_x \bar{x}_s + g_y \bar{y}_s .$$

In order to derive these formulas, it must be assumed that the partial derivatives of $\bar{x}, \bar{y}$, and g are continuous. The proof technique extends to the case where $\mathbf{f}$ is a function of n variables and g is a scalar function of m variables. Thus, given $\mathbf{f}(x_1, \ldots, x_n)$ from an open set $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$, and $g(y_1, \ldots, y_m)$ from an open set in $\mathbf{R}^m$ to $\mathbf{Re}$ so that $g\circ\mathbf{f}$ is defined, we have the next theorem.

Chain Rule Theorem

If the partial derivatives of $\bar{f}_1, \ldots, \bar{f}_m$ and g are continuous, then for each j, $j = 1, 2, \ldots, n$,

$$(g \circ \mathbf{f})_{x_j} = (g_{y_1} \circ \mathbf{f})(\bar{f}_1)_{x_j} + \cdots + (g_{y_m} \circ \mathbf{f})(\bar{f}_m)_{x_j}.$$

Example 3.3 Given $\mathbf{f}(r, s) = \langle rs^2, re^s \rangle$ and $g(x, y) = xy$, then

$$g_x = y; \bar{x} = rs^2; \bar{x}_r = s^2; \bar{x}_s = 2rs;$$
$$g_y = x; \bar{y} = re^s; \bar{y}_r = e^s; \bar{y}_s = re^s.$$

By the chain rule theorem,

$$\begin{aligned} g_r &= \bar{y}s^2 + \bar{x}e^s = rs^2e^s + rs^2e^s = 2rs^2e^s, \\ g_s &= \bar{y}(2rs) + \bar{x}(re^s) = (re^s)(2rs) + rs^2(re^s) \\ &= r^2se^s(2 + s). \end{aligned}$$

The same answers could have been obtained by forming $g \circ \mathbf{f} = (rs^2)(re^s) = r^2s^2e^s$ and applying the usual techniques of finding partial derivatives.

Questions

1. The equation $f_{x_i}(\mathbf{u}_o) = f_{\mathbf{v}_o}(\mathbf{u}_o)$ holds true where $\mathbf{v}_o =$ ________.
2. The chain rule theorem gives a vector derivative formula for the operation of ________.
3. The chain rule formula for $(g \circ \mathbf{f})_{x_j}$ requires that the ________ of $\bar{f}_1, \ldots, \bar{f}_m$ and g be continuous.

Problems

1. Do Problem Sets A and B at the end of the chapter.

4. Gradient; Directional Derivatives

In this section we shall derive a formula expressing vector derivatives of scalar functions in terms of partial derivatives. This formula introduces the *gradient function*, which is a vector function having many of the properties of the derivative.

Again f will denote a scalar function having as its domain an open set $\mathbf{U}$ in $\mathbf{R}^n$; then f is *continuously differentiable*, or *of class* C^1, provided all the partial derivatives of f are continuous. The class C^1 property is preserved by the operations of addition, multiplication by a scalar, product, quotient, and composition. For instance, if $f(x, y)$ and $g(x, y)$ are of class C^1, then $(f+g)_x = f_x + g_x$ and $(f+g)_y = f_y + g_y$ are continuous, since the sum of two continuous functions is continuous. Also each coordinate function on $\mathbf{R}^n$ is of class C^1, since each partial derivative function is the constant function 1 or 0. Since all elementary functions of one variable have continuous derivatives wherever defined, it follows as for the continuity property in Chapter IX that the scalar functions constructed from these elementary functions using the various operations are also of class C^1.

We have seen that vector derivatives may be computed by the formula

$$f_{\mathbf{v}_o}(\mathbf{u}_o) = \frac{d(f \circ \lambda)}{dt}(0).$$

If f is of class C^1, then the chain rule can be applied to the right side of this equality. The case for $f(x, y)$ where $\mathbf{u}_o = \langle x_o, y_o \rangle$ and $\mathbf{v}_o = \langle x_1, y_1 \rangle$ will be considered. Then,

$$\lambda(t) = \mathbf{u}_o + t\mathbf{v}_o = \langle x_o + tx_1, y_o + ty_1 \rangle.$$

Application of the chain rule gives

$$\begin{aligned}
\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) &= \frac{d(f \circ \lambda)}{dt}(0) \\
&= (f_x \circ \lambda)\frac{d(x_o + x_1 t)}{dt} + (f_y \circ \lambda)\frac{d(y_o + y_1 t)}{dt}\bigg|_{t=0} \\
&= f_x(\mathbf{u}_o)x_1 + f_y(\mathbf{u}_o)y_1 \\
&= \langle f_x(\mathbf{u}_o), f_y(\mathbf{u}_o) \rangle \cdot \mathbf{v}_o .
\end{aligned}$$

The vector $\langle f_x(\mathbf{u}_o), f_y(\mathbf{u}_o) \rangle$ is called the *gradient* of f at $\mathbf{u}_o$. Extending to the general case, we find the following definition.

Definition of Gradient

If $f(x_1, \ldots, x_n)$ has for its domain an open set $\mathbf{U}$ in $\mathbf{R}^n$ and $\mathbf{u}_o$ is in $\mathbf{U}$, then the gradient *of f at $\mathbf{u}_o$ is the vector*

$$\nabla f(\mathbf{u}_o) = \langle f_{x_1}(\mathbf{u}_o), \ldots, f_{x_n}(\mathbf{u}_o) \rangle .$$

The conclusion of the previous paragraph may now be formalized.

Gradient Formula

If f is of class C^1, *then* $f_{\mathbf{v}_o}(\mathbf{u}_o) = \nabla f(\mathbf{u}_o) \cdot \mathbf{v}_o$.

This formula provides an effective method for computing vector derivatives.

Example 4.1 Given $f(x, y) = xy^2$, $\mathbf{u}_o = \langle 2, 1 \rangle$, and $\mathbf{v}_o = \langle -3, 1 \rangle$, then

$$\nabla f(\mathbf{u}_o) = \langle y^2, 2xy \rangle |_{\langle 2,1 \rangle} = \langle 1, 4 \rangle$$

and, hence, $f_{\mathbf{v}_o}(\mathbf{u}_o) = \langle 1, 4 \rangle \cdot \langle -3, 1 \rangle = 1$.

The gradient formula yields the following corollary, which will lead to a merger of vector-derivative and linear-function theory in the next chapter.

Proposition 4.1 If f is of class C^1, then

$$f_{\mathbf{v}_o + \mathbf{v}_1}(\mathbf{u}_o) = f_{\mathbf{v}_o}(\mathbf{u}_o) + f_{\mathbf{v}_1}(\mathbf{u}_0).$$

It is assumed that $\mathbf{v}_o$ and $\mathbf{v}_1$ are in $\mathbf{R}^n$. A proof is

$$\begin{aligned} f_{\mathbf{v}_o + \mathbf{v}_1}(\mathbf{u}_o) &= \nabla f(\mathbf{u}_o) \cdot (\mathbf{v}_o + \mathbf{v}_1) \\ &= \nabla f(\mathbf{u}_o) \cdot \mathbf{v}_o + \nabla f(\mathbf{u}_o) \cdot \mathbf{v}_1 \\ &= f_{\mathbf{v}_o}(\mathbf{u}_o) + f_{\mathbf{v}_1}(\mathbf{u}_o). \end{aligned}$$

If f is of class C^1, then there is associated with each domain vector $\mathbf{u}_o$ the gradient vector $\nabla f(\mathbf{u}_o)$. This defines the *gradient function* ∇f; it is a vector function having the same domain as f. The gradient function possesses many of the properties of the derivative.

Proposition 4.2

(a) $\nabla(f + g) = \nabla f + \nabla g$,

(b) $\nabla(cf) = c(\nabla f)$,

(c) $\nabla(fg) = f(\nabla g) + (\nabla f)g$, and

(d) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{(\nabla f)g - f(\nabla g)}{g^2}$.

For proofs of (a) and (b) see Proofs, exercise 1. A proof of (c) for the case $f(x, y)$, $g(x, y)$ is

$$\begin{aligned}\nabla(fg) &= \langle (fg)_x, (fg)_y \rangle, \\ &= \langle fg_x + f_x g, fg_y + f_y g \rangle, \\ &= \langle fg_x, fg_y \rangle + \langle f_x g, f_y g \rangle, \\ &= f\langle g_x, g_y \rangle + \langle f_x, f_y \rangle g, \\ &= f(\nabla g) + (\nabla f)g.\end{aligned}$$

The chain-rule property of the gradient follows from the chain rule theorem of the previous section when $m = 1$. For instance, given $f(r, s)$ and $g(x)$, then

$$\begin{aligned}\nabla(g\circ f) &= \langle (g\circ f)_r, (g\circ f)_s \rangle \\ &= \left\langle \left(\frac{dg}{dx}\circ f\right) f_r, \left(\frac{dg}{dx}\circ f\right) f_s \right\rangle \\ &= \left(\frac{dg}{dx}\circ f\right) \langle f_r, f_s \rangle \\ &= \left(\frac{dg}{dx}\circ f\right) \nabla f.\end{aligned}$$

More generally, we have for $f(x_1, \ldots, x_n)$ and $g(y)$ the following formula.

Proposition 4.3

$$\nabla(g\cdot f) = \left(\frac{dg}{dy}\circ f\right)\nabla f.$$

Example 4.2 The gradient of $h(\langle r, s\rangle) = |\langle r, s\rangle| = \sqrt{r^2 + s^2}$ can be obtained by writing $h = g\circ f$, where $f = r^2 + s^2$ and $g = \sqrt{x}$. Thus,

$$\nabla h = \left(\frac{1}{2\sqrt{x}}\bigg|_{x = r^2 + s^2}\right) \langle 2r, 2s\rangle = \frac{\langle r, s\rangle}{\sqrt{r^2 + s^2}}.$$

This proves for $n = 2$ that if $h(\mathbf{u}) = |\mathbf{u}|$, then

$$\nabla h(\mathbf{u}) = \frac{\mathbf{u}}{|\mathbf{u}|}, \quad \mathbf{u} \neq \mathbf{0}.$$

The gradient has valuable geometric and physical interpretations. Before considering these we introduce the *directional derivative*. It has been previously observed that if the domain of f is an open set in $\mathbf{R}^2$ and $|\mathbf{v}_0| = 1$, then $f_{\mathbf{v}_0}(\mathbf{u}_0)$ can be interpreted as the slope of the surface graph of f at $\mathbf{u}_0$ in the direction $\mathbf{v}_0$. If $\mathbf{v}_0 \neq \mathbf{0}$, then $\mathbf{v}_0/|\mathbf{v}_0|$ has norm 1 and the same direction as $\mathbf{v}_0$. This suggests the following definition.

Definition of Directional Derivative

The directional derivative *of f at $\mathbf{u}_o$ in the direction of $\mathbf{v}_o$, $\mathbf{v}_o \neq \mathbf{0}$, is*

$$\frac{df}{d\mathbf{v}_o}(\mathbf{u}_o) = \nabla f(\mathbf{u}_o) \cdot \frac{\mathbf{v}_o}{|\mathbf{v}_o|}.$$

Therefore, when $n = 2$ the directional derivative may be interpreted as a slope. We now fix $\mathbf{u}_o$ and regard $df/d\mathbf{v}(\mathbf{u}_o)$ as a function of $\mathbf{v}$. It is natural to ask in which direction $df/d\mathbf{v}(\mathbf{u}_o)$ is a maximum. For $n = 2$ this means the direction in which the graph of f is the steepest. In the case of a variable temperature on a flat plate, the corresponding problem is the direction, at a fixed point, in which the temperature changes most rapidly. The next result says the maximum rate of change occurs in the direction of the gradient, and its maximum value is the norm of the gradient.

Proposition 4.4 The maximum value of $\frac{df}{d\mathbf{v}}(\mathbf{u}_o)$ is $|\nabla f(\mathbf{u}_o)|$; it occurs for $\mathbf{v} = \nabla f(\mathbf{u}_o)$.

For the proof we first note that if $\mathbf{v} = \nabla f(\mathbf{u}_o)$, then

$$\frac{df}{d\mathbf{v}}(\mathbf{u}_o) = \nabla f(\mathbf{u}_o) \cdot \frac{\nabla f(\mathbf{u}_o)}{|\nabla f(\mathbf{u}_o)|} = |\nabla f(\mathbf{u}_o)|.$$

Next, letting $\mathbf{v} = \mathbf{v}_o$, $\mathbf{v}_o$ arbitrary, the definition of directional derivative yields, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|\frac{df}{d\mathbf{v}_o}(\mathbf{u}_o)\right| &= \left|\nabla f(\mathbf{u}_o) \cdot \frac{\mathbf{v}_o}{|\mathbf{v}_o|}\right|, \\ &\leq |\nabla f(\mathbf{u}_o)| \left|\frac{\mathbf{v}_o}{|\mathbf{v}_o|}\right|, \\ &= |\nabla f(\mathbf{u}_o)|. \end{aligned}$$

Questions

1. If the partial derivatives of f are continuous, then f is said to be ________ or ________.
2. The ________ gives the slope of a surface graph.
3. The gradient formula states that $f_{\mathbf{v}_o}(\mathbf{u}_o) =$ ________.
4. The gradient behaves like the ________ in first-year calculus.
5. The maximum value of the directional derivative of f at $\mathbf{u}_o$ is ________.

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Given $f(x, y) = x^2y$ and $\mathbf{u}_o = \langle 1, 3\rangle$, show that $\nabla f(\mathbf{u}_o)$ is orthogonal to the tangent vector of $x^2y = 3$ at $(1, 3)$.
2. Given $\mathbf{v}_o = \langle b_1, b_2\rangle$ and $f(x, y) = xy^2$, find a relationship between b_1 and b_2 so that

   (a) $f_{\mathbf{v}_o}(\langle 1, 5\rangle) = 4$, (b) $f_{\mathbf{v}_o}(\langle 1, 2\rangle - 3\mathbf{v}_o) = 2$.

Proofs

1. Prove (a) $\nabla(f + g) = \nabla f + \nabla g$, (b) $\nabla(cf) = c\nabla f$.
2. Given $f(x, y)$, $\mathbf{u}_o$, and $\mathbf{v} = \langle\cos\theta, \sin\theta\rangle$, where $f_x(\mathbf{u}_o) \neq 0$:
   (a) Express $df/d\mathbf{v}(\mathbf{u}_o)$ as a function $F(\theta)$ of θ.
   (b) Show by methods of elementary calculus that the maximum value of $F(\theta)$ occurs when $\theta = \arctan f_y(\mathbf{u}_o)/f_x(\mathbf{u}_o)$.
   (c) Show that the value of θ in (b) agrees with the direction vector $\mathbf{v} = \nabla f(\mathbf{u}_o)$ in Proposition 4.4.
3. Given $\mathbf{u}_o$ in the domain of $g(x, y)$, show that $\nabla g(\mathbf{u}_o)$ is orthogonal to a tangent vector to the curve $g(x, y) = g(\mathbf{u}_o)$ at $\mathbf{u}_o$. (*Hint*: Let

$$\mathbf{f}(r) = \langle\bar{x}(r), \bar{y}(r)\rangle$$

describe the curve, and apply the chain rule to the equation

$$g \circ \mathbf{f}(r) = g(\mathbf{u}_o).)$$

5. Approximation Theorem

If $f(x)$ is a differentiable function of one variable and c is in the domain of f, then the graph of f near c is approximated by the tangent line at $(c, f(c))$ (see Figure 10.7). This tangent line is the graph of

$$g_c(x) = f(c) + \frac{df}{dx}(c)(x - c).$$

The values of x near c give

$$f(x) \approx f(c) + \frac{df}{dx}(c)(x - c),$$

where "$\approx$" means "approximately equal." The degree of approximation can be made precise

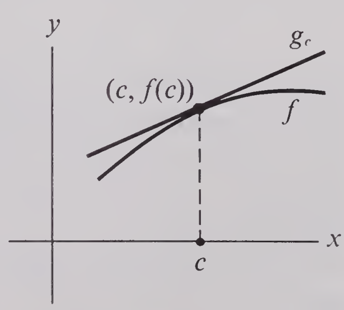


Figure 10.7

by a simple manipulation of the derivative definition

$$\frac{df}{dx}(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

to give

$$\lim_{x \to c} \frac{f(x) - f(c) - \dfrac{df}{dx}(c)(x - c)}{x - c} = 0.$$

Example 5.1 The number $\sqrt{9.0001}$ may be approximated from the expression

$$f(x) \approx f(c) + \frac{df}{dx}(c)(x - c)$$

by letting $f(x) = \sqrt{x}$, $c = 9$, $x = 9.0001$. Thus,

$$\sqrt{9.0001} \approx \sqrt{9} + \frac{1}{2\sqrt{9}}(.0001) \approx 3.00002$$

(see Figure 10.8).

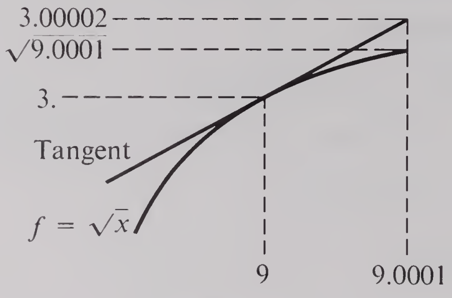


Figure 10.8

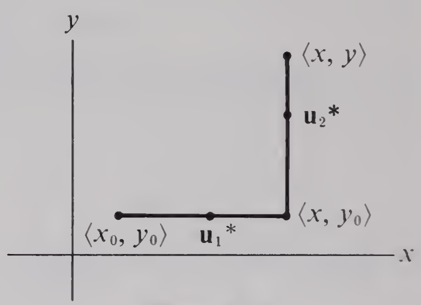


Figure 10.9

We next consider an extension of this approximation relation to $f(x, y)$. If $\mathbf{u}_o = \langle x_o, y_o \rangle$ and $\mathbf{u} = \langle x, y \rangle$ are in the domain of f, then (see Figure 10.9)

$$f(\mathbf{u}) - f(\mathbf{u}_o) = [f(x, y_o) - f(x_o, y_o)] + [f(x, y) - f(x, y_o].$$

By the n-variable mean value theorem and Proposition 2.2(b), there exist $\mathbf{u}_1{}^*$ and $\mathbf{u}_2{}^*$, lying respectively on the segments

$$[\langle x_o, y_o \rangle, \langle x, y_o \rangle] \quad \text{and} \quad [\langle x, y_o \rangle, \langle x, y \rangle],$$

such that

$$f(x, y_o) - f(x_o, y_o) = f_{\langle x - x_o, 0 \rangle}(\mathbf{u}_1{}^*) = f_x(\mathbf{u}_1{}^*)(x - x_o)$$

and

$$f(x, y) - f(x, y_o) = f_{\langle 0,\, y - y_o \rangle}(\mathbf{u}_2{}^*) = f_y(\mathbf{u}_2{}^*)(y - y_o).$$

If f_x and f_y are continuous and $\langle x, y\rangle$ is near $\langle x_o, y_o\rangle$, then

$$f_x(\mathbf{u}_1^*) \approx f_x(\mathbf{u}_o); \qquad f_y(\mathbf{u}_2^*) \approx f_y(\mathbf{u}_o).$$

Substituting gives the approximation relation

$$\begin{aligned} f(\mathbf{u}) - f(\mathbf{u}_o) &\approx f_x(\mathbf{u}_o)(x - x_o) + f_y(\mathbf{u}_o)(y - y_o) \\ &= \nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o) \\ &= f_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o)\,. \end{aligned}$$

An understanding of the usefulness of this approximation requires some precise statement about the nature of its error. This is given by the following result, which asserts that the order of error is less than $|\mathbf{u} - \mathbf{u}_o|$. A proof is in the appendix.

Approximation Theorem for Scalar Functions

If $f(x_1, \ldots, x_n)$ is of class C^1, *then*

$$\lim_{\mathbf{u}\to\mathbf{u}_o} \frac{f(\mathbf{u}) - f(\mathbf{u}_o) - f_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o)}{|\mathbf{u} - \mathbf{u}_o|} = 0.$$

The approximating formula,

$$f(\mathbf{u}) \approx f(\mathbf{u}_o) + \nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o),$$

may be used to estimate the differences in f corresponding to small differences in the coordinate variables.

Example 5.2 An approximation for the number $e^{.001} \sin .0001$ may be obtained by letting

$$f(x, y) = e^x \sin y, \quad \mathbf{u}_o = \langle 0, 0\rangle, \quad \mathbf{u} = \langle .001, .0001\rangle.$$

Then

$$\begin{aligned} e^{.001} \sin .0001 &\approx e^0 \sin 0 + \langle e^0 \sin 0, e^0 \cos 0\rangle \cdot \langle .001, .0001\rangle \\ &= .0001. \end{aligned}$$

In first-year calculus it is shown that for functions of one variable the existence of a derivative implies continuity. It might then be supposed that for functions of several variables the existence of vector derivatives implies continuity. However, there are examples which show that a function f may have $f_{\mathbf{v}}(\mathbf{u}_o)$ exist for all $\mathbf{v}$ at a fixed point $\mathbf{u}_o$, and yet fail to be continuous at $\mathbf{u}_o$. The missing element lies in the sufficiency requirement for the approximation theorem. For functions of one variable the mere *existence* of the derivative is sufficient; for functions of more than one variable a stronger requirement, such

as continuity of partial derivatives, is needed. We shall now show in the next proposition that this condition is sufficient.

Proposition 5.1 If f is of class C^1, then f is continuous.

The continuity of f will be proved at $\mathbf{u}_o$. By the approximation theorem there exists $\delta_1 > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta_1$, then

$$|f(\mathbf{u}) - f(\mathbf{u}_o) - \nabla f(\mathbf{u}_o)\cdot(\mathbf{u} - \mathbf{u}_o)| < |\mathbf{u} - \mathbf{u}_o|.$$

Letting $\varepsilon > 0$ and defining, for reasons to be understood later in the proof,

$$\delta = \text{minimum}\left\{\delta_1, \frac{\varepsilon}{1 + |\nabla f(\mathbf{u}_o)|}\right\},$$

we seek to show that δ satisfies the definition of continuity at $\mathbf{u}_o$ for f and the given ε. If $\mathbf{u}_1$ is chosen to satisfy $|\mathbf{u}_1 - \mathbf{u}_o| < \delta$, then setting

$$a = f(\mathbf{u}_1) - f(\mathbf{u}_o) \quad \text{and} \quad b = \nabla f(\mathbf{u}_o)\cdot(\mathbf{u}_1 - \mathbf{u}_o)$$

and applying the real-number inequality, $|a| \le |a - b| + |b|$ gives (see Proofs, exercise 1),

$$\begin{aligned}
|f(\mathbf{u}_1) - f(\mathbf{u}_o)| &= |a| \\
&\le |a - b| + |b| \\
&< |\mathbf{u}_1 - \mathbf{u}_o| + |\nabla f(\mathbf{u}_o)\cdot(\mathbf{u}_1 - \mathbf{u}_o)| \\
&\le |\mathbf{u}_1 - \mathbf{u}_o| + |\nabla f(\mathbf{u}_o)|\,|\mathbf{u}_1 - \mathbf{u}_o| \\
&\le |\mathbf{u}_1 - \mathbf{u}_o|(1 + |\nabla f(\mathbf{u}_o)|) \\
&< \frac{\varepsilon}{1 + |\nabla f(\mathbf{u}_o)|}(1 + |\nabla f(\mathbf{u}_o)|) \\
&= \varepsilon,
\end{aligned}$$

completing the proof.

Questions

1. The approximation theorem says that $f(\mathbf{u})$ is approximated near $\mathbf{u}_o$ by ________.

2. A function f is continuous provided ________.
   (a) all vector derivatives exist,
   (b) all partial derivatives exist and are equal,
   (c) all partial derivatives are continuous.

Problems

1. Do Problem Set D at the end of the chapter.

Exercises

1. Given $f(x, y) = xe^y$ and $\mathbf{u}_o = \langle 1, 0\rangle$, $\mathbf{u} = \langle a, b\rangle$, show that if $|\mathbf{u} - \mathbf{u}_o|$ is small, then $f(\mathbf{u})$ can be approximated by $a + b$.

Proofs

1. Justify each step in the chain of inequalities in the proof of Proposition 5.1.

2. Given $f(x, y) = (xy)/(x^2 + y^2)$, with $f(0, 0) = 0$:
   (a) Show that f_x and f_y exist at $\langle 0, 0\rangle$.
   (b) Show f is not continuous at $\langle 0, 0\rangle$.

Problems

A. *Partial Derivatives; Chain Rule*

Given $g(x, y)$, the partial derivatives g_x and g_y of g may be found by the following rules.

A.1 (a) g_x is obtained by taking the (ordinary) derivative of g with respect to x, treating y as a constant.
(b) g_y is obtained by taking the derivative of g with respect to y, treating x as a constant.

1. Find g_x and g_y if $g(x, y) =$
   (a) x^2y^3, (b) $x \cos xy$, (c) $\ln (x + \cos y)$.

Given $g(x, y, z)$, then g_x, g_y, and g_z are found by a procedure similar to that in A.1. In each case, all but one variable is treated as a constant while applying the derivative operation.

2. Find g_x, g_y, and g_z if $g(x, y, z) =$
   (a) $x^2y^3z^4$, (b) e^{xy^2+z}.

If g is a scalar function and $\mathbf{f}$ a vector function so that $g \circ \mathbf{f}$ is defined, then the *chain-rule formula* gives the partial derivatives of $g \circ \mathbf{f}$ in terms of the partial derivatives of g and the coordinate functions of $\mathbf{f}$. We cite a particular case.

A.2 If $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, then
(a) $(g \circ \mathbf{f})_r = (g_x \circ \mathbf{f})\bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_r$,
(b) $(g \circ \mathbf{f})_s = (g_x \circ \mathbf{f})\bar{x}_s + (g_y \circ \mathbf{f})\bar{y}_s$.

An abbreviated form of A.2(a) is

$$g_r = g_x \bar{x}_r + g_y \bar{y}_r.$$

3. By analogy, write the chain-rule formula for g_r given

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}$$

   and

$$g(x, y, z).$$

4. Given $\mathbf{f}(r, s) = r^2s\mathbf{i} + (r + s)\mathbf{j}$ and $g(x, y) = 2x + y^2$:
   (a) Find $g \circ \mathbf{f}$.
   (b) From (a) find $(g \circ \mathbf{f})_r$.
   (c) Find $(g \circ \mathbf{f})_r$ by the chain-rule formula and compare it with the result of (b).
5. Using the chain-rule formula,
   (a) find g_r at $(r, s) = (1, 0)$ if $\mathbf{f}(r, s) = re^s\mathbf{i} + (r^2 + s^2)\mathbf{j}$ and $g(x, y) = x^2 + y$,
   (b) find g_s at $(r, s) = (1, \pi)$ if

$$f(r, s) = r^2\mathbf{i} + (r + s)\mathbf{j} + \cos s\, \mathbf{k}$$

   and

$$g(x, y, z) = xy - z.$$

6. Given that $g(x, y)$ is arbitrary and $\mathbf{f}(r, \theta) = r\cos\theta\ \mathbf{i} + r\sin\theta\,\mathbf{j}$, find
   (a) g_r, (b) g_θ.
7. Given $g(x, y, z)$ and $\mathbf{f}(r, \theta, z) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\mathbf{k}$, find (a) g_r, (b) g_θ, (c) g_z.
8. Given $g(x, y, z)$ and

$$\mathbf{f}(\rho, \phi, \theta) = \rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j} + \rho\cos\phi\,\mathbf{k},$$

   find
   (a) g_ρ,
   (b) g_ϕ,
   (c) g_θ.

9. Given $\mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$ and $g(x, y)$, find g_x and g_y in terms of g_r and g_θ as follows:
   (a) Using Problem 6, find a 2×2 matrix A such that
   $$\begin{bmatrix} g_r \\ g_\theta \end{bmatrix} = A \begin{bmatrix} g_x \\ g_y \end{bmatrix}.$$
   (b) Multiply the matrix equation in (a) by A^{-1} and solve for g_x and g_y.
10. Given $\mathbf{f}(r, s) = (r^2 - s)\mathbf{i} + (r + s^2)\mathbf{j}$ and $g(x, y)$, find g_x and g_y in terms of g_r and g_s.

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11. Find g_x and g_y if $g(x, y) =$
    (a) x^2y^6, (b) $\arctan\dfrac{y}{x}$, (c) $\sin^2(xy^2)$.
12. Find g_x, g_y, and g_z if $g(x, y, z) =$
    (a) xy^2e^{xz}, (b) $x\ln(yz^2)$.
13. Given $g(x, y)$ and $\mathbf{f}(r, s, t) = \bar{x}(r, s, t)\mathbf{i} + \bar{y}(r, s, t)\mathbf{j}$, give the abbreviated chain-rule formula for g_r.
14. Using the chain rule find g_r at $(r, s) = (1, 2)$ if
    $$\mathbf{f}(r, s) = rs\mathbf{i} + (r - s^2)\mathbf{j} + 2r^2\mathbf{k} \quad \text{and} \quad g(x, y, z) = x^2yz.$$
15. Given $\mathbf{f}(r, s) = r^2s\mathbf{i} + (r - 2s)\mathbf{j}$ and $g(x, y)$, find
    (a) g_r, (b) g_s, (c) g_x and g_y in terms of g_r and g_s.

B. *Implicit Differentiation*

From elementary calculus we know that if $g(x, y) = 0$ is satisfied by (x_0, y_0), then, with possible certain exceptions, there is a real function $h(x)$ such that $g(x, h(x)) = 0$ for all x in some open interval containing x_0. For example, if $g(x, y) = x^2 + y^2 - 1$, then such a function h can be found provided $y_0 \neq 0$; if $y_0 > 0$, set $h(x) = \sqrt{1 - x^2}$, and if $y_0 < 0$, set $h(x) = -\sqrt{1 - x^2}$. It may be possible to compute the derivative of h at x_0 even though an explicit formula for h cannot be obtained. The technique is called *implicit differentiation*, and the procedure is to substitute $y = h(x)$ and differentiate $g(x, h(x)) = 0$ with respect to x using the chain-rule formula. Then $dh/dx(x_0)$ may be evaluated from the resulting equation.

1. Given that $x^3 + y^3 - 2 = 0$ is satisfied by $y = h(x)$ near $(1, 1)$, find $dh/dx(1)$ using the following procedure.
   (a) Substitute $y = h(x)$ in the given equation.
   (b) Apply the chain rule to the equation in (a).
   (c) From (b) solve for $dh/dx(1)$.

2. Given that $xe^y + y^2 - 2 = 0$ is satisfied by $y = h(x)$ near (2,0), find $dh/dx(2)$.
3. Given that the pair of equations

$$x^2 + y^2 + rs - 6 = 0 \quad \text{and} \quad xy - rs^2 - 1 = 0$$

have solutions $x = \bar{x}(r, s)$, $y = \bar{y}(r, s)$ near $(r, s, x, y) = (1, 1, 1, 2)$, find

$$\bar{x}_r, \quad \bar{x}_s, \quad \bar{y}_r, \quad \text{and} \quad \bar{y}_s$$

at $(r, s) = (1, 1)$ using the following steps.
   (a) Substitute $x = \bar{x}(r, s)$ and $y = \bar{y}(r, s)$ into the given equations and by the chain rule obtain two pairs of equations with the desired unknowns as coefficients.
   (b) Solve the system in (a) at $(r, s, \bar{x}, \bar{y}) = (1, 1, 1, 2)$.
4. Given that

$$2x^2 + y^2 - rs - 4 = 0 \quad \text{and} \quad xy - rs = 0$$

have solutions $x = \bar{x}(r, s)$, $y = \bar{y}(r, s)$ near $(r, s, x, y) = (1, 2, 1, 2)$; find

$$\bar{x}_r, \quad \bar{x}_s, \quad \bar{y}_r, \quad \text{and} \quad \bar{y}_s$$

at $(r, s) = (1, 2)$.

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5. Given that $r^2 \cos x - se^x = 0$ has as a solution $x = \bar{x}(r, s)$ near $(r, s, x) = (1, 1, 0)$, find $\bar{x}_r$ and $\bar{x}_s$ at $(r, s) = (1, 1)$.
6. Given that

$$rs^2 + xe^s + r\cos y = 0 \quad \text{and} \quad r\sin y + xs^2 = 0$$

have as a common solution $x = \bar{x}(r, s)$, $y = \bar{y}(r, s)$ near $(r, s, x, y) = (2, 0, 2, \pi)$, find

$$\bar{x}_r, \quad \bar{x}_s, \quad \bar{y}_r, \quad \text{and} \quad \bar{y}_s$$

at $(r, s) = (2, 0)$.

C. *Gradient Vectors*

Given $f(x, y)$, then the *gradient* of f is the vector function

C.1 $\quad \nabla f(x, y) = f_x\mathbf{i} + f_y\mathbf{j}.$

Similarly, the gradient of $f(x, y, z)$ is

C.2 $\quad \nabla f(x, y, z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}.$

1. Find ∇f:
   (a) $f(x, y) = x^2y^3$, (b) $f(x, y) = e^{xy^2}$,
   (c) $f(x, y, z) = x(y^2 + z)$, (d) $f(x, y, z) = e^x \sin yz$.
2. Find $\nabla f(1, 3)$ in 1(a) and (b).
3. Find $\nabla f(0, 0, 2)$ in 1(c) and (d).

The *vector derivative* of f at $\mathbf{u}_o$ with respect to $\mathbf{v}_o$ may be computed by the formula

C.3 $$f_{\mathbf{v}_o}(\mathbf{u}_o) = \nabla f(\mathbf{u}_o) \cdot \mathbf{v}_o.$$

4. Find $f_{\mathbf{v}_o}(\mathbf{u}_o)$:
   (a) $f(x, y) = xy^2$, $\mathbf{u}_o = \langle 1, 2\rangle$, $\mathbf{v}_o = \langle 3, -1\rangle$,
   (b) $f(x, y) = e^{xy}$, $\mathbf{u}_o = \mathbf{j}$, $\mathbf{v}_o = 2\mathbf{i} + 3\mathbf{j}$,
   (c) $f(x, y, z) = x \ln yz$, $\mathbf{u}_o = \langle 2, 4, 1\rangle$, $\mathbf{v}_o = \langle 0, 2, 3\rangle$.

The *directional derivative* of $\mathbf{f}$ at $\mathbf{u}_o$ in the direction of $\mathbf{v}_o$, $\mathbf{v}_o \neq \mathbf{0}$, is

C.4 $$\frac{df}{d\mathbf{v}_o}(\mathbf{u}_o) = \nabla f(\mathbf{u}_o) \cdot \frac{\mathbf{v}_o}{|\mathbf{v}_o|}.$$

5. Find $df/d\mathbf{v}_o(\mathbf{u}_o)$ in 4(a), (b), (c).

For a fixed $\mathbf{u}_o$, the maximum value of $df/d\mathbf{v}(\mathbf{u}_o)$ occurs when $\mathbf{v} = \nabla f(\mathbf{u}_o)$; the maximum value of $df/d\mathbf{v}(\mathbf{u}_o)$ is then $|\nabla f(\mathbf{u}_o)|$.

6. For each of the following f and $\mathbf{u}_o$ find first a vector for $\mathbf{v}$ which maximizes $df/d\mathbf{v}(\mathbf{u}_o)$ and then the maximum value of $df/d\mathbf{v}(\mathbf{u}_o)$.
   (a) $f(x, y) = x^2y^3$, $\mathbf{u}_o = \langle 1, 3\rangle$;
   (b) $f(x, y) = x \sin xy$, $\mathbf{u}_o = \mathbf{i} + \pi\mathbf{j}$;
   (c) $f(x, y, z) = x^2ye^z$, $\mathbf{u}_o = \langle 2, 1, 0\rangle$.

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7. Find $\nabla f(\mathbf{u}_o)$, $f_{\mathbf{v}_o}(\mathbf{u}_o)$, and $df/d\mathbf{v}_o(\mathbf{u}_o)$ in each case below.
   (a) $f(x, y) = x + 3y$, $\mathbf{u}_o = \langle 1, 2\rangle$, $\mathbf{v}_o = \langle 3, 2\rangle$;
   (b) $f(x, y) = x^2 - 2xy + y^3$, $\mathbf{u}_o = \mathbf{i} + \mathbf{j}$, $\mathbf{v}_o = 2\mathbf{i}$;
   (c) $f(x, y, z) = \cos xz + \sin xy$, $\mathbf{u}_o = \langle 1, 0, \pi\rangle$, $\mathbf{v}_o = \langle 1, 2, 1\rangle$.
8. For f and $\mathbf{u}_o$ in 7(a), (b), and (c) find a vector $\mathbf{v}$ which maximizes $df/d\mathbf{v}(\mathbf{u}_o)$; also find the maximum value of $df/d\mathbf{v}(\mathbf{u}_o)$.

D. *Approximations*

The difference between the values of a function $f(x, y)$ between nearby points (x_o, y_o) and (x_1, y_1) is given approximately by

D.1 $$f(x_1, y_1) - f(x_o, y_o) \approx f_x(x_o, y_o)(x_1 - x_o) + f_y(x_o, y_o)(y_1 - y_o).$$

The approximation in D.1 is often abbreviated

$$\Delta f \approx f_x \Delta x + f_y \Delta y,$$

where the symbol "Δ" denotes a small change or difference.

1. Approximate $[(.001)^2 + (8.0003)^2]^{1/3}$ using the method outlined below.
   (a) Let $f(x, y) = (x^2 + y^2)^{1/3}$, $(x_o, y_o) = (0, 8)$, $(x_1, y_1) = (.001, 8.0003)$ and evaluate f, f_x, and f_y at (x_o, y_o).
   (b) Use D.1 to obtain the desired value.
2. Approximate $[7(1.001)^3 + (2.999)^2]^{1/4}$.

A similar approximation formula exists for $f(x, y, z)$. In abbreviated form it reads

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z.$$

3. Approximate $[(5.0001)^2 + (6.001)^2 + 3(1.001)^4]^{2/3}$.

The approximation formulas are often used to estimate the maximum possible error in f produced by measured errors in x, y, and z.

4. The sides of a rectangle are measured and found to be 4 and 7 with a possible error of .1 each. By how much can the area of the rectangle differ from 28?
5. By measurement, a cylinder is found to have base radius 10 and height 4 with a possible error of .1 in each measurement. What is the largest possible volume of the cylinder?
6. A racer in the Indianapolis 500 mile race was clocked at an average speed of 150 mph. If the total distance measurement has a possible error of .1 mile and the possible time error is 1 second, what is the maximum possible error in the average speed?

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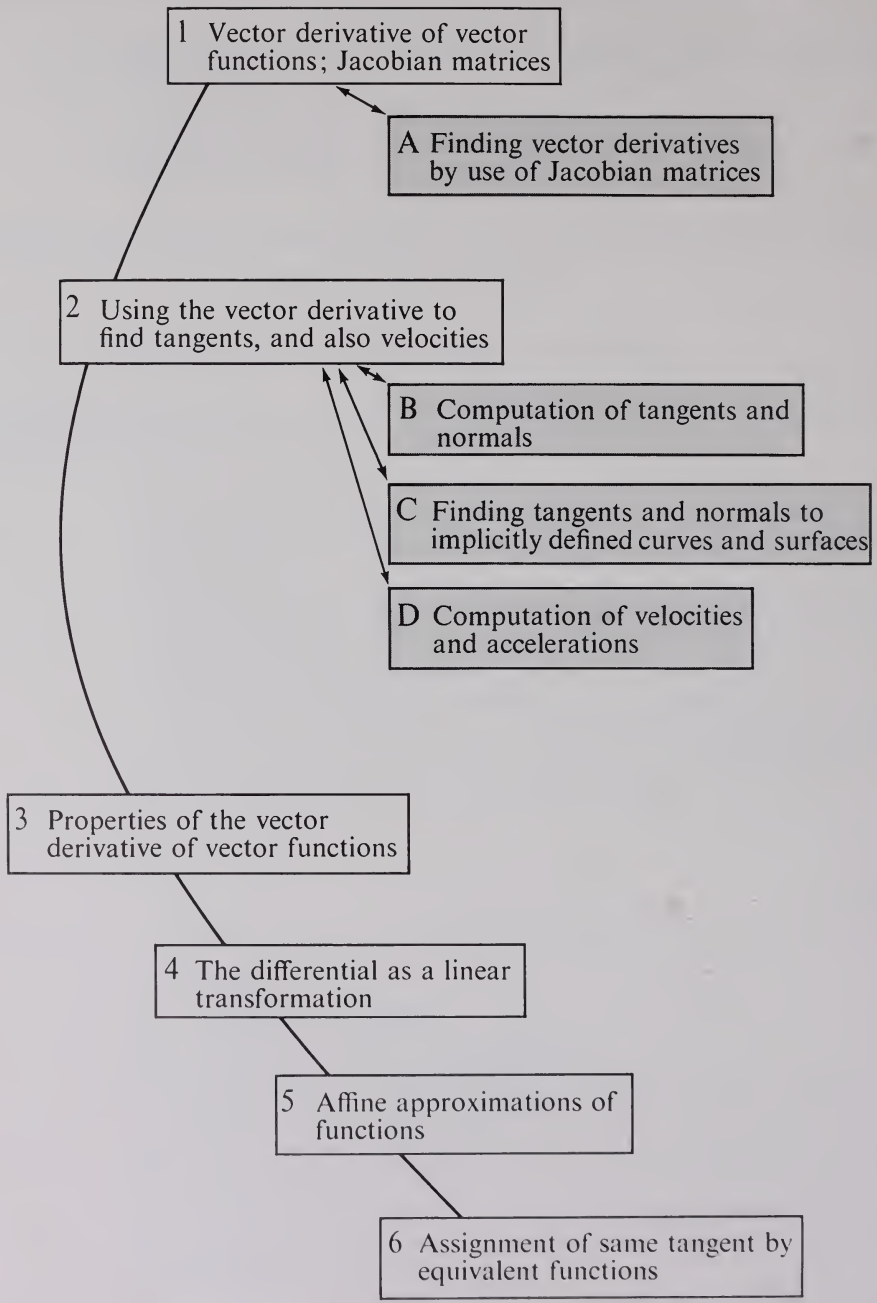
7. Approximate
   (a) $\sqrt{(3.001)^2 + 5(1.999)^3}$,
   (b) $[(2.0001)^2 + (.999)^3 + 3(1.0001)^2]^{1/3}$.
8. The hypotenuse of a right triangle is computed from measured legs of lengths 3 and 4. If the measurements have a possible error of .01 each, what is the maximum possible error in the computed hypotenuse?
9. The average speed of an airplane was computed to be 600 miles per hour over a distance of 2400 miles. If the possible errors in distance and time measurement are 1 mile and 1 minute, respectively, what is the maximum possible error in the average speed?


1 Vector derivative of vector functions; Jacobian matrices
A Finding vector derivatives by use of Jacobian matrices
2 Using the vector derivative to find tangents, and also velocities
B Computation of tangents and normals
C Finding tangents and normals to implicitly defined curves and surfaces
D Computation of velocities and accelerations
3 Properties of the vector derivative of vector functions
4 The differential as a linear transformation
5 Affine approximations of functions
6 Assignment of same tangent by equivalent functions

XI

Vector Derivatives of Vector Functions; Differentials

In this chapter we introduce vector derivatives of vector functions. Many of the definitions and results are a natural extension of the vector derivative theory of scalar functions. Vector derivatives of vector functions have numerous applications, such as in the study of tangent lines and planes, and in physics problems on velocity and acceleration. They can be related to linear functions, called *differentials*. The matrices of these linear functions, called *Jacobian matrices*, may be used to perform many useful computations.

1. Vector Derivatives of Vector Functions

In the previous chapter it was observed that the major results for scalar functions, such as the chain rule, gradient formula, and approximation theorem, required continuity of partial derivatives. In the extension to vector functions this property again plays an underlying role. Let $\mathbf{f}$ denote a vector function having for its domain an open set $\mathbf{U}$ in $\mathbf{R}^n$ and its range $\mathbf{R}^m$. Then $\mathbf{f}$ is

defined to be *continuously differentiable*, or *of class* C^1, provided that all of its coordinate functions are of class C^1. For example,

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$$

is of class C^1 if and only if $\bar{x}_r$, $\bar{x}_s$, $\bar{y}_r$, and $\bar{y}_s$ are all continuous.

Proposition 1.1 If $\mathbf{f}$ is of class C^1, then $\mathbf{f}$ is continuous.

The proof follows from the corresponding property for scalar functions and the coordinate theorem for continuity. The next result implies that the class C^1 functions from $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$ form a vector space.

Proposition 1.2 If $\mathbf{f}$ and $\mathbf{g}$ are of class C^1, then so are
(a) $\mathbf{f} + \mathbf{g}$, and
(b) $c\mathbf{f}$.

We prove (a) for the case in which $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ and $\mathbf{g}(r, s) = \bar{x}'(r, s)\mathbf{i} + \bar{y}'(r, s)\mathbf{j}$. By hypotheses, $\bar{x}_r$, $\bar{x}_s$, $\bar{x}_r{}'$, and $\bar{x}_s{}'$ are all continuous and, therefore,

$$(\bar{x} + \bar{x}')_r = \bar{x}_r + \bar{x}'_r \quad \text{and} \quad (\bar{x} + \bar{x}')_s = \bar{x}_s + \bar{x}'_s$$

are continuous, since addition preserves continuity. Similarly,

$$(\bar{y} + \bar{y}')_r = \bar{y}_r + \bar{y}'_r \quad \text{and} \quad (\bar{y} + \bar{y}')_s = \bar{y}_s + \bar{y}'_s$$

are continuous, and from

$$\mathbf{f} + \mathbf{g} = (\bar{x} + \bar{x}')\,\mathbf{i} + (\bar{y} + \bar{y}')\mathbf{j}$$

it follows that $\mathbf{f} + \mathbf{g}$ is of class C^1 as desired.

We next observe that the C^1 property is preserved by other operations.

Proposition 1.3 If f, g, $\mathbf{f}$ and $\mathbf{g}$ are of class C^1, then so is

(a) fg, (b) $\dfrac{f}{g}$, (c) $f\mathbf{g}$,

(d) $\dfrac{\mathbf{f}}{g}$, (e) $\mathbf{f} \cdot \mathbf{g}$, and (f) $\mathbf{f} \times \mathbf{g}$.

The proof of (a) for $f(x, y)$, $g(x, y)$ proceeds from the equalities,

$$(fg)_x = fg_x + f_x g \quad \text{and} \quad (fg)_y = fg_y + f_y g\,.$$

Since the C^1 functions f and g are also continuous, the preservation of continuity by the sum and product operations gives the desired conclusion. Other proofs are similar.

The theory in this chapter is based almost entirely on the C^1 property, and we shall in this chapter assume all given functions are of class C^1.

Definition of Vector Derivative of a Vector Function

The vector derivative *of* **f** *at* $\mathbf{u}_o$ *with respect to* $\mathbf{v}_o$ *is*

$$\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{f}(\mathbf{u}_o)}{t}.$$

This definition is an evident extension of that for $f_{\mathbf{v}_o}(\mathbf{u}_o)$. If $\mathbf{f}(x)$ is a vector function of one variable, then the symbol

$$\frac{d\mathbf{f}}{dx}$$

will often be used to denote the vector derivative with respect to the unit vector $\langle 1\rangle$. Given $\mathbf{f}(x, y)$, then the *partial derivatives of f* are $\mathbf{f}_x = \mathbf{f}_\mathbf{i}$ and $\mathbf{f}_y = \mathbf{f}_\mathbf{j}$. More generally, the ith partial derivative of $\mathbf{f}(x_1, \ldots, x_n)$ is $\mathbf{f}_{x_i} = \mathbf{f}_{\mathbf{e}_i}$.

The assumption that **f** is of class C^1 implies that the limit necessarily exists in the vector derivative definition. Furthermore, the vector derivative of **f** can be found from the vector derivatives of the coordinate functions of **f**. For instance, given $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, then

$$\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \bar{x}_{\mathbf{v}_o}(\mathbf{u}_o)\mathbf{i} + \bar{y}_{\mathbf{v}_o}(\mathbf{u}_o)\mathbf{j}.$$

A proof is

$$\begin{aligned}
&\bar{x}_{\mathbf{v}_o}(\mathbf{u}_o)\mathbf{i} + \bar{y}_{\mathbf{v}_o}(\mathbf{u}_o)\mathbf{j} \\
&\quad = \left(\lim_{t\to 0} \frac{\bar{x}(\mathbf{u}_o + t\mathbf{v}_o) - \bar{x}(\mathbf{u}_o)}{t}\right)\mathbf{i} + \left(\lim_{t\to 0} \frac{\bar{y}(\mathbf{u}_o + t\mathbf{v}_o) - \bar{y}(\mathbf{u}_o)}{t}\right)\mathbf{j} \\
&\quad = \lim_{t\to 0} \left(\frac{\bar{x}(\mathbf{u}_o + t\mathbf{v}_o) - \bar{x}(\mathbf{u}_o)}{t}\mathbf{i} + \frac{\bar{y}(\mathbf{u}_o + t\mathbf{v}_o) - \bar{y}(\mathbf{u}_o)}{t}\mathbf{j}\right) \\
&\quad = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{f}(\mathbf{u}_o)}{t} \\
&\quad = \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o).
\end{aligned}$$

The general result for $\mathbf{f} = \langle \bar{f}_1, \ldots, \bar{f}_m\rangle$ is expressed in the following theorem.

Coordinate Theorem for Vector Derivatives

$$\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \langle (\bar{f}_1)_{\mathbf{v}_o}(\mathbf{u}_o), \ldots, (\bar{f}_m)_{\mathbf{v}_o}(\mathbf{u}_o)\rangle.$$

Example 1.1 Given $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} = r^2s^3\mathbf{i} + (2r - s^2)\mathbf{j}$ and $\mathbf{u}_o = \mathbf{i} - 2\mathbf{j}$, then $\bar{x}_r = 2rs^3$, $\bar{y}_r = 2$. Hence,

$$\mathbf{f}_r(\mathbf{u}_o) = 2rs^3\mathbf{i} + 2\mathbf{j}|_{\mathbf{i}-2\mathbf{j}} = -16\mathbf{i} + 2\mathbf{j}.$$

We shall now derive a matrix method for computing $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$. Given

$$\mathbf{f}(r, s) = \langle \bar{x}(r, s), \bar{y}(r, s) \rangle \quad \text{and} \quad \mathbf{u}_o,$$

then by the gradient formula

$$\begin{aligned} \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) &= \langle \bar{x}_{\mathbf{v}_o}(\mathbf{u}_o), \bar{y}_{\mathbf{v}_o}(\mathbf{u}_o) \rangle \\ &= \langle \nabla\bar{x}(\mathbf{u}_o) \cdot \mathbf{v}_o, \nabla\bar{y}(\mathbf{u}_o) \cdot \mathbf{v}_o \rangle. \end{aligned}$$

The right side of this equality may be written in matrix form as the product of a 2×2 matrix with row vectors $\nabla\bar{x}(\mathbf{u}_o)$ and $\nabla\bar{y}(\mathbf{u}_o)$ and the column matrix $\mathbf{v}_o$.

$$\begin{bmatrix} \nabla\bar{x}(\mathbf{u}_o) \\ \nabla\bar{y}(\mathbf{u}_o) \end{bmatrix} \begin{bmatrix} \mathbf{v}_o \end{bmatrix}.$$

The product is a column matrix which corresponds to the vector $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$.

Example 1.2 Let

$$\begin{aligned} \mathbf{f} = \langle \bar{x}, \bar{y} \rangle &= \langle rs^2, 3r - s \rangle, \\ \mathbf{u}_o &= \langle 1, 3 \rangle, \\ \mathbf{v}_o &= \langle 2, -1 \rangle. \end{aligned}$$

Then

$$\begin{aligned} \nabla\bar{x}(\mathbf{u}_o) &= \langle s^2, 2rs \rangle|_{\langle 1, 3 \rangle} = \langle 9, 6 \rangle, \\ \nabla\bar{y}(\mathbf{u}_o) &= \langle 3, -1 \rangle. \end{aligned}$$

Hence, from

$$\begin{bmatrix} 9 & 6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix},$$

we obtain $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \langle 12, 7 \rangle$.

In general, applying the gradient formula to the coordinate theroem for vector derivatives gives the following result.

Proposition 1.4

$$\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \langle \nabla\bar{f}_1(\mathbf{u}_o) \cdot \mathbf{v}_o, \ldots, \nabla\bar{f}_m(\mathbf{u}_o) \cdot \mathbf{v}_o \rangle.$$

From Proposition 1.4 we may now derive a matrix method for computing $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$. The *Jacobian matrix*, $J_{\mathbf{f}}(\mathbf{u}_o)$, of $\mathbf{f}$ at $\mathbf{u}_o$ is defined to be the $m \times n$ matrix whose ith row is $\nabla\bar{f}_i(\mathbf{u}_o)$; hence the jth column is $\mathbf{f}_{x_j}(\mathbf{u}_o)$. Multiplication of $J_{\mathbf{f}}(\mathbf{u}_o)$

by the vector $\mathbf{v}_o$, identified with its column matrix, may be symbolized as shown below.

$$J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{v}_o = \begin{bmatrix} \nabla\bar{f}_1(\mathbf{u}_o) \\ \nabla\bar{f}_2(\mathbf{u}_o) \\ \vdots \\ \nabla\bar{f}_m(\mathbf{u}_o) \end{bmatrix} \begin{bmatrix} \\ \mathbf{v}_o \\ \\ \end{bmatrix} = \begin{bmatrix} \nabla\bar{f}_1(\mathbf{u}_o)\cdot\mathbf{v}_o \\ \nabla\bar{f}_2(\mathbf{u}_o)\cdot\mathbf{v}_o \\ \vdots \\ \nabla\bar{f}_m(\mathbf{u}_o)\cdot\mathbf{v}_o \end{bmatrix}.$$

Comparison of this matrix equality with Proposition 1.4 gives the next formula.

Proposition 1.5

$$\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{v}_o.$$

Questions

1. By definition, $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t\to 0}$ ______.
2. $d\mathbf{f}/dx$ is the vector derivative of $\mathbf{f}(x)$ with respect to ______.
3. The ith row of $J_{\mathbf{f}}(\mathbf{u}_o)$ is the vector ______.
4. The vector derivative $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$ exists provided that ______.
   (a) $\mathbf{f}$ if continuous,
   (b) all partial derivatives of $\mathbf{f}$ exist,
   (c) $\mathbf{f}$ is of class C^1.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. Prove that if $\mathbf{f} = \langle \bar{f}_1, \bar{f}_2 \rangle$ is of class C^1, then so is $c\mathbf{f}$.

2. Applications of the Vector Derivative

In this section we shall apply vector derivatives to a study of

(I) tangents and normals to curves and surfaces described by vector equations, and
(II) velocity and acceleration vectors associated with an object moving in space.

The development will be informal, in that there will be no attempt to find a precise relationship between tangents and the sets from which they are derived;

this is reserved for the section on affine approximations. On the other hand, we shall introduce certain properties of vector functions which are sufficient for them to produce meaningful tangents.

The tangent line to a differentiable function $f(x)$ at $x = c$ is the line through $(c, f(c))$ with slope $df/dx(c)$. It has as its equation

$$y = f(c) + \frac{df}{dx}(c)(x - c).$$

Since

$$\frac{df}{dx}(c) = \lim_{t \to 0} \frac{f(c+t) - f(c)}{t},$$

this tangent line may be regarded as the limit as t approaches 0 of the line

$$y_t = f(c) + \frac{f(c+t) - f(c)}{t}(x - c),$$

which passes through $(c, f(c))$ and $(c+t, f(c+t))$ (see Figure 11.1). It is this notion of the tangent line as a limiting line which we shall use in the study of tangent lines to curves defined by vector equations.

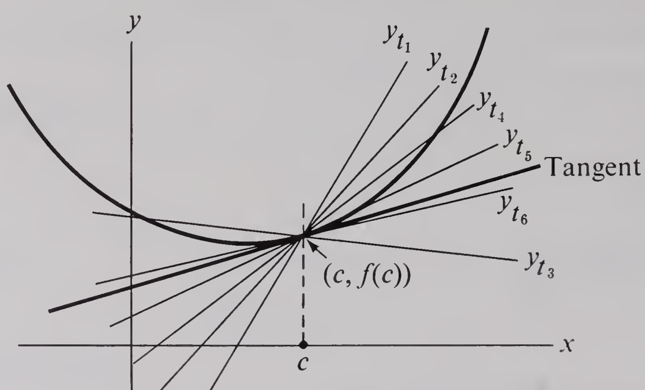


Figure 11.1

Let $\mathscr{C}$ be a curve in the Cartesian plane described by a vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$. It will be assumed that

(a) $\mathbf{f}$ is of class C^1,
(b) $\mathbf{f}$ is injective, and
(c) $\dfrac{d\bar{x}}{dr}$ and $\dfrac{d\bar{y}}{dr}$ never vanish simultaneously $\left(\text{that is, } \left(\dfrac{d\bar{x}}{dr}\right)^2 + \left(\dfrac{d\bar{y}}{dr}\right)^2 \neq 0 \text{ for each } r\right)$.

The class C^1 condition in (a) ensures that $\mathscr{C}$ has no breaks, jumps, or abrupt changes in direction (corners). A curve with this C^1 property is called *smooth*. The injective property in (b) implies that the curve $\mathscr{C}$ does not cross itself. Such a

curve is called *simple*. Property (c) is not so much a property of $\mathscr{C}$ as of the vector equation used to describe it, and the need for this condition will be seen later in this section.

We seek the tangent line to $\mathscr{C}$ at $\mathbf{f}(r_o)$. If $t \neq 0$, then the line through $\mathbf{f}(r_o)$ and $\mathbf{f}(r_o + t)$ is known from vector algebra to be (see Figure 11.2)

$$\mathbf{f}(r_o) + r(\mathbf{f}(r_o + t) - \mathbf{f}(r_o)).$$

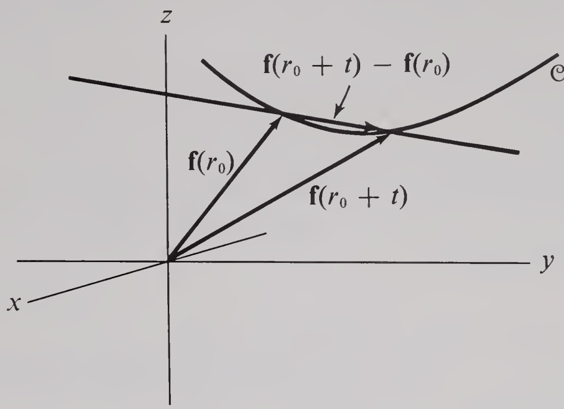


Figure 11.2

Since multiplication of the direction vector by the scalar $1/t$ does not change the line, it may also be written

$$\mathbf{f}(r_o) + r\left(\frac{\mathbf{f}(r_o + t) - \mathbf{f}(r_o)}{t}\right).$$

Letting t approach 0 and observing that

$$\lim_{t \to 0} \frac{\mathbf{f}(r_o + t) - \mathbf{f}(r_o)}{t} = \frac{d\mathbf{f}}{dr}(t_o),$$

we arrive at a reasonable definition of the *tangent line* to $\mathscr{C}$ at $\mathbf{f}(r_o)$; it is

$$\mathbf{f}(r_o) + r\frac{d\mathbf{f}}{dr}(r_o).$$

Using the coordinate theorem for vector derivatives, this line may be written

$$\mathbf{f}(r_o) + r\left(\frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j}\right).$$

Example 2.1 Let $\mathscr{C}$ be the curve described by $\mathbf{f}(r) = r^2\mathbf{i} + r^3\mathbf{j}$. Then from

$$\mathbf{f}(2) = \langle 4, 8 \rangle, \qquad \frac{d\bar{x}}{dr}(2) = 4, \qquad \frac{d\bar{y}}{dr}(2) = 12$$

the tangent line to $\mathscr{C}$ at $(4, 8)$ is

$$(4\mathbf{i} + 8\mathbf{j}) + r(4\mathbf{i} + 12\mathbf{j}) = (4\mathbf{i} + 8\mathbf{j}) + r(\mathbf{i} + 3\mathbf{j}).$$

The necessity for $d\bar{x}/dr$ and $d\bar{y}/dr$ not to vanish simultaneously at r_o is now apparent, for if it does then

$$\frac{d\mathbf{f}}{dr}(r_o) = \mathbf{0},$$

and $\mathbf{f}(r_o) + r\, d\mathbf{f}/dr(r_o)$ is not a line. The next example shows that a curve $\mathscr{C}$ may be described by a class C^1 injective vector function which fails to give a tangent line to $\mathscr{C}$, even when the tangent line exists and can be obtained by an alternate vector function.

Example 2.2 Let $\mathscr{C}$ be the graph of $y = x$. Then $\mathscr{C}$ is described by each of the vector equations $\mathbf{g}(r) = r\mathbf{i} + r\mathbf{j}$ and $\mathbf{f}(r) = r^3\mathbf{i} + r^3\mathbf{j}$. Using $\mathbf{g}$, the tangent line at the origin is easily found to be $r(\mathbf{i} + \mathbf{j})$. However $dr^3/dr = 3r^2 = 0$ when $r = 0$ implies that $\mathbf{f}$ fails to give a tangent line to $\mathscr{C}$ at the origin.

The *normal line* to $\mathscr{C}$ at $\mathbf{f}(r_o)$ is the line which contains $\mathbf{f}(r_o)$ and is perpendicular to the tangent line. If $\mathbf{f}(r_o) + r(a\mathbf{i} + b\mathbf{j})$ denotes the normal line, then the perpendicularity condition implies that

$$\left(\frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j}\right) \cdot (a\mathbf{i} + b\mathbf{j}) = 0.$$

There are infinitely many possibilities for a and b. A convenient choice is

$$a = \frac{d\bar{y}}{dr}(r_o), \qquad b = -\frac{d\bar{x}}{dr}(r_o).$$

This gives, for the normal line to $\mathscr{C}$ at $\mathbf{f}(r_o)$,

$$\mathbf{f}(r_o) + r\left(\frac{d\bar{y}}{dr}(r_o)\mathbf{i} - \frac{d\bar{x}}{dr}(r_o)\mathbf{j}\right).$$

Example 2.3 The curve $\mathscr{C}$ in Example 2.1 has tangent line $(4\mathbf{i} + 8\mathbf{j}) + r(\mathbf{i} + 3\mathbf{j})$. Hence, its normal line is $(4\mathbf{i} + 8\mathbf{j}) + r(3\mathbf{i} - \mathbf{j})$.

We next let $\mathscr{C}$ be a curve in Cartesian space described by the vector equation

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}.$$

As for curves in the plane, it will be assumed that

(a) $\mathbf{f}$ is of class C^1,
(b) $\mathbf{f}$ is injective, and
(c) $\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2 + \left(\frac{d\bar{z}}{dr}\right)^2 \neq 0$ for each r.

By an analysis similar to the plane case, the *tangent line* is seen to be given by

$$\mathbf{f}(r_o) + r\frac{d\mathbf{f}}{dr}(r_o),$$

which may be written

$$\mathbf{f}(r_o) + r\left(\frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j} + \frac{d\bar{z}}{dr}(r_o)\mathbf{k}\right).$$

The *normal plane* to $\mathscr{C}$ at $\mathbf{f}(r_o)$ is the plane which contains $\mathbf{f}(r_o)$ and is perpendicular to the tangent line. Thus, the normal plane is $\mathbf{f}(r_o) + r\mathbf{u}_1 + s\mathbf{u}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are nonparallel (independent) vectors orthogonal to $d\mathbf{f}/dr(r_o)$. A convenient choice that works in most cases is

$$\mathbf{u}_1 = \frac{d\bar{z}}{dr}(r_o)\mathbf{i} - \frac{d\bar{x}}{dr}(r_o)\mathbf{k} \quad \text{and} \quad \mathbf{u}_2 = \frac{d\bar{z}}{dr}(r_o)\mathbf{j} - \frac{d\bar{y}}{dr}(r_o)\mathbf{k}.$$

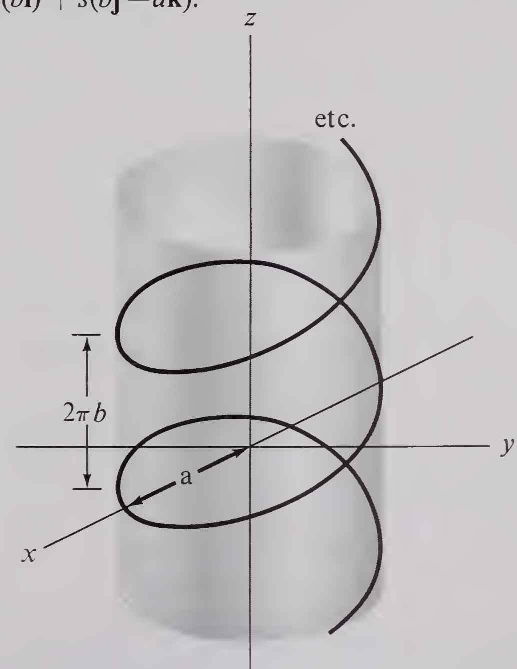
Example 2.4 The curve $\mathscr{C}$ with vector equation

$$\mathbf{f}(\theta) = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j} + b\theta\mathbf{k}$$

is called a *helix* (see Figure 11.3). From

$$d\mathbf{f}/d\theta = -a\sin\theta\mathbf{i} + a\cos\theta\mathbf{j} + b\mathbf{k}$$

the tangent line at $\mathbf{f}(0) = \langle a, 0, 0\rangle$ is $a\mathbf{i} + r(a\mathbf{j} + b\mathbf{k})$. The normal plane is $a\mathbf{i} + r(b\mathbf{i}) + s(b\mathbf{j} - a\mathbf{k})$.



Helix

Figure 11.3

We next consider a surface $\mathscr{S}$ described by

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}.$$

It will be assumed that

(a) $\mathbf{f}$ is of class C^1,
(b) $\mathbf{f}$ is injective, and
(c) $\{\mathbf{f}_r(r, s), \mathbf{f}_s(r, s)\}$ is independent for each (r, s).

The *tangent plane* to $\mathscr{S}$ at $\mathbf{f}(r_o, s_o)$ will be obtained as a limit of planes containing $\mathbf{f}(r_o, s_o)$ and nearby points on $\mathscr{S}$. If t_1 and t_2 are nonzero, then from vector algebra the plane through $\mathbf{f}(r_o, s_o)$, $\mathbf{f}(r_o + t_1, s_o)$, and $\mathbf{f}(r_o, s_o + t_2)$ may be written (see Figure 11.4)

$$\mathbf{f}(r_o, s_o) + r\left(\frac{\mathbf{f}(r_o + t_1, s_o) - \mathbf{f}(r_o, s_o)}{t_1}\right) + s\left(\frac{\mathbf{f}(r_o, s_o + t_2) - \mathbf{f}(r_o, s_o)}{t_2}\right).$$

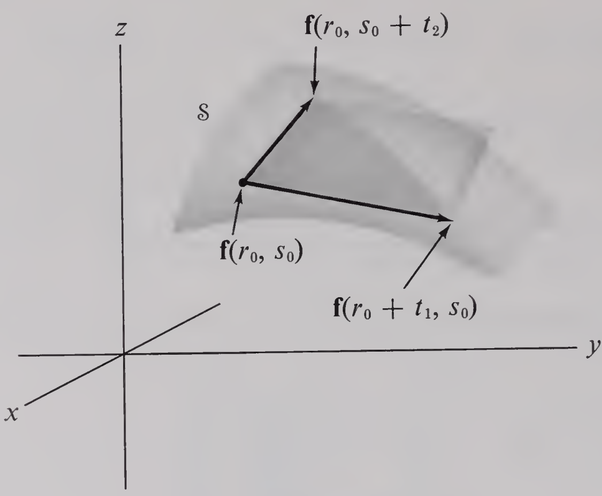


Figure 11.4

Taking the limit as t_1 and t_2 approach 0 gives the limit plane,

$$\mathbf{f}(r_o, s_o) + r\mathbf{f}_r(r_o, s_o) + s\mathbf{f}_s(r_o, s_o).$$

This is defined to be the *tangent plane* to $\mathscr{S}$ at $\mathbf{f}(r_o, s_o)$. The coordinate theorem for vector derivatives may be used to give

$$\mathbf{f}_r = \bar{x}_r\mathbf{i} + \bar{y}_r\mathbf{j} + \bar{z}_r\mathbf{k},$$
$$\mathbf{f}_s = \bar{x}_s\mathbf{i} + \bar{y}_s\mathbf{j} + \bar{z}_s\mathbf{k}.$$

The reason for the imposed requirement in (c) that $\{\mathbf{f}_r(r, s), \mathbf{f}_s(r, s)\}$ be everywhere independent is evident from the tangent plane formula, which could otherwise yield a line or point. The *normal line* to $\mathscr{S}$ at $\mathbf{f}(r_o, s_o)$ has direction perpendicular to each of the direction vectors $\mathbf{f}_r(r_o, s_o)$, $\mathbf{f}_s(r_o, s_o)$ of the tangent

plane. Since the cross-product operation produces a vector perpendicular to two given vectors, it follows that the normal line to $\mathscr{S}$ at $\mathbf{f}(r_o, s_o)$ is

$$\mathbf{f}(r_o, s_o) + r(\mathbf{f}_r(r_o, s_o) \times \mathbf{f}_s(r_o, s_o)).$$

Example 2.5 Let $\mathscr{S}$ be the ellipsoidal surface which is obtained by revolving the curve $\mathbf{g}(\phi) = 2\sin\phi\mathbf{i} + 4\cos\phi\mathbf{k}$ in the xz plane about the z axis (see Figure 11.5). The resulting surface is described by

$$\mathbf{f}(\phi, \theta) = 2\sin\phi\cos\theta\mathbf{i} + 2\sin\phi\sin\theta\mathbf{j} + 4\cos\phi\mathbf{k}.$$

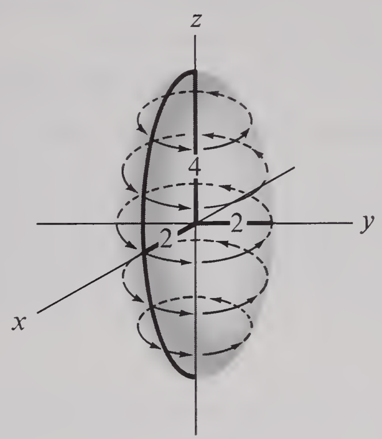


Figure 11.5

If $0 < \phi < \pi$ and $0 \leq \theta < 2\pi$, then $\mathbf{f}$ is injective. From

$$\mathbf{f}_\phi = 2\cos\phi\cos\theta\mathbf{i} + 2\cos\phi\sin\theta\mathbf{j} - 4\sin\phi\mathbf{k},$$
$$\mathbf{f}_\theta = -2\sin\phi\sin\theta\mathbf{i} + 2\sin\phi\cos\theta\mathbf{j},$$

the tangent plane at $\mathbf{f}(\pi/4, \pi/4) = \langle 1, 1, 2\sqrt{2}\rangle$ is

$$(\mathbf{i} + \mathbf{j} + 2\sqrt{2}\mathbf{k}) + r(\mathbf{i} + \mathbf{j} - 2\sqrt{2}\mathbf{k}) + s(-\mathbf{i} + \mathbf{j}).$$

The normal line is

$$(\mathbf{i} + \mathbf{j} + 2\sqrt{2}\mathbf{k}) + r(\mathbf{i} + \mathbf{j} - 2\sqrt{2}\mathbf{k}) \times (-\mathbf{i} + \mathbf{j})$$
$$= (\mathbf{i} + \mathbf{j} + 2\sqrt{2}\mathbf{k}) + r(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 2\mathbf{k}).$$

The surface $\mathscr{S}$, explicitly described by $z = f(x, y)$, has the vector equation $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$. Substitution into the tangent plane formula gives at (x_o, y_o, z_o) on $\mathscr{S}$ the tangent plane

$$(x_o\mathbf{i} + y_o\mathbf{j} + z_o\mathbf{k}) + r(\mathbf{i} + f_x(x_o, y_o)\mathbf{k}) + s(\mathbf{j} + f_y(x_o, y_o)\mathbf{k}).$$

This vector equation yields the parametric equations,

$$x = x_o + r,$$
$$y = y_o + s,$$
$$z = z_o + rf_x(x_o, y_o) + sf_y(x_o, y_o);$$

and elimination of r and s gives the tangent plane in the form

$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o).$$

Example 2.6 Given the surface $z = f(x, y) = x^2y^3$, from $f(2, 1) = 4$, $f_x(2, 1) = 4$, and $f_y(2, 1) = 12$ we find from the previous formula that the tangent plane at $(2, 1, 4)$ is

$$z - 4 = 4(x - 2) + 12(y - 1).$$

In standard form, this plane is $4x + 12y - z - 16 = 0$.

It is worth noting that the imposed injective condition on $\mathbf{f}$ is unnecessarily harsh. The study of a tangent at a point $\mathbf{f}(\mathbf{u}_o)$ involves only those $\mathbf{u}$ near $\mathbf{u}_o$. Therefore, the assumption that $\mathbf{f}$ be injective can be replaced by the much weaker condition that $\mathbf{f}$ be injective on some open interval (for curves) or open disc (for surfaces) centered at $\mathbf{u}_o$.

We shall now study an object moving in space. It will be assumed that a Cartesian axis system has been imposed on space, and that the path of the object is described by a class C^1 position vector function $\mathbf{f}(t) = \bar{x}(t)\mathbf{i} + \bar{y}(t)\mathbf{j} + \bar{z}(t)\mathbf{k}$, where t denotes time. If t_o is a fixed time and $s \neq 0$, then the change in position between t_o and $t_o + s$ is $\mathbf{f}(t_o + s) - \mathbf{f}(t_o)$. Dividing by s gives the *average rate of change of position*,

$$\frac{\mathbf{f}(t_o + s) - \mathbf{f}(t_o)}{s}.$$

Taking the limit as s approaches 0 gives, by definition, the *velocity*

$$\mathbf{v}(t_o) = \frac{d\mathbf{f}}{dt}(t_o).$$

The *velocity function* of the object is therefore, $\mathbf{v} = d\mathbf{f}/dt$. The *acceleration* $\mathbf{a}$ is the velocity of the velocity; it is defined by $\mathbf{a} = d\mathbf{v}/dt$. Using the coordinate theorem for vector derivatives, we summarize:

(a) position vector function $\quad \mathbf{f}(t) = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$,

(b) velocity vector function $\quad \mathbf{v}(t) = \dfrac{d\bar{x}}{dt}\mathbf{i} + \dfrac{d\bar{y}}{dt}\mathbf{j} + \dfrac{d\bar{z}}{dt}\mathbf{k}$,

(c) acceleration vector function $\quad \mathbf{a}(t) = \dfrac{d^2\bar{x}}{dt^2}\mathbf{i} + \dfrac{d^2\bar{y}}{dt^2}\mathbf{j} + \dfrac{d^2\bar{z}}{dt^2}\mathbf{k}$.

If the movement of an object having mass m is produced by a physical force $\mathbf{F}$, then the physics principle $\mathbf{F} = m\mathbf{a}$ applies.

Example 2.7 It is desired to find the position vector, $\mathbf{f}(t) = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$, of an object which has mass 1/12 and is acted upon by a force $\mathbf{F}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$; the position and velocity at time $t = 0$ are given by $\mathbf{f}(0) = \mathbf{k}$ and $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. From $\mathbf{F}(t) = m\, d\mathbf{v}/dt$, it is seen that

$$\frac{1}{12}\frac{d^2\bar{x}}{dt^2} = t$$

and hence, $d\bar{x}/dt = 6t^2 + c$ for some constant c. From $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ it follows that

$$2 = \frac{d\bar{x}}{dt}(0) = c.$$

Proceeding similarly with the second and third coordinates gives

$$\mathbf{v} = (6t^2 + 2)\mathbf{i} + (4t^3 + 1)\mathbf{j} + (3t^4 + 3)\mathbf{k}.$$

Continuing, $d\bar{x}/dt = 6t^2 + 2$ gives $\bar{x} = 2t^3 + 2t + c'$, and application of $\mathbf{f}(0) = \mathbf{k}$ yields $0 = \bar{x}(0) = c'$. Following the same steps with $\bar{y}$ and $\bar{z}$,

$$\mathbf{f}(t) = (2t^3 + 2t)\mathbf{i} + (t^4 + t)\mathbf{j} + (\tfrac{3}{5}t^5 + 3t + 1)\mathbf{k}.$$

Questions

1. The graph of $\mathbf{f}(r)$ does not cross itself provided $\mathbf{f}$ is ______.
2. If $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ assigns a tangent line at $\mathbf{f}(r_o)$, then $d\bar{x}/dr(r_o)$ and $d\bar{y}/dr(r_o)$ must satisfy ______.
3. If the tangent line at $\mathbf{u}_o$ to a plane curve $\mathscr{C}$ has direction vector $a\mathbf{i} + b\mathbf{j}$, then a direction vector of the normal line is ______.
4. Direction vectors of the tangent at $\mathbf{f}(r_o, s_o)$ to the surface described by $\mathbf{f}(r, s)$ are ______ and ______.
5. The rate of change of the velocity is called the ______.

Problems

1. Do Problem Sets B, C, and D at the end of the chapter.

Exercises

1. Find the tangent and normal line to

$$\frac{(y-4)^2}{3} - \frac{(x-3)^2}{2} = 1$$

at (5, 7). (*Hint*: First show that $d(\cosh r)/dr = \sinh r$ and $d(\sinh r)/dr = \cosh r$.)

Proofs

1. Prove that each normal line to the sphere of radius a, centered at $(0, 0, 0)$, passes through the origin.

2. Prove that each tangent plane to the conical surface obtained by revolving the line $y = x$, $x \geq 0$ about the x axis passes through the origin (the vertex has no tangent).

3. Properties of the Vector Derivative of Vector Functions

In this section various properties of vector derivatives are extended to vector functions, and new properties are derived. The extensions are generally made by use of the coordinate theorem for vector derivatives. All given functions will be assumed to be of class C^1. The *vector derivative function* $\mathbf{f}_{\mathbf{v}_0}$ of $\mathbf{f}$ is defined to be the function having the same domain and range as $\mathbf{f}$, and rule given by $\mathbf{f}_{\mathbf{v}_0}(\mathbf{u}_0)$.

Proposition 3.1

(a) $(\mathbf{f} + \mathbf{g})_{\mathbf{v}_0} = \mathbf{f}_{\mathbf{v}_0} + \mathbf{g}_{\mathbf{v}_0}$,
(b) $(c\mathbf{f})_{\mathbf{v}_0} = c\mathbf{f}_{\mathbf{v}_0}$.

For the proof of (b), when $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, if $\mathbf{u}_0$ is arbitrary in the domain of $\mathbf{f}$, then

$$\begin{aligned}(c\mathbf{f}_{\mathbf{v}_0})(\mathbf{u}_0) &= (c\bar{x})_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{i} + (c\bar{y})_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{j}\\ &= c\bar{x}_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{i} + c\bar{y}_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{j}\\ &= c(\bar{x}_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{i} + \bar{y}_{\mathbf{v}_0}(\mathbf{u}_0)\mathbf{j})\\ &= c\mathbf{f}_{\mathbf{v}_0}(\mathbf{u}_0).\end{aligned}$$

Similar proofs give the additional properties that follow.

Proposition 3.2

(a) $\mathbf{f}_{\mathbf{v}_0 + \mathbf{v}_1} = \mathbf{f}_{\mathbf{v}_0} + \mathbf{f}_{\mathbf{v}_1}$,
(b) $\mathbf{f}_{c\mathbf{v}_0} = c\mathbf{f}_{\mathbf{v}_0}$.

Application of the vector derivative definition to other vector space operations gives our next proposition.

Proposition 3.3

(a) $(f\mathbf{g})_{\mathbf{v}_o} = f\mathbf{g}_{\mathbf{v}_o} + f_{\mathbf{v}_o}\mathbf{g}$,

(b) $\left(\dfrac{\mathbf{f}}{g}\right)_{\mathbf{v}_o} = \dfrac{\mathbf{f}_{\mathbf{v}_o}g - \mathbf{f}g_{\mathbf{v}_o}}{g^2}$,

(c) $(\mathbf{f} \cdot \mathbf{g})_{\mathbf{v}_o} = \mathbf{f} \cdot \mathbf{g}_{\mathbf{v}_o} + \mathbf{f}_{\mathbf{v}_o} \cdot \mathbf{g}$,

(d) $(\mathbf{f} \times \mathbf{g})_{\mathbf{v}_o} = \mathbf{f} \times \mathbf{g}_{\mathbf{v}_o} + \mathbf{f}_{\mathbf{v}_o} \times \mathbf{g}$.

For the proof of (d), we begin with

$$(\mathbf{f} \times \mathbf{g})_{\mathbf{v}_o}(\mathbf{u}_o) = \lim_{t \to 0} \frac{\mathbf{f} \times \mathbf{g}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{f} \times \mathbf{g}(\mathbf{u}_o)}{t}.$$

Adding and subtracting $\mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) \times \mathbf{g}(\mathbf{u}_o)$ from the numerator on the right side of the equality and using the properties $\mathbf{f} \times \mathbf{g}(\mathbf{u}) = \mathbf{f}(\mathbf{u}) \times \mathbf{g}(\mathbf{u})$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ converts the numerator to

$$\begin{aligned}&[\mathbf{f} \times \mathbf{g}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) \times \mathbf{g}(\mathbf{u}_o)] + [\mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) \times \mathbf{g}(\mathbf{u}_o) - \mathbf{f} \times \mathbf{g}(\mathbf{u}_o)]\\ &= \mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) \times [\mathbf{g}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{g}(\mathbf{u}_o)] + [\mathbf{f}(\mathbf{u}_o + t\mathbf{v}_o) - \mathbf{f}(\mathbf{u}_o)] \times \mathbf{g}(\mathbf{u}_o).\end{aligned}$$

Dividing by t and taking the limit as t approaches 0 gives the desired result.

We next extend the approximation theorem and chain rule.

Approximation Theorem for Vector Functions

$$\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_o) - \mathbf{f}_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o)|}{|\mathbf{u} - \mathbf{u}_o|} = 0.$$

A proof will be indicated for $\mathbf{f} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$. Let Δ be the function defined by the equation

$$\Delta(\mathbf{u}) = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_o) - \mathbf{f}_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o).$$

Its coordinate functions are

$$\begin{aligned}\Delta_x(\mathbf{u}) &= \bar{x}(\mathbf{u}) - \bar{x}(\mathbf{u}_o) - \bar{x}_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o),\\ \Delta_y(\mathbf{u}) &= \bar{y}(\mathbf{u}) - \bar{y}(\mathbf{u}_o) - \bar{y}_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o).\end{aligned}$$

Then

$$\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{\Delta_x(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|} = \lim_{\mathbf{u} \to \mathbf{u}_o} \frac{\Delta_y(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|} = 0$$

by the approximation theorem for scalar functions, and, hence,

$$\begin{aligned}\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{|\Delta(\mathbf{u})|}{|\mathbf{u} - \mathbf{u}_o|} &= \lim_{\mathbf{u} \to \mathbf{u}_o} \left| \frac{\Delta_x(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|}\mathbf{i} + \frac{\Delta_y(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|}\mathbf{j} \right|\\ &= \left| \lim_{\mathbf{u} \to \mathbf{u}_o} \left(\frac{\Delta_x(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|}\mathbf{i} + \frac{\Delta_y(\mathbf{u})}{|\mathbf{u} - \mathbf{u}_o|}\mathbf{j} \right) \right|\\ &= |0\mathbf{i} + 0\mathbf{j}|\\ &= 0.\end{aligned}$$

The continuity of the *norm function* $g(\mathbf{u}) = |\mathbf{u}|$ is used here (see Proofs, exercise 2).

The chain-rule formula for $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ and $g(x, y)$ at $\mathbf{u}_o$ is

$$(g\circ\mathbf{f})_r(\mathbf{u}_o) = g_x(\mathbf{f}(\mathbf{u}_o))\bar{x}_r(\mathbf{u}_o) + g_y(\mathbf{f}(\mathbf{u}_o))\bar{y}_r(\mathbf{u}_o).$$

Since $\mathbf{f}_r = \bar{x}_r\mathbf{i} + \bar{y}_r\mathbf{j}$, this may be written

$$(g\circ\mathbf{f})_r(\mathbf{u}_o) = \nabla g(\mathbf{f}(\mathbf{u}_o)) \cdot \mathbf{f}_r(\mathbf{u}_o).$$

Application of the gradient formula gives

$$(g\circ\mathbf{f})_r(\mathbf{u}_o) = g_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)),$$

where $\mathbf{w}_o = \mathbf{f}_r(\mathbf{u}_o)$. This conclusion can be extended to arbitrary vector derivatives of composites of vector functions. A proof is in the appendix.

Chain Rule for Vector Functions

$$(\mathbf{g}\circ\mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) = \mathbf{g}_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)), \quad \textit{where } \mathbf{w}_o = \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o).$$

We shall now give this formula a matrix interpretation. Applying $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = J_\mathbf{f}(\mathbf{u}_o)\mathbf{v}_o$ to our chain-rule formula gives

$$\begin{aligned} \mathbf{g}_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)) &= (J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o))\mathbf{w}_o \\ &= (J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o))\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) \\ &= J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o))J_\mathbf{f}(\mathbf{u}_o)\mathbf{v}_o. \end{aligned}$$

This proves our next result.

Proposition 3.4 $\quad (\mathbf{g}\circ\mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) = J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o))J_\mathbf{f}(\mathbf{u}_o)\mathbf{v}_o.$

We have obtained an effective technique for computing vector derivatives of the composition of functions.

Example 3.1 Let

$$\begin{aligned} \mathbf{f}(r, \theta) &= \langle r\cos\theta, r\sin\theta\rangle, \\ \mathbf{g}(x, y) &= \langle xy^2, 2x + y\rangle, \\ \mathbf{u}_o &= \langle 1, 0\rangle, \quad \text{and} \\ \mathbf{v}_o &= \langle 2, 1\rangle. \end{aligned}$$

The Jacobian matrices $J_\mathbf{f}(\mathbf{u}_o)$ and $J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o))$ are, respectively,

$$J_\mathbf{f}(\mathbf{u}_o) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}_{\langle 1,0\rangle} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$J_\mathbf{g}(\mathbf{f}(\mathbf{u}_o)) = \begin{bmatrix} y^2 & 2xy \\ 2 & 1 \end{bmatrix}_{\langle 1,0\rangle} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

By application of Proposition 3.4,

$$\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

gives $(\mathbf{g} \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) = \langle 0, 5 \rangle$.

Questions

1. The chain rule for vector functions says that

$$(\mathbf{g} \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) = \mathbf{g}_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)), \text{ where } \mathbf{w}_o = \underline{\qquad\quad}.$$

2. A property of vector derivatives is $(\mathbf{f} \cdot \mathbf{g})_{\mathbf{v}_o} = \underline{\qquad\quad}$.

Exercises

1. Find $(\mathbf{g} \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o)$ if

   (a) $\mathbf{f}(r, s) = (r - s)\mathbf{i} + rs\,\mathbf{j}$, $\mathbf{g}(x, y) = (x^2 + y^2)\mathbf{i} + xy\mathbf{k}$, $\mathbf{u}_o = \langle 1, 3 \rangle$, and $\mathbf{v}_o = \langle 2, -5 \rangle$;

   (b) $\mathbf{f}(r, s) = rs\mathbf{i} - r\mathbf{j} + s^2\mathbf{k}$, $\mathbf{g}(x, y, z) = xz\mathbf{i} + yz\mathbf{j}$, $\mathbf{u}_o = \langle 1, 1 \rangle$, and $\mathbf{v}_o = \langle 0, 2 \rangle$.

Proofs

1. Prove for $\mathbf{f} = \langle \bar{f}_1, \bar{f}_2 \rangle$, $\mathbf{g} = \langle \bar{g}_1, \bar{g}_2 \rangle$ that

   (a) $(\mathbf{f} + \mathbf{g})_{\mathbf{v}_o} = \mathbf{f}_{\mathbf{v}_o} + \mathbf{g}_{\mathbf{v}_o}$,

   (b) $(\mathbf{f} \cdot \mathbf{g})_{\mathbf{v}_o} = \mathbf{f} \cdot \mathbf{g}_{\mathbf{v}_o} + \mathbf{f}_{\mathbf{v}_o} \cdot \mathbf{g}$.

2. Prove that the scalar function $g(\mathbf{u}) = |\mathbf{u}|$ from $\mathbf{R}^n$ to $\mathbf{Re}$ is continuous. (*Hint*: Use the inequality $||\mathbf{v}| - |\mathbf{u}|| \le |\mathbf{v} - \mathbf{u}|$.)

4. The Differential

The introduction of Jacobian matrices for computing vector derivatives and the correspondence between matrices and linear functions suggest a possible linkage between vector derivatives and linear functions. In this section we establish the connection with the *differential*, which gives valuable insight into the nature of class C^1 functions.

Let $\mathbf{f}$ be a class C^1 function from $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$. The vector derivative symbol $\mathbf{f}_{\mathbf{v}}(\mathbf{u})$ suggests that $\mathbf{f}_{\mathbf{v}}$ is a fixed function with variable domain element $\mathbf{u}$. Reversing the fixed and variable roles of $\mathbf{u}$ and $\mathbf{v}$ gives us the differential.

Definition of Differential

The differential *of* $\mathbf{f}$ *at* $\mathbf{u}_o$ *is the function* $D_{\mathbf{u}_o}\mathbf{f}$ *from* $\mathbf{R}^n$ *to* $\mathbf{R}^m$ *which has the rule*

$$D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}) = \mathbf{f}_{\mathbf{v}}(\mathbf{u}_o).$$

Thus there exists, at each domain element $\mathbf{u}_o$ of $\mathbf{f}$, a function $D_{\mathbf{u}_o}\mathbf{f}$.

Proposition 4.1 $D_{\mathbf{u}_o}\mathbf{f}$ is linear.

For the proof let $\mathbf{v}_1$ and $\mathbf{v}_2$ be arbitrary in $\mathbf{R}^n$. Then

$$\begin{aligned} D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{f}_{\mathbf{v}_1+\mathbf{v}_2}(\mathbf{u}_o) \\ &= \mathbf{f}_{\mathbf{v}_1}(\mathbf{u}_o) + \mathbf{f}_{\mathbf{v}_2}(\mathbf{u}_o) \\ &= D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}_1) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}_2). \end{aligned}$$

A similar proof shows that $D_{\mathbf{u}_o}\mathbf{f}(c\mathbf{v}_1) = cD_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}_1)$ (see Proofs, exercise 1). Our next result shows how the differential corresponds to the Jacobian matrix.

Proposition 4.2 Mat $D_{\mathbf{u}_o}\mathbf{f} = J_{\mathbf{f}}(\mathbf{u}_o)$.

For the proof, we recall from vector algebra that the column vectors of $D_{\mathbf{u}_o}\mathbf{f}$ are $D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_1), \ldots, D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_n)$. Since $D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_i) = \mathbf{f}_{\mathbf{e}_i}(\mathbf{u}_o) = \mathbf{f}_{x_i}(\mathbf{u}_o)$, it follows that Mat $D_{\mathbf{u}_o}\mathbf{f} = \langle \mathbf{f}_{x_1}(\mathbf{u}_o), \ldots, \mathbf{f}_{x_n}(\mathbf{u}_o)\rangle = J_{\mathbf{f}}(\mathbf{u}_o)$.

Example 4.1 If $\mathbf{f}(x, y) = \langle x + 2y, x^2y\rangle$, $\mathbf{u}_o = \langle 1, 3\rangle$, and $\mathbf{v} = \langle x, y\rangle$, then

$$J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2xy & x^2 \end{bmatrix}_{\langle 1,\, 3\rangle} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 6x + \ y \end{bmatrix}$$

shows $D_{\mathbf{u}_o}\mathbf{f}(\langle x, y\rangle) = \langle x + 2y, 6x + y\rangle$.

The geometric meaning of the differential will now be considered. The Jacobian matrix of $f(x)$ at $x = c$ is the 1×1 matrix $[df/dx(c)]$; this corresponds to the linear function $y = df/dx(c)x$. Hence, for a real function $f(x)$ of one variable,

the graph of the differential $D_c f(x)$ is the straight line through the origin and parallel to the tangent line at $(c, f(c))$ of the graph of f (see Figure 11.6). Given $f(x, y)$, then the Jacobian matrix at $\mathbf{u}_o$ is the 1×2 matrix $[f_x(\mathbf{u}_o) \ f_y(\mathbf{u}_o)]$; this corresponds to the linear function $z = f_x(\mathbf{u}_o)x + f_y(\mathbf{u}_o)y$, whose graph passes through the origin and is parallel to the plane tangent to the graph of $\mathbf{f}$ at $(\mathbf{u}_o, f(\mathbf{u}_o))$ by the tangent-plane formula which precedes Example 2.6 in Section 2.

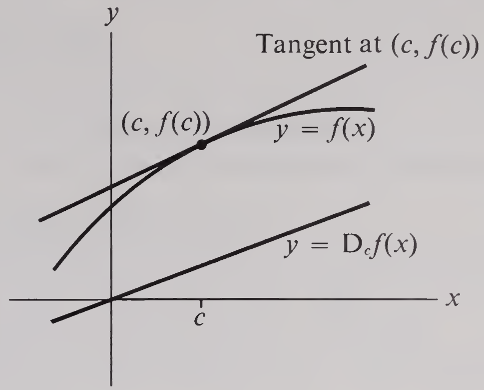


Figure 11.6

The defining equality $D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}) = \mathbf{f}_\mathbf{v}(\mathbf{u}_o)$ establishes a duality between the differential and vector derivative which yields corresponding properties. Therefore from each property of the vector derivative we may obtain a property of the differential; instances are given by the following proposition.

Proposition 4.3

(a) $D_{\mathbf{u}_o}(\mathbf{f} + \mathbf{g}) = D_{\mathbf{u}_o}\mathbf{f} + D_{\mathbf{u}_o}\mathbf{g}$,

(b) $D_{\mathbf{u}_o}c\mathbf{f} = cD_{\mathbf{u}_o}\mathbf{f}$,

(c) $D_{\mathbf{u}_o}fg = f(\mathbf{u}_o)D_{\mathbf{u}_o}g + (D_{\mathbf{u}_o}f)g(\mathbf{u}_o)$,

(d) $D_{\mathbf{u}_o}\dfrac{f}{g} = \dfrac{(D_{\mathbf{u}_o}f)g(\mathbf{u}_o) - f(\mathbf{u}_o)(D_{\mathbf{u}_o}g)}{g^2}$.

A proof of Proposition 4.3(a) is

$$\begin{aligned} D_{\mathbf{u}_o}(\mathbf{f} + \mathbf{g})(\mathbf{v}) &= D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}) + D_{\mathbf{u}_o}\mathbf{g}(\mathbf{v}) \\ &= \mathbf{f}_\mathbf{v}(\mathbf{u}_o) + \mathbf{g}_\mathbf{v}(\mathbf{u}_o) \\ &= D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}) + D_{\mathbf{u}_o}\mathbf{g}(\mathbf{v}) \\ &= (D_{\mathbf{u}_o}\mathbf{f} + D_{\mathbf{u}_o}\mathbf{g})(\mathbf{v}). \end{aligned}$$

For (b) see Proofs, exercise 2. Other corresponding results are set forth in the following theorems.

Mean Value Theorem for Differentials

If f is a C^1 function with domain $\mathbf{U}$ *in* $\mathbf{R}^n$, *and* $[\mathbf{u}_o, \mathbf{u}_1]$ *is contained in* $\mathbf{U}$, *then there exits* $\mathbf{u}_o{}^*$ *in* $[\mathbf{u}_o, \mathbf{u}_1]$ *such that*

$$f(\mathbf{u}_1) - f(\mathbf{u}_o) = D_{\mathbf{u}_o{}^*} f(\mathbf{u}_1 - \mathbf{u}_o).$$

Chain Rule for Differentials

$$D_{\mathbf{u}_o}(\mathbf{g} \circ \mathbf{f}) = (D_{\mathbf{f}(\mathbf{u}_o)}\mathbf{g}) \circ D_{\mathbf{u}_o}\mathbf{f}.$$

Approximation Theorem for Differentials

$$\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u} - \mathbf{u}_o)|}{|\mathbf{u} - \mathbf{u}_o|} = 0.$$

The chain rule for differentials follows directly from Proposition 3.4 and the correspondence between $D_{\mathbf{u}_o}\mathbf{f}$ and $J_{\mathbf{f}}(\mathbf{u}_o)$. This correspondence also yields the following properties for Jacobian matrices.

Proposition 4.4

(a) $J_{c\mathbf{f}}(\mathbf{u}_o) = cJ_{\mathbf{f}}(\mathbf{u}_o)$,
(b) $J_{\mathbf{f}+\mathbf{g}}(\mathbf{u}_o) = J_{\mathbf{f}}(\mathbf{u}_o) + J_{\mathbf{g}}(\mathbf{u}_o)$,
(c) $J_{\mathbf{g} \circ \mathbf{f}}(\mathbf{u}_o) = J_{\mathbf{g}}(\mathbf{f}(\mathbf{u}_o))J_{\mathbf{f}}(\mathbf{u}_o)$.

Example 4.2 Given that $\mathbf{f}(r, s) = \langle rs, r - s, r + 3s \rangle$, $\mathbf{g}(x, y, z) = \langle xy^2, 3z \rangle$, and $\mathbf{u}_o = \langle 2, 1 \rangle$, then

$$\mathbf{f}(\mathbf{u}_o) = \langle 2, 1, 5 \rangle,$$

$$J_{\mathbf{f}}(\mathbf{u}_o) = \begin{bmatrix} s & r \\ 1 & -1 \\ 1 & 3 \end{bmatrix}_{\langle 2,\,1 \rangle} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix},$$

and

$$J_{\mathbf{g}}(\mathbf{f}(\mathbf{u}_o)) = \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & 0 & 3 \end{bmatrix}_{\langle 2,\,1,\,5 \rangle} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Hence from Proposition 4.4(c), Mat $D_{\mathbf{u}_o}(\mathbf{g} \circ \mathbf{f})$ is given by

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 9 \end{bmatrix}.$$

Questions

1. The linear function which corresponds to the vector derivative function is called the ______.
2. The matrix of the differential is the ______ matrix.
3. In geometric instances, the graph of the differential is ______.
   (a) a tangent,
   (b) parallel to a tangent,
   (c) perpendicular to a tangent.
4. The chain rule for differentials is $D_{\mathbf{u}_o}(\mathbf{g} \circ \mathbf{f}) =$ ______.
5. The Jacobian matrix property corresponding to the chain rule for differentials is ______.

Exercises

1. Given that $\mathbf{f}(r, s) = rs^2\mathbf{i} - r\mathbf{j}$, $\mathbf{g}(x, y) = (x - y)\mathbf{i} + xy\mathbf{j}$, $\mathbf{u}_o = \langle 1, 3\rangle$, and $\mathbf{v}_o = \langle -2, 1\rangle$ find
   (a) Mat $D_{\mathbf{u}_o}\mathbf{f}$, (b) Mat $D_{\mathbf{f}(\mathbf{u}_o)}\mathbf{g}$, (c) $D_{\mathbf{u}_o}(\mathbf{g} \circ \mathbf{f})(\mathbf{v}_o)$.
2. Repeat Exercise 1 for the conditions $\mathbf{f}(r, s, t) = rt\mathbf{i} - r^2s\mathbf{j}$, $\mathbf{g}(x, y) = xy^2\mathbf{i} + (x - y)\mathbf{j}$, $\mathbf{u}_o = \langle 1, 0, 1\rangle$, and $\mathbf{v}_o = \langle 2, 1, 0\rangle$.

Proofs

1. Prove that $D_{\mathbf{u}_o}\mathbf{f}(c\mathbf{v}) = cD_{\mathbf{u}_o}\mathbf{f}(\mathbf{v})$.
2. Prove that $D_{\mathbf{u}_o}c\mathbf{f} = cD_{\mathbf{u}_o}\mathbf{f}$.
3. Prove that if $\mathbf{f} = \langle \bar{f}_1, \bar{f}_2\rangle$ is linear, then $D_{\mathbf{u}_o}\mathbf{f} = \mathbf{f}$. (*Hint*: Use the property $f_{\mathbf{v}_o}(\mathbf{u}_o) = f(\mathbf{v}_o)$ if f is linear; also $\mathbf{f}$ is linear if and only if $\bar{f}_1$ and $\bar{f}_2$ are each linear.)

5. *Affine Approximations*

In Section 2 of this chapter we obtained equations for tangent lines and tangent planes; however, there was no attempt to describe a precise relationship between a given curve or surface and its tangents. In this section we shall, with suitable hypotheses, obtain for a fixed domain element $\mathbf{u}_o$ of a vector function $\mathbf{f}$ an affine function which approximates $\mathbf{f}$ near $\mathbf{u}_o$. The image of this affine function gives a tangent to the image of $\mathbf{f}$ in certain geometric situations,

and a precise formulation of the relationship between these images follows from the approximation theorem.

Let $\mathbf{f}$ be a class C^1 function from an open set $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$. If $\mathbf{u}_o$ is in $\mathbf{U}$ and $\mathbf{v}$ in $\mathbf{R}^n$ is near $\mathbf{u}_o$, then the approximation theorem for differentials gives us

$$\mathbf{f}(\mathbf{v}) \approx \mathbf{f}(\mathbf{u}_o) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v} - \mathbf{u}_o).$$

Since $D_{\mathbf{u}_o}\mathbf{f}$ is linear, the right side of this relation is

$$(\mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o)) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v});$$

this describes an affine function which approximates $\mathbf{f}$ near $\mathbf{u}_o$.

Definition of Affine Approximation

The affine approximation *of* $\mathbf{f}$ *at* $\mathbf{u}_o$ *is the affine function* $\mathbf{A}$ *from* $\mathbf{R}^n$ *to* $\mathbf{R}^m$ *which satisfies the equality,*

$$\mathbf{A}(\mathbf{v}) = (\mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o)) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v}).$$

An immediate corollary of the approximation theorem is given in the following proposition.

Proposition 5.1 $$\lim_{\mathbf{v}\to\mathbf{u}_o} \frac{|\mathbf{f}(\mathbf{v}) - \mathbf{A}(\mathbf{v})|}{|\mathbf{v} - \mathbf{u}_o|} = 0.$$

The definition of $\mathbf{A}$ implies that $\mathbf{A}$ has the constant matrix

$$\langle \mathbf{f}(\mathbf{u}_o)\rangle - \text{Mat } D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o) = \mathbf{f}(\mathbf{u}_o) - J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{u}_o$$

and the linear matrix

$$\text{Mat } D_{\mathbf{u}_o}\mathbf{f} = J_{\mathbf{f}}(\mathbf{u}_o).$$

For the definition of the constant and linear matrix, see Chapter V, Section 1.

Example 5.1 Let $\mathbf{f}(x, y) = \langle x^2, x - 3y, 2y^2\rangle$, $\mathbf{u}_o = \langle 1, 2\rangle$, and $\mathbf{v} = \langle x, y\rangle$. Then $\mathbf{f}(\mathbf{u}_o) = \langle 1, -5, 8\rangle$ and

$$J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{u}_o = \begin{bmatrix} 2x & 0 \\ 1 & -3 \\ 0 & 4y \end{bmatrix}_{\langle 1, 2\rangle} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 16 \end{bmatrix}.$$

The constant and linear matrices of $\mathbf{A}$ are, respectively,

$$\begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 1 & -3 \\ 1 & 8 \end{bmatrix}.$$

The behavior of affine approximations with respect to function operations is considered next. Let $\mathbf{A_f}$ and $\mathbf{A_g}$ respectively denote the affine approximations of $\mathbf{f}$ and $\mathbf{g}$ at $\mathbf{u}_o$.

Proposition 5.2
(a) $\mathbf{A_{f+g}} = \mathbf{A_f} + \mathbf{A_g}$,
(b) $\mathbf{A}_{c\mathbf{f}} = c\mathbf{A_f}$.

For (a) see Proofs, exercise 1. A proof of (b) is

$$\begin{aligned} \mathbf{A}_{c\mathbf{f}}(\mathbf{v}) &= (c\mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}c\mathbf{f}(\mathbf{u}_o)) + D_{\mathbf{u}_o}c\mathbf{f}(\mathbf{v}) \\ &= c[(\mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o)) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{v})] \\ &= c\mathbf{A_f}(\mathbf{v}). \end{aligned}$$

Similarly, if $\mathbf{A_f}$ is the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$ and $\mathbf{A_g}$ of $\mathbf{g}$ at $\mathbf{f}(\mathbf{u}_o)$, then the affine approximation $\mathbf{A}_{\mathbf{g}\circ\mathbf{f}}$ of $\mathbf{g}\circ\mathbf{f}$ at $\mathbf{u}_o$ is given by the following relationship.

Proposition 5.3 $\quad \mathbf{A}_{\mathbf{g}\circ\mathbf{f}} = \mathbf{A_g} \circ \mathbf{A_f}$.

From vector algebra it follows that if $\mathbf{A_f}$ has constant matrix C_1 and linear matrix B_1, and $\mathbf{A_g}$ has constant matrix C_2 and linear matrix B_2, then

(a) $\mathbf{A_{f+g}}$ has constant matrix $C_1 + C_2$ and linear matrix $B_1 + B_2$,
(b) $\mathbf{A}_{c\mathbf{f}}$ has constant matrix cC_1 and linear matrix cB_1, and
(c) $\mathbf{A}_{\mathbf{g}\circ\mathbf{f}}$ has constant matrix $C_2 + B_2C_1$ and linear matrix B_2B_1.

Example 5.2 Given $\mathbf{f}(r, s) = \langle rs, r - 3s\rangle$, $\mathbf{g}(x, y) = \langle x^2y, x + y^2\rangle$, and $\mathbf{u}_o = \langle 2, 1\rangle$, then the constant and linear matrices of $\mathbf{A_f}$ at $\mathbf{u}_o$ and $\mathbf{A_g}$ at $\mathbf{f}(\mathbf{u}_o) = \langle 2, -1\rangle$ are, respectively,

$$C_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -4 & 4 \\ 1 & -2 \end{bmatrix}.$$

Thus the constant and linear matrices of $\mathbf{A}_{\mathbf{g}\circ\mathbf{f}}$ at $\mathbf{u}_o$ are

$$\begin{bmatrix} 8 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} -4 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -20 \\ -1 & 8 \end{bmatrix}.$$

It has previously been noted that in geometric instances the image graph of $D_{\mathbf{u}_o}\mathbf{f}$ is parallel to the tangent to the image of $\mathbf{f}$ at $\mathbf{f}(\mathbf{u}_o)$. If $\mathbf{A}$ is the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$, then $\mathbf{A}$ differs from $D_{\mathbf{u}_o}\mathbf{f}$ by a constant function, and hence, its image graph is parallel to that of $D_{\mathbf{u}_o}\mathbf{f}$.

Furthermore, from

$$\mathbf{A}(\mathbf{u}_o) = (\mathbf{f}(\mathbf{u}_o) - D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o)) + D_{\mathbf{u}_o}\mathbf{f}(\mathbf{u}_o) = \mathbf{f}(\mathbf{u}_o),$$

it is seen that, in geometric instances, the image graph of $\mathbf{A}$ contains $\mathbf{f}(\mathbf{u}_o)$ and is therefore the tangent to the image of $\mathbf{f}$ at $\mathbf{f}(\mathbf{u}_o)$. We shall now formalize for the case in which $\mathbf{f}$ is a class C^1 function from $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$ where $n < m$. To avoid degenerate cases, it will also be assumed that $D_{\mathbf{u}_o}\mathbf{f}$ is injective.

Then, by Chapter VI, Section 3, the image of $\mathbf{A}$ is the n-plane

$$\mathbf{f}(\mathbf{u}_o) + \mathrm{Sp}\,\{D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_1), \ldots, D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_n)\}.$$

From $D_{\mathbf{u}_o}\mathbf{f}(\mathbf{e}_i) = \mathbf{f}_{x_i}(\mathbf{u}_o)$, we arrive at the following definition.

Definition of Tangent n-Plane

The tangent n-plane *to the* **f**-*image of* **U** *at* $\mathbf{f}(\mathbf{u}_o)$ *is*

$$\mathbf{f}(\mathbf{u}_o) + \mathrm{Sp}\,\{\mathbf{f}_{x_1}(\mathbf{u}_o), \ldots, \mathbf{f}_{x_n}(\mathbf{u}_o)\}.$$

It may be recalled that $\mathbf{f}_{x_1}(\mathbf{u}_o), \ldots, \mathbf{f}_{x_n}(\mathbf{u}_o)$ are the column vectors of $J_\mathbf{f}(\mathbf{u}_o)$. The injective condition on $D_{\mathbf{u}_o}\mathbf{f}$ is equivalent to the equality $r(J_\mathbf{f}(\mathbf{u}_o)) = n$, where r denotes rank.

Example 5.3 Let $\mathbf{f}(x, y) = \langle xy, x - y, x + 2y, 3x^2\rangle$ and $\mathbf{u}_o = \langle 1, 2\rangle$. Then $\mathbf{f}(\mathbf{u}_o) = \langle 2, -1, 5, 3\rangle$ and

$$J_\mathbf{f}(\mathbf{u}_o) = \begin{bmatrix} y & x \\ 1 & -1 \\ 1 & 2 \\ 6x & 0 \end{bmatrix}_{\langle 1, 2\rangle} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \\ 6 & 0 \end{bmatrix}.$$

Hence, the tangent 2-plane to the image of $\mathbf{f}$ at $\mathbf{f}(\mathbf{u}_o)$ is

$$\langle 2, -1, 5, 3\rangle + \mathrm{Sp}\,\{\langle 2, 1, 1, 6\rangle, \langle 1, -1, 2, 0\rangle\}.$$

Questions

1. The precise relationship between $\mathbf{f}$ near $\mathbf{u}_o$ and its affine approximation at $\mathbf{u}_o$ is established by the ______ theorem.

2. If $\mathbf{A}$ is the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$, then $\mathbf{A}(\mathbf{0}) =$ ______; $\mathbf{A}(\mathbf{u}_o) =$ ______.

3. The linear matrix of the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$ is ______.

4. The image of the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$ is the ______ to the $\mathbf{f}$-image at $\mathbf{f}(\mathbf{u}_o)$.

5. If $\mathbf{A}$ is the affine approximation of $\mathbf{f}$ at $\mathbf{u}_o$, then $\lim\limits_{\mathbf{v}\to\mathbf{u}_o}$ ______ $= 0$.

6. If $\mathbf{f}$ assigns a tangent at $\mathbf{f}(\mathbf{u}_o)$, then $D_{\mathbf{u}_o}\mathbf{f}$ is required to be ______.

Exercises

1. Find the constant and linear matrices of the affine approximations $\mathbf{A}_\mathbf{f}$, $\mathbf{A}_\mathbf{g}$, $\mathbf{A}_{\mathbf{f}+\mathbf{g}}$, and $\mathbf{A}_{3\mathbf{f}}$ at $\mathbf{u}_o$ given the following conditions.
   (a) $\mathbf{f}(x, y) = \langle x^2 - y, x + 2y\rangle$, $\mathbf{g}(x, y) = \langle 3x + y, x + 4y\rangle$, and $\mathbf{u}_o = \langle 1, 3\rangle$;
   (b) $\mathbf{f}(x, y) = \langle x, y^2, x - y^2\rangle$, $\mathbf{g}(x, y) = \langle x + y, y^2, xy^2\rangle$, and $\mathbf{u}_o = \langle 2, -1\rangle$.

2. Find the constant and linear matrices of the affine approximations $\mathbf{A}_\mathbf{f}$ at $\mathbf{u}_o$, $\mathbf{A}_\mathbf{g}$ at $\mathbf{f}(\mathbf{u}_o)$, and $\mathbf{A}_{\mathbf{g}\circ\mathbf{f}}$ at $\mathbf{u}_o$ given
   $\mathbf{f}(r, s) = \langle r - s, rs\rangle$, $\mathbf{g}(x, y) = \langle xy, x - y, x^2 + y\rangle$, and $\mathbf{u}_o = \langle 1, 3\rangle$.

3. Given $\mathbf{f}(x, y, z) = \langle xz, x - y - z, yz, x + y\rangle$ and $\mathbf{u}_o = \langle 1, 0, 1\rangle$, find the tangent 2-plane to the image of $\mathbf{f}$ at $\mathbf{f}(\mathbf{u}_o)$.

Proofs

1. Prove that $\mathbf{A}_{\mathbf{f}+\mathbf{g}} = \mathbf{A}_\mathbf{f} + \mathbf{A}_\mathbf{g}$.

6. *Equivalence of Functions*

In the previous section we found a general method of assigning tangents to sets in $\mathbf{R}^m$ which were suitably described as the image of an open set in $\mathbf{R}^n$, $n < m$. It has already been observed that a set may be the image of infinitely many vector functions, and it is natural to inquire if any two such functions necessarily assign the same tangent n-plane at a given point. From Example 2.2 it is seen that the answer to this question is negative. However, the assignment of tangent n-planes is not so haphazard as this example might suggest.

Example 6.1 The functions $\mathbf{f}(\theta) = \langle \cos\theta, \sin\theta \rangle$, $0 < \theta < \pi$, and $\mathbf{g}(r) = \langle r, \sqrt{1-r^2} \rangle$, $-1 < r < 1$ each have as their image graph the upper half of the circle $x^2 + y^2 = 1$. The tangent planes $\mathbf{P_f}$ and $\mathbf{P_g}$ assigned by $\mathbf{f}$ and $\mathbf{g}$ at

$$\mathbf{f}\left(\frac{\pi}{3}\right) = \mathbf{g}\left(\frac{1}{2}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

are given by

$$\mathbf{P_f} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + \mathrm{Sp}\left\{\left\langle -\sin\frac{\pi}{3}, \cos\frac{\pi}{3} \right\rangle\right\}$$

$$= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + \mathrm{Sp}\left\{\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle\right\},$$

$$\mathbf{P_g} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + \mathrm{Sp}\left\{\left\langle 1, -\frac{1}{\sqrt{3}} \right\rangle\right\}.$$

Since $\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = -\frac{\sqrt{3}}{2}\left\langle 1, -\frac{1}{\sqrt{3}} \right\rangle$, it follows that $\mathbf{P_f} = \mathbf{P_g}$.

We shall now investigate conditions which are sufficient for two vector functions to assign the same tangent n-plane at a common image point. Our approach is informal, but it introduces some ideas which pertain to the structure theory of various types of sets, such as curves and surfaces. Let $\mathbf{f}$ and $\mathbf{g}$ each be class C^1 functions with respective domains $\mathbf{U_f}$ and $\mathbf{U_g}$ in $\mathbf{R}^n$, and having the same image $\mathbf{U}'$ in $\mathbf{R}^m$, $n < m$. It is assumed that $D_{\mathbf{u}_0}\mathbf{f}$ and $D_{\mathbf{v}_0}\mathbf{g}$ are injective for all domain elements $\mathbf{u}_0$, $\mathbf{v}_0$ of $\mathbf{f}$ and $\mathbf{g}$ respectively. In order to simplify the discussion we shall make an additional assumption that $\mathbf{f}$ and $\mathbf{g}$ are injective. Then, a unique injective function $\mathbf{h}$ from $\mathbf{U_f}$ to $\mathbf{U_g}$ may be defined as follows (see Figure 11.7).

$$\mathbf{h}(\mathbf{u}) = \mathbf{v} \text{ if and only if } \mathbf{f}(\mathbf{u}) = \mathbf{g}(\mathbf{v}).$$

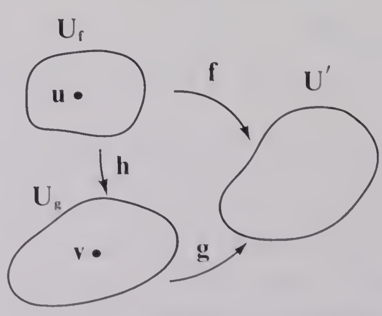


Figure 11.7

Thus $\mathbf{h}$ relates vectors which have the same image. It is evident that

$$\mathbf{f} = \mathbf{g} \circ \mathbf{h} \quad \text{and} \quad \mathbf{g} = \mathbf{f} \circ \mathbf{h}^{-1}.$$

We say $\mathbf{f}$ is *equivalent* to $\mathbf{g}$, written $\mathbf{f} \sim \mathbf{g}$, provided $\mathbf{h}$ and $\mathbf{h}^{-1}$ are each of class C^1. The next result asserts that $\sim$ is an equivalence relation (see Proofs, exercise 1).

Proposition 6.1

(a) $\mathbf{f} \sim \mathbf{f}$,
(b) If $\mathbf{f} \sim \mathbf{g}$, then $\mathbf{g} \sim \mathbf{f}$, and
(c) If $\mathbf{f} \sim \mathbf{g}$ and $\mathbf{g} \sim \mathbf{h}$, then $\mathbf{f} \sim \mathbf{h}$.

The use of the symbol $\mathbf{h}$ in (c) should not be confused with its use in defining equivalence.

Example 6.2 Let $\mathbf{f}(\theta) = \langle \cos\theta, \sin\theta \rangle$, $0 < \theta < \pi$, and $\mathbf{g}(r) = \langle r, \sqrt{1 - r^2} \rangle$, $-1 < r < 1$, as in Example 6.1. If $h(\theta) = \cos\theta$, then

$$\mathbf{g} \circ h(\theta) = \mathbf{g}(\cos\theta) = \langle \cos\theta, \sqrt{1 - \cos^2\theta} \rangle = \langle \cos\theta, \sin\theta \rangle = \mathbf{f}(\theta);$$

hence, $\mathbf{f} = \mathbf{g} \circ h$. Since $h(\theta) = \cos\theta$ is of class C^1 on $(0, \pi)$ and $h^{-1}(r) = \text{arc}\cos r$ is of class C^1 on $(-1, 1)$, it follows that $\mathbf{f} \sim \mathbf{g}$.

Example 6.3 Let $\mathbf{f}(r) = \langle r^3, r^3 \rangle$, $-1 < r < 1$ and $\mathbf{g}(s) = \langle s, s \rangle$, $-1 < s < 1$ (see Example 2.2). Then $\mathbf{f} = \mathbf{g} \circ h$ where $h(r) = r^3$ and $h^{-1}(s) = s^{1/3}$. In this case h is of class C^1, but h^{-1} is not of class C^1; since $dh^{-1}/ds(0)$ does not exist. Therefore, $\mathbf{f}$ and $\mathbf{g}$ are not equivalent.

In most instances it is difficult, for a given $\mathbf{f}$ and $\mathbf{g}$, to find a formula for their relating function $\mathbf{h}$. We shall not attempt here to find a general method for deciding if a given $\mathbf{f}$ and $\mathbf{g}$ are equivalent. One useful criterion comes from the identity $\mathbf{h}^{-1} \circ \mathbf{h} = \mathbf{id}$, where $\mathbf{id}$ is the identity function. This implies that

$$(\det J_{\mathbf{h}^{-1}}\mathbf{h}(\mathbf{u}_0))(\det J_{\mathbf{h}}(\mathbf{u}_0)) = \det \mathbf{id} = 1.$$

Hence, $\mathbf{f} \sim \mathbf{g}$ implies $J_{\mathbf{h}}$ and $J_{\mathbf{h}^{-1}}$ each have a nonzero determinant everywhere. It can be shown, though the proof is beyond our scope, that if $\mathbf{h}$ is of class C^1 and $\det J_{\mathbf{h}} \neq 0$ everywhere, then $\mathbf{h}^{-1}$ is also of class C^1 and, hence, $\mathbf{f} \sim \mathbf{g}$.

In most elementary cases any two vector functions used to describe a curve or surface are equivalent. This lends strength to the following result, which says that equivalent functions assign the same tangent n-plane at common image points; the proof is in the appendix.

Proposition 6.2 Let $\mathbf{f}(\mathbf{u}_o) = \mathbf{g}(\mathbf{v}_o)$ and $\mathbf{f}(x_1, \ldots, x_n) \sim \mathbf{g}(y_1, \ldots, y_n)$. If $D_{\mathbf{u}_o}\mathbf{f}$ is injective, then $D_{\mathbf{v}_o}\mathbf{g}$ is also injective and

$$\mathbf{f}(\mathbf{u}_o) + \mathrm{Sp}\{\mathbf{f}_{x_1}(\mathbf{u}_o), \ldots, \mathbf{f}_{x_n}(\mathbf{u}_o)\} = \mathbf{g}(\mathbf{v}_o) + \mathrm{Sp}\{\mathbf{g}_{y_1}(\mathbf{v}_o), \ldots, \mathbf{g}_{y_n}(\mathbf{v}_o)\}.$$

Questions

1. Two distinct vector functions with the same image set _______ assign the same tangents at an image point.
   (a) must,
   (b) may but not necessarily,
   (c) cannot.
2. Two injective class C^1 vector functions are equivalent if their relating vector function $\mathbf{h}$ satisfies _______.
3. If $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are injective and $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$, then $\mathbf{g} = \mathbf{f} \circ$ _______.
4. Two vector functions, with the same image set, assign the same tangent at an image point provided they are _______.

Exercises

1. Let $\mathbf{f}(r) = \langle r, r^2\rangle$, $0 < r < 1$, and $\mathbf{g}(s) = \langle\sqrt{s}, s\rangle$, $0 < s < 1$.
   (a) Show that $\mathbf{f}$ and $\mathbf{g}$ assign the same tangent 1-plane at $\langle 1/2, 1/4\rangle$.
   (b) Find h so that $\mathbf{f} = \mathbf{g} \circ h$, and show $\mathbf{f} \sim \mathbf{g}$.
2. Let $\mathbf{f}(\theta) = \langle\sin\theta, -\cos\theta\rangle$, $0 < \theta < \pi$ and $\mathbf{g}(r) = \langle\sqrt{1 - r^2}, r\rangle$, $-1 < r < 1$.
   (a) Show that $\mathbf{f}$ and $\mathbf{g}$ assign the same tangent 1-plane at $\langle\sqrt{3}/2, 1/2\rangle$.
   (b) Find h so that $\mathbf{f} = \mathbf{g} \circ h$, and show $\mathbf{f} \sim \mathbf{g}$.
3. Let $\mathbf{f}(r) = \langle 1 + r^{1/3}, 1 + r^{2/3}\rangle$, $-1 < r < 1$, and $\mathbf{g}(s) = \langle s, s^2 - 2s + 2\rangle$, $0 < s < 2$.
   (a) Sketch the image sets of $\mathbf{f}$ and $\mathbf{g}$.
   (b) Find h so that $\mathbf{f} = \mathbf{g} \circ h$, and show that $\mathbf{f}$ and $\mathbf{g}$ are not equivalent.
   (c) Show that $\mathbf{f}$ does not assign a tangent at $\langle 1, 1\rangle$, whereas $\mathbf{g}$ does.

Proofs

1. Prove that $\mathbf{f} \sim \mathbf{g}$ defines an equivalence relation.
2. Prove that if $\mathbf{f}$ and $\mathbf{g}$ are linear injective functions from $\mathbf{R}^2$ to $\mathbf{R}^3$ with $\mathbf{f}(\mathbf{R}^2) = \mathbf{g}(\mathbf{R}^2)$, then $\mathbf{f} \sim \mathbf{g}$. (*Hint*: Show the function $\mathbf{h}$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$ is linear and has rank 2.)

Problems

A. *Jacobian Matrices*

Associated with each vector function is a matrix, called its *Jacobian matrix*, whose entries are functions. A special instance is shown below.

A.1 The Jacobian matrix of $\mathbf{f}(x, y) = \bar{f}_1(x, y)\mathbf{i} + \bar{f}_2(x, y)\mathbf{j}$ is

$$J_{\mathbf{f}} = \begin{bmatrix} (\bar{f}_1)_x & (\bar{f}_1)_y \\ (\bar{f}_2)_x & (\bar{f}_2)_y \end{bmatrix}.$$

1. Find $J_{\mathbf{f}}$ if $\mathbf{f}(x, y) =$
   (a) $xy^3\mathbf{i} - (x^2 + y)\mathbf{j}$, (b) $ye^x\mathbf{i} - xy^2\mathbf{j}$.
2. Given $\mathbf{u}_o = \langle 0, 2\rangle$, evaluate $J_{\mathbf{f}}(\mathbf{u}_o)$ in 1(a) and (b).

Jacobian matrices of other vector functions are defined similarly. In each case, the rows of $J_{\mathbf{f}}$ are the gradients of the coordinate functions of $\mathbf{f}$.

3. Find $J_{\mathbf{f}}$ and $J_{\mathbf{f}}(\mathbf{u}_o)$ for each function and vector below.

   (a) $\mathbf{f}(x, y, z) = x^2yz\mathbf{i} - xz\mathbf{j}$, $\mathbf{u}_o = \langle 1, 1, 2\rangle$,
   (b) $\mathbf{f}(x, y) = xy\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$, $\mathbf{u}_o = \langle 1, 3\rangle$,
   (c) $\mathbf{f}(x, y, z) = (x + z)\mathbf{i} + y^2z\mathbf{j} - xy\mathbf{k}$, $\mathbf{u}_o = \langle 1, 1, 0\rangle$.

The *vector derivative* of $\mathbf{f}$ at $\mathbf{u}_o$ with respect to $\mathbf{v}_o$ is computed by the formula

A.2 $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) = J_{\mathbf{f}}(\mathbf{u}_o)\mathbf{v}_o$.

In A.2 the vector $\mathbf{v}_o$ is written as a column matrix.

4. Find $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$ for the given function and vectors.
   (a) $\mathbf{f}(x, y) = xy\mathbf{i} - y^2\mathbf{j}$, $\mathbf{u}_o = \langle 1, 2\rangle$, $\mathbf{v}_o = \langle 3, -2\rangle$;
   (b) $\mathbf{f}(x, y, z) = xz\mathbf{i} + y^2\mathbf{j}$, $\mathbf{u}_o = \langle 1, 0, 3\rangle$, $\mathbf{v}_o = \langle 2, -1, 1\rangle$;
   (c) $\mathbf{f}(x, y) = x^2y\mathbf{i} + (x - y)\mathbf{j} - x^3y\mathbf{k}$, $\mathbf{u}_o = \langle 1, 1\rangle$, $\mathbf{v}_o = \langle 3, 0\rangle$;
   (d) $\mathbf{f}(x, y, z) = xz\mathbf{i} - yz\mathbf{j} + x^2y\mathbf{k}$, $\mathbf{u}_o = \langle 1, 1, 0\rangle$, $\mathbf{v}_o = \langle 0, 2, 1\rangle$.

Review

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5. Find $J_{\mathbf{f}}$, $J_{\mathbf{f}}(\mathbf{u}_o)$, and $\mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$ given the following conditions.
   (a) $\mathbf{f}(x, y) = x^2y\mathbf{i} + (2x - y)\mathbf{j}$, $\mathbf{u}_o = \langle 1, 1\rangle$, $\mathbf{v}_o = \langle 2, 3\rangle$;
   (b) $\mathbf{f}(x, y) = (x - 3y)\mathbf{i} + xy\mathbf{j} - y^3\mathbf{k}$, $\mathbf{u}_o = \langle 2, 3\rangle$, $\mathbf{v}_o = \langle -1, 2\rangle$;
   (c) $\mathbf{f}(x, y, z) = xyz^2\mathbf{i} - (3x + y)\mathbf{j}$, $\mathbf{u}_o = \langle 1, 2, 0\rangle$, $\mathbf{v}_o = \langle 0, 3, 1\rangle$.

B. *Tangents and Normals to Curves and Surfaces*

Given the curve $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ in the Cartesian plane, the *tangent line* to $\mathbf{f}(r_o)$ is the line

B.1 $$\mathbf{f}(r_o) + r\left(\frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j}\right).$$

The *normal line* at $\mathbf{f}(r_o)$ is perpendicular to the tangent line. It is given by the expression

B.2 $$\mathbf{f}(r_o) + r\left(\frac{d\bar{y}}{dr}(r_o)\mathbf{i} - \frac{d\bar{x}}{dr}(r_o)\mathbf{j}\right).$$

1. Find the tangent and normal lines at the indicated point.
   (a) $\mathbf{f}(r) = r^2\mathbf{i} + 3r\mathbf{j}$, $(25, 15)$,
   (b) $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $(1/2, \sqrt{3}/2)$.
2. By using a suitable vector equation, find the tangent and normal lines to each curve.
   (a) $y - 2 = 4(x - 2)^2$ at $(3, 6)$,
   (b) $(x - 4)^2/2 + y^2/8 = 1$ at $(3, 2)$.
3. Using the standard vector equation, find the tangent and normal lines to the curve
   (a) $f(x) = 3e^{2x}$ at $x = 0$, and (b) $f(x) = \cos^2 x$ at $x = 0$.

Given the curve $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$ in space, the *tangent line* at $\mathbf{f}(r_o)$ is the line

B.3 $$\mathbf{f}(r_o) + r\left(\frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j} + \frac{d\bar{z}}{dr}(r_o)\mathbf{k}\right).$$

The *normal plane* at $\mathbf{f}(r_o)$ is perpendicular to the tangent line. Its direction vectors may be chosen as any two independent (noncollinear) vectors orthogonal to the direction vector of the tangent line. A convenient choice that usually works gives the following expression for the normal plane:

B.4 $$\mathbf{f}(r_o) + r\left(\frac{d\bar{y}}{dr}(r_o)\mathbf{i} - \frac{d\bar{x}}{dr}(r_o)\mathbf{j}\right) + s\left(\frac{d\bar{z}}{dr}(r_o)\mathbf{i} - \frac{d\bar{x}}{dr}(r_o)\mathbf{k}\right).$$

4. Find the tangent line and normal plane to the curve

   (a) $\mathbf{f}(r) = e^r\mathbf{i} + \ln(r+1)\mathbf{j} + r^2\mathbf{k}$ at $(1, 0, 0)$,
   (b) $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + \theta\mathbf{k}$ at $(0, 1, \pi/2)$.

Given the surface $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}$, the *tangent plane* at $\mathbf{f}(r_o, s_o)$ is

B.5 $$\mathbf{f}(r_o, s_o) + r\mathbf{f}_r(r_o, s_o) + s\mathbf{f}_s(r_o, s_o),$$

where

$$\mathbf{f}_r = \bar{x}_r\mathbf{i} + \bar{y}_r\mathbf{j} + \bar{z}_r\mathbf{k} \quad \text{and} \quad \mathbf{f}_s = \bar{x}_s\mathbf{i} + \bar{y}_s\mathbf{j} + \bar{z}_s\mathbf{k}.$$

The *normal line* at $\mathbf{f}(r_o, s_o)$ is perpendicular to the tangent plane. Its direction vector is given by the cross product of direction vectors of the tangent plane. Thus, from B.5 the normal line is

B.6 $$\mathbf{f}(r_o, s_o) + r(\mathbf{f}_r(r_o, s_o) \times \mathbf{f}_s(r_o, s_o)).$$

5. Find the tangent plane and normal line to the surfaces

   (a) $\mathbf{f}(r, s) = (r^2 + 2rs)\mathbf{i} + (r - 3s)\mathbf{j} + (r^3 + 4rs)\mathbf{k}$ at $\mathbf{f}(1, 2) = (5, -5, 9)$,
   (b) $\mathbf{f}(r, s) = e^{rs}\mathbf{i} + e^{2r^2s}\mathbf{j} - e^{s^2}\mathbf{k}$ at $\mathbf{f}(0, 2)$.

6. Using the standard vector equation, find the tangent plane and normal line to the surfaces

   (a) $z = xy^3$ at $(1, 3, 27)$, and (b) $z = e^{x+y^2}$ at $(-1, 1, 1)$.

7. Find the tangent plane and normal line to each surface of revolution described below at the indicated point.

   (a) $y = x^2$ about the x axis, $(1, \sqrt{3}/2, -1/2)$;
   (b) $z = e^y$ about the z axis, $(\sqrt{2}/2, \sqrt{2}/2, e)$.

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8. Find the tangent and normal lines to the following curves at the indicated point.

   (a) $(x-2)^2 + (y-4)^2 = 10$ at $(3, 7)$,
   (b) $x - 8 = 2(y-2)^2$ at $(16, 4)$,
   (c) $f(x) = e^x \cos x$ at $(0, 1)$.

9. Find the tangent line and normal plane to the curves

   (a) $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j} + r^3\mathbf{k}$ at $(3, 9, 27)$,
   (b) $\mathbf{f}(r) = \arctan r\mathbf{i} + e^r\mathbf{j} + e^{2r}\mathbf{k}$ at $(\pi/4, e, e^2)$.

10. Find the tangent plane and normal line to each surface at the indicated point.

   (a) $\mathbf{f}(r, s) = r^3s\mathbf{i} + (r^2 + s^2)\mathbf{j} + 4rs\mathbf{k}$, $\mathbf{f}(2, 1)$,
   (b) $z = e^{xy^2}$, $(0, 1, 1)$,
   (c) the surface of revolution of $z=3x$ about the x axis $(2, 3\sqrt{2}, -3\sqrt{2})$.

C. Tangents and Normals to Implicitly Defined Curves and Surfaces

If a curve or surface is defined implicitly, then the tangent and normal at a given point may often be obtained by the following steps.

C.1 (a) Assume a standard vector equation of the curve or surface near the given point,
(b) substitute from the vector equation of (a) into the implicit equation (or equations),
(c) obtain partial derivatives by implicit differentiation, and
(d) find the tangent and normal by formulas of the previous section.

1. Determine the tangent and normal lines at $(0, 1)$ to the curve defined implicitly by $xe^{xy} - y^2 + 1 = 0$ by the following procedure.

   (a) Let $\mathbf{f}(r) = r\mathbf{i} + \bar{y}(r)\mathbf{j}$ satisfy the given equation near $(0, 1)$.
   (b) Substitute $x = r$, $y = \bar{y}(r)$ into the given implicit equation.
   (c) Find $d\bar{y}/dr(0)$ by implicit differentiation.
   (d) From (a) and (c) obtain the tangent and normal lines.

2. Find the tangent and normal lines at $(1, 1)$ to the curve $x^3y + xy^3 - 2 = 0$.

3. Find the tangent line and normal plane to the curve formed by the intersection of the surfaces $x^2 + y^2 + z^2 - 6 = 0$ and $x + 3y + z - 6 = 0$ at $(1, 1, 2)$. (*Hint*: Let $\mathbf{f}(r) = r\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$ describe the curve, and find $d\bar{y}/dr(1)$ and $d\bar{z}/dr(1)$ by implicit differentiation.)

4. Find the tangent and normal to the curve of intersection of the surfaces $x^3 + y^2 + 4z - 9 = 0$ and $x + 2y + z - 6 = 0$ at $(1, 2, 1)$.

5. Find the tangent plane and normal line at $(1, 1, 1)$ to the surface $x^2 + y^2 + z^2 - 3 = 0$. (*Hint*: Let $\mathbf{f}(r, s) = r\mathbf{i} + s\mathbf{j} + \bar{z}(r, s)\mathbf{k}$ describe the surface, and find $\bar{z}_r$ and $\bar{z}_s$ by implicit differentiation.)

6. Find the tangent plane and normal line at $(1, 0, 2)$ to the surface $xy^2 + x^3z - yz - 2 = 0$.

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7. Find the tangent and normal, at the indicated point, to the surface described.

   (a) $x^2y^5 - x^3y + 4 = 0$, $(2, 1)$;
   (b) the intersection of $x^3 + y^3 + z^3 - 10 = 0$ and $x + 3y + z - 6 = 0$, $(1, 1, 2)$;
   (c) $x^3y + y^3z^2 - xz^3 + 1 = 0$, $(1, 0, 1)$.

D. *Velocity and Acceleration*

We shall let $\mathbf{f}(t) = \bar{x}(t)\mathbf{i} + \bar{y}(t)\mathbf{j}$ denote the position at time t of a point particle moving in the Cartesian plane. Then the *velocity* $\mathbf{v}$ and *acceleration* $\mathbf{a}$ of the point are given by the following formulas.

D.1 $$\mathbf{v} = \frac{d\bar{x}}{dt}\mathbf{i} + \frac{d\bar{y}}{dt}\mathbf{j} \qquad \left(= \frac{d\mathbf{f}}{dt}\right),$$

$$\mathbf{a} = \frac{d^2\bar{x}}{dt^2}\mathbf{i} + \frac{d^2\bar{y}}{dt^2}\mathbf{j} \qquad \left(= \frac{d\mathbf{v}}{dt}\right).$$

The *speed* is $|\mathbf{v}| = \sqrt{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2}$.

Similarly, in Cartesian space, if the position vector is described by $\mathbf{f}(t) = \bar{x}(t)\mathbf{i} + \bar{y}(t)\mathbf{j} + \bar{z}(t)\mathbf{k}$, then the velocity and acceleration are given by

D.2 $$\mathbf{v} = \frac{d\bar{x}}{dt}\mathbf{i} + \frac{d\bar{y}}{dt}\mathbf{j} + \frac{d\bar{z}}{dt}\mathbf{k},$$

$$\mathbf{a} = \frac{d^2\bar{x}}{dt^2}\mathbf{i} + \frac{d^2\bar{y}}{dt^2}\mathbf{j} + \frac{d^2\bar{z}}{dt^2}\mathbf{k}.$$

The speed is $|\mathbf{v}| = \sqrt{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2}$.

1. Find the velocity, acceleration, and speed as a function of time t for each position vector function.

   (a) $\mathbf{f}(t) = t^2\mathbf{i} - t^3\mathbf{j}$,
   (b) $\mathbf{f}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j}$,
   (c) $\mathbf{f}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j} + e^{3t}\mathbf{k}$.

2. In 1(a) find the velocity, acceleration, and speed when $t = 1$.

If the movement of an object having mass m is produced by a variable vector force $\mathbf{F}(t)$, then from physics we have the equality,

D.3 $\mathbf{F} = m\mathbf{a}$.

3. Find, as follows, the position vector $\mathbf{f}(t)$ of an object which has mass 10 and which is acted upon by a force $\mathbf{F}(t) = 100t^3\mathbf{i} - 10\mathbf{j} + 20t\mathbf{k}$; assume that the initial position is $\mathbf{f}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and initial velocity is $\mathbf{v}(0) = 10\mathbf{j}$.

   (a) Apply integration to $\mathbf{a} = \mathbf{F}/m$ to obtain an equation for $\mathbf{v}(t)$ involving an arbitrary constant vector $\mathbf{C}$; determine $\mathbf{C}$ using $\mathbf{v}(0)$.

   (b) Apply integration to the expression for $\mathbf{v}(t)$ to obtain an equation for $\mathbf{f}(t)$ involving an arbitrary constant vector $\mathbf{C}'$; determine $\mathbf{C}'$ using $\mathbf{f}(0)$.

4. An object having mass 2 is acted upon by a force $\mathbf{F}(t) = 2\mathbf{i} + 2t\mathbf{j} - 32\mathbf{k}$; it has initial position $200\mathbf{k}$ and initial velocity $-10\mathbf{k}$. Find at time $t = 1$ the

   (a) acceleration, (b) velocity, and (c) position of the object.

Other types of velocity and acceleration problems will now be considered.

5. A plane is moving in a northeasterly direction at a rate of 500 mph, and is rising at a rate of 10 mph. If, at time $t = 0$, the plane's position is 2 miles high, 200 miles east, and 100 miles north of an observer, using a suitable coordinate system (positive x axis = east) with origin at the observer, determine the

   (a) velocity, (b) position, and (c) speed of the plane at an arbitrary time t.

6. Repeat Problem 5 if the plane is moving in a northerly direction at 400 mph, is rising at a rate of 5 mph, and is located 3 miles high and 200 miles east of the observer at time $t = 0$.

7. A bug crawls radially outward at 3 inches per minute on a wheel spinning at a rate of 1 radian per minute.

   (a) Find the polar coordinates of the bug's position at time t. Assume that the bug starts at the origin in the direction of the polar axis.
   (b) From (a) obtain a vector equation for the bug's position $\mathbf{f}(t)$ at time t.
   (c) Find the velocity and speed of the bug at an arbitrary time t.

8. A bug crawls at 1 inch per minute toward the vertex of a cone which has a base radius of 4 inches and altitude of 3 inches. The cone is turning at 2 radians per minute about an axis through the vertex and perpendicular to the base of the cone.

(a) Obtain a vector equation of the bug's position, using a circle in the xy plane and centered at the origin as the base. Assume that the bug starts at the base circle and on the x axis.
(b) From (a) find the velocity and speed of the bug at time t.

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9. Find the velocity, acceleration, and speed as a function of time t for the given position vector function.

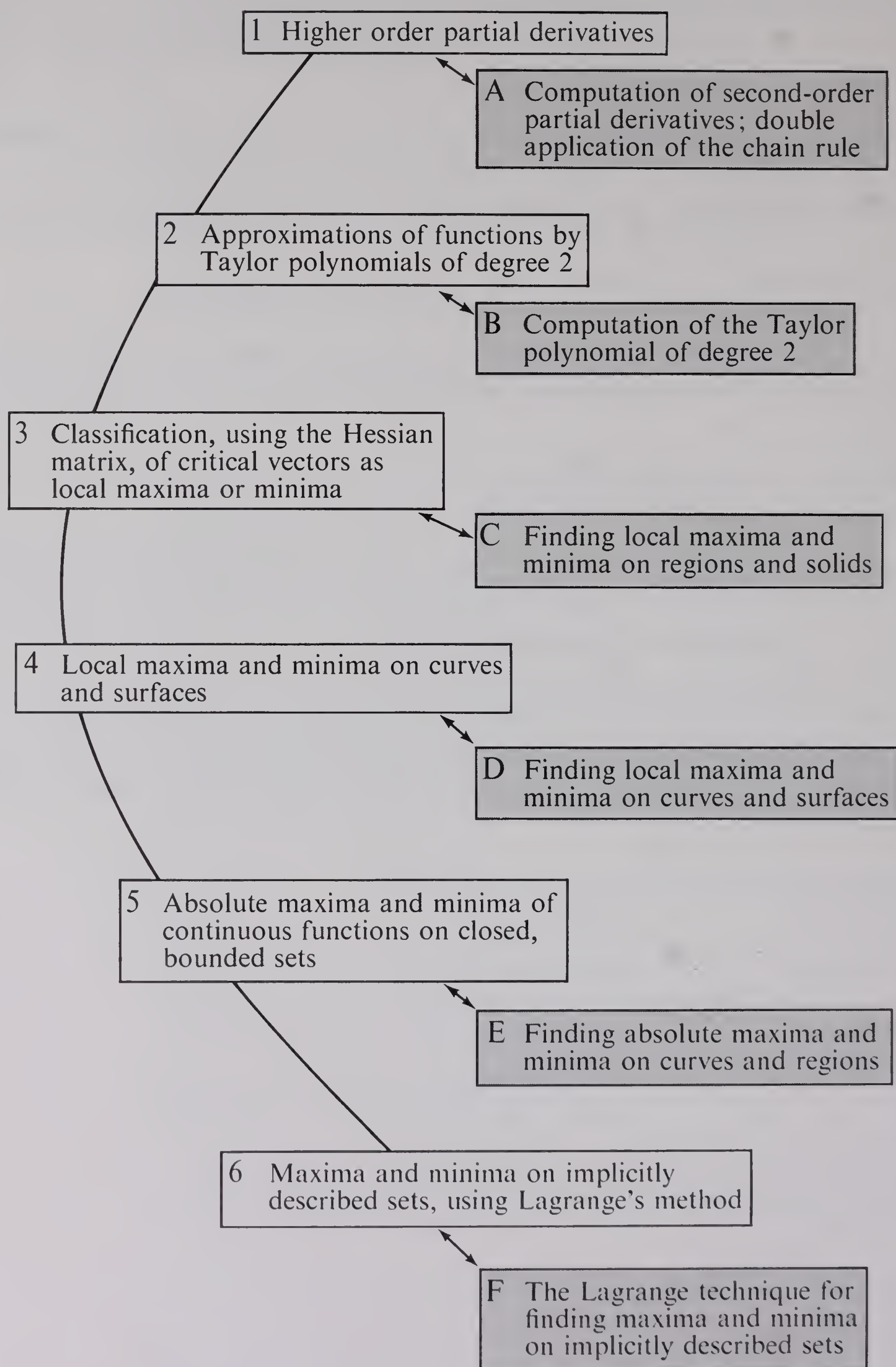
   (a) $\mathbf{f}(t) = t\mathbf{i} + t^3\mathbf{j}$, (b) $\mathbf{f}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 4t\mathbf{k}$.

10. An object having mass 1 is acted upon by a force $\mathbf{F}(t) = 2t\mathbf{i} - t\mathbf{j} + \mathbf{k}$. The initial position of the object is $\mathbf{i} - \mathbf{j}$ and the initial velocity $\mathbf{k}$. Find the

    (a) velocity, and
    (b) position of the object at time $t = 2$.

11. A plane is moving due south at a rate of 600 mph and is falling at a rate of 10 mph. If the plane is 5 miles high and 400 miles west of an observer at time $t = 0$, find an equation of the velocity and position of the plane at time t, relative to the observer.

12. A bug crawls vertically at 4 inches per minute along the surface of a cylinder of radius 3 which is revolving about its axis at 1 radian per minute. Using a suitable coordinate system, find a vector equation of the bug's (a) position and (b) velocity. Also find (c) the speed at time t.


1 Higher order partial derivatives
A Computation of second-order partial derivatives; double application of the chain rule
2 Approximations of functions by Taylor polynomials of degree 2
B Computation of the Taylor polynomial of degree 2
3 Classification, using the Hessian matrix, of critical vectors as local maxima or minima
C Finding local maxima and minima on regions and solids
4 Local maxima and minima on curves and surfaces
D Finding local maxima and minima on curves and surfaces
5 Absolute maxima and minima of continuous functions on closed, bounded sets
E Finding absolute maxima and minima on curves and regions
6 Maxima and minima on implicitly described sets, using Lagrange's method
F The Lagrange technique for finding maxima and minima on implicitly described sets

XII

Maxima and Minima

A set of real numbers is *bounded* if it is contained in some finite interval $[a, b]$. The *completeness* property of the system of real numbers says that every bounded set B of real numbers has a *least upper bound* d which is the smallest number greater than or equal to every number in the set B, and a *greatest lower bound* c which is the largest number less than or equal to every number in B. If d is in B, then d is the largest number or *maximum* of B; if c is in B, then c is the smallest number or *minimum* of B. For example, if $B = [0, 1)$, then the least upper bound is 1 and the greatest lower bound is 0. In this case, B does not have a maximum, since 1 is not in B; the number 0 is the minimum of B.

We associate with each scalar function f, having domain X, the real number set $f(X)$, which is the f-image of X. Then *f is bounded* according to whether or not $f(X)$ is bounded, and the *maximum* and *minimum values of f* are defined to be the maximum and minimum values of $f(X)$ whenever they exist. For example, if X denotes the points on a flat plate and $f(\mathbf{u})$, $\mathbf{u}$ in X, denotes the temperature at $\mathbf{u}$, then the maximum and minimum of f are respectively the temperatures at the hottest and coldest points on the plate. If X denotes the earth's surface and $f(\mathbf{u})$, $\mathbf{u}$ in X, is the elevation at $\mathbf{u}$, then the maximum and minimum of f are respectively the elevation at the highest and lowest points on the earth's surface.

Frequently *local maxima* and *minima* are of more interest than the *absolute* maximum and minimum, discussed in the preceding paragraph. A point $\mathbf{u}_0$ in the

domain X of f, where f is not necessarily bounded, gives a local maximum of f if $f(\mathbf{u}_0) \geq f(\mathbf{u})$ for all $\mathbf{u}$ sufficiently near $\mathbf{u}_0$; a similar definition holds for local minimum. Thus each "hot spot" on a flat plate and each hilltop on the earth's surface gives a local maximum.

In this chapter we develop techniques for finding local and absolute maxima and minima of scalar functions of several variables. Although our methods may be regarded as an extension of those employed in the study of maxima and minima in first-year calculus, the extension is not necessarily evident even after the methods have been developed. In order to fill this gap, we now offer a preview of the several-variable technique by looking at the one-variable procedure in a modification of the form usually seen in elementary calculus.

Let $f(x)$ be a function and assume that d^3f/dx^3 exists. It is known that local maxima and minima occur among values, called *critical values*, where the first derivative is zero. We seek to classify the critical values; each gives either (I) a local maximum, (II) a local minimum, or (III) an inflection point (neither (I) nor (II)). Consider the Taylor polynomial of degree 2 of f about a critical value c,

$$P_2(x) = f(c) + \frac{df}{dx}(c)(x-c) + \frac{d^2f}{dx^2}(c)\frac{(x-c)^2}{2!}.$$

Taylor's theorem says that $P_2(x)$ approximates $f(x)$ for x near c. Since c is a critical point, $df/dx(c) = 0$ and the approximation may be written

$$f(x) \approx f(c) + \frac{d^2f}{dx^2}(c)\frac{(x-c)^2}{2!},$$

or, in another way,

$$f(x) - f(c) \approx \frac{d^2f}{dx^2}(c)\frac{(x-c)^2}{2!}.$$

If this approximation is sufficiently good, and this can be proved using Taylor's theorem, then $f(x) - f(c)$ and

$$\frac{d^2f}{dx^2}(c)\frac{(x-c)^2}{2!}$$

have the same sign for x near c. This implies two things.

(I) If $d^2f/dx^2(c) < 0$, then $f(x) < f(c)$ for x near c, and hence, f has a local maximum at c.

(II) If $d^2f/dx^2(c) > 0$, then $f(x) > f(c)$ for x near c, and hence, f has a local minimum at c.

If $df^2/dx^2(c) = 0$, then nothing can be concluded from this analysis; further investigation must be made.

As may be guessed from the foregoing discussion, it will be expedient to develop background material on higher ordered vector derivatives and Taylor polynomials of functions of several variables. This will be done in the first two sections of this chapter.

1. Partial Derivatives of Higher Order

In first-year calculus the second-order derivative function of $f(x)$ is defined by

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

Second-order partial derivatives of a vector function $\mathbf{f}$ from an open set $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$ are defined similarly. Thus, the second-order partial derivative function of $\mathbf{f}$ with respect to x and y is

$$\mathbf{f}_{xy} = (\mathbf{f}_x)_y;$$

a common symbolism used elsewhere is $\partial^2\mathbf{f}/\partial x\, \partial y$. Application of the coordinate theorem for vector derivatives to $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ gives

$$\mathbf{f}_{rs} = (\mathbf{f}_r)_s = (\bar{x}_r\mathbf{i} + \bar{y}_r\mathbf{j})_s = \bar{x}_{rs}\mathbf{i} + \bar{y}_{rs}\mathbf{j}.$$

More generally, the second-order partial derivative function of $\mathbf{f}(x_1, \ldots, x_n)$ with respect to x_i and x_j is $\mathbf{f}_{x_i x_j} = (\mathbf{f}_{x_i})_{x_j}$, and the following statement holds.

Proposition 1.1

$$\mathbf{f}_{x_ix_j} = \langle(\bar{f}_1)_{x_ix_j}, \ldots, (\bar{f}_m)_{x_ix_j}\rangle.$$

Example 1.1 (a) Given $f(x, y) = x^2y^3$, then $f_x = 2xy^3$ and $f_y = 3x^2y^2$. The four second-order partial derivatives are

$$f_{xx} = 2y^3, \qquad f_{xy} = 6xy^2, \qquad f_{yx} = 6xy^2, \qquad f_{yy} = 6x^2y.$$

(b) Given $\mathbf{f}(x, y) = \langle xy^2, 3x^2 - y^3\rangle$, then

$$\mathbf{f}_x = \langle y^2, 6x\rangle \quad \text{and} \quad \mathbf{f}_{xy} = \langle 2y, 0\rangle.$$

Higher-order partial derivative functions, when they exist, are defined inductively by the equation

$$\mathbf{f}_{x_{i_1}x_{i_2}\cdots x_{i_k}} = (\mathbf{f}_{x_{i_1}x_{i_2}\cdots x_{i_{(k-1)}}})_{x_{i_k}}.$$

Example 1.2 The eight third-order partial derivatives of $\mathbf{f}(x, y)$ are $\mathbf{f}_{xxx}$, $\mathbf{f}_{xxy}$, $\mathbf{f}_{xyx}$, $\mathbf{f}_{xyy}$, $\mathbf{f}_{yxx}$, $\mathbf{f}_{yxy}$, $\mathbf{f}_{yyx}$, and $\mathbf{f}_{yyy}$. In Example 1.1(b) from $\mathbf{f}_{xy} = \langle 2y, 0\rangle$, we have $\mathbf{f}_{xyx} = \langle 0, 0\rangle$ and $\mathbf{f}_{xyy} = \langle 2, 0\rangle$.

A function $\mathbf{f}$ is *of class* C^2 if all second-order partial derivative functions of $\mathbf{f}$ exist and are continuous. More generally, $\mathbf{f}$ is *of class* C^k if all kth-order partial derivative functions exist and are continuous. Further, $\mathbf{f}$ is *of class* C^∞ if $\mathbf{f}$ is of

class C^k for each positive integer k. It can be shown the C^k property is preserved by addition, multiplication by a scalar, and composition, as well as the other function operations studied thus far. If a function is of class C^{k+1}, then it is also of class C^k. All functions formed from the rational, trigonometric, logarithmic, exponential functions of one variable, and coordinate functions using our various function operations are of class C^∞ on the domain of definition.

We now study in detail the second-order partial derivative of the composition $g \circ \mathbf{f}$, considering the C^2 functions $\mathbf{f}(r, s)$ and $g(x, y)$ first. By the chain rule, we can write

$$(g \circ \mathbf{f})_r = (g_x \circ \mathbf{f})\bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_r .$$

Applying the product rule of vector differentiation to this equality gives

$$(g \circ \mathbf{f})_{rs} = (g_x \circ \mathbf{f})\bar{x}_{rs} + (g_x \circ \mathbf{f})_s \bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_{rs} + (g_y \circ \mathbf{f})_s \bar{y}_r .$$

Again using the chain rule,

$$(g_x \circ \mathbf{f})_s = (g_{xx} \circ \mathbf{f})\bar{x}_s + (g_{xy} \circ \mathbf{f})\bar{y}_s$$

and

$$(g_y \circ \mathbf{f})_s = (g_{yx} \circ \mathbf{f})\bar{x}_s + (g_{yy} \circ \mathbf{f})\bar{y}_s .$$

Substituting the latter two equalities into the previous equality gives, after rearrangement,

$$\begin{aligned}(g \circ \mathbf{f})_{rs} = [(g_x \circ \mathbf{f})\bar{x}_{rs} + (g_y \circ \mathbf{f})\bar{y}_{rs}] + [(g_{xx} \circ \mathbf{f})\bar{x}_s\bar{x}_r + (g_{xy} \circ \mathbf{f})\bar{y}_s\bar{x}_r \\ + (g_{yx} \circ \mathbf{f})\bar{x}_s\bar{y}_r + (g_{yy} \circ \mathbf{f})\bar{y}_s\bar{y}_r].\end{aligned}$$

The foregoing procedure, rather than the resulting formula, should be learned.

Example 1.3 Given $\mathbf{f}(r, \theta) = \langle r\cos\theta, r\sin\theta\rangle$ and $g(x, y)$, then $\bar{x}(r, \theta) = r\cos\theta$, $\bar{y}(r, \theta) = r\sin\theta$, and, in abbreviated notation,

$$\begin{aligned}
g_r &= g_x \cos\theta + g_y \sin\theta, \\
g_{r\theta} &= g_x(-\sin\theta) + g_{x\theta}\cos\theta + g_y\cos\theta + g_{y\theta}\sin\theta, \\
g_{x\theta} &= g_{xx}x_\theta + g_{xy}y_\theta \\
&= g_{xx}(-r\sin\theta) + g_{xy}(r\cos\theta), \\
g_{y\theta} &= g_{yx}(-r\sin\theta) + g_{yy}(r\cos\theta), \\
g_{r\theta} &= (-\sin\theta)g_x + (\cos\theta)g_y + r\sin\theta\cos\theta(g_{yy} - g_{xx}) \\
&\quad + (r\cos^2\theta)g_{xy} - (r\sin^2\theta)g_{yx} .
\end{aligned}$$

The previously derived formula for $(g \circ \mathbf{f})_{rs}$ may be written as

$$(g \circ \mathbf{f})_{rs} = (\nabla g \circ \mathbf{f}) \cdot \mathbf{f}_{rs} + [\bar{x}_s \, \bar{y}_s](H_g \circ \mathbf{f})\begin{bmatrix}\bar{x}_r \\ \bar{y}_r\end{bmatrix},$$

where

$$H_g = \begin{bmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{bmatrix}.$$

The matrix H_g is called the *Hessian matrix* of g. Its entries are scalar functions. For $g(x_1, \ldots, x_m)$ it is defined by

$$H_g = \begin{bmatrix} g_{x_1x_1} & g_{x_2x_1} & \cdots & g_{x_mx_1} \\ g_{x_1x_2} & g_{x_2x_2} & \cdots & g_{x_mx_2} \\ \vdots & \vdots & & \vdots \\ g_{x_1x_m} & g_{x_2x_m} & \cdots & g_{x_mx_m} \end{bmatrix}.$$

Thus, the ith row of H_g is $(\nabla g)_{x_i}$. We now state an extension of our result for the case where $\mathbf{f}$ is a class C^2 function from $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$ and g is such that the composition $g \circ \mathbf{f}$ is defined.

Proposition 1.2

$$(g \circ \mathbf{f})_{x_ix_j} = (\nabla g \circ \mathbf{f}) \cdot \mathbf{f}_{x_ix_j} + \langle \mathbf{f}_{x_j} \rangle^*(H_g \circ \mathbf{f})\langle \mathbf{f}_{x_i} \rangle.$$

The symbol (*) denotes matrix transpose; thus, $\langle \mathbf{f}_{x_i} \rangle$ is a column matrix and $\langle \mathbf{f}_{x_j} \rangle^*$ a row matrix.

Example 1.4 The solution in Example 1.3 may be written in the form

$$\begin{aligned} g_{r\theta} &= \langle g_x, g_y \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \\ &\quad + [-r\sin\theta \quad r\cos\theta] \begin{bmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \end{aligned}$$

It may be observed in Example 1.1(a) that $f_{xy} = f_{yx}$. With suitable conditions on $\mathbf{f}$ this holds in general; a proof is in the appendix.

Theorem on Commutativity of Partial Differentiation

If $\mathbf{f}$ is a class C^2 function from $\mathbf{U}$ in $\mathbf{R}^n$ to $\mathbf{R}^m$, then

$$\mathbf{f}_{x_ix_j} = \mathbf{f}_{x_jx_i}.$$

It is a consequence of this theorem that if $\mathbf{f}$ is of class C^k, then partial differentiation of $\mathbf{f}$ with respect to any k coordinate variables is independent of the order of differentiation. Thus, if $\mathbf{f}(x, y)$ is of class C^3, then $\mathbf{f}_{xxy} = \mathbf{f}_{xyx} = \mathbf{f}_{yxx}$. The following corollary may be seen by inspection.

Proposition 1.3 H_g is symmetric.

Questions

1. If g is of class C^2, then g ______ be of class C^3.
   (a) must,
   (b) cannot,
   (c) may or may not.
2. If H_g is the Hessian matrix of $g(x, y, z)$, then the 23-entry of H_g is ______.
3. The Hessian matrix is always ______.
   (a) invertible,
   (b) symmetric,
   (c) orthogonal.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. Using the Hessian matrix find $(g \circ \mathbf{f})_{rt}$ if $\mathbf{f}(r, s, t) = \langle r^2 s, rt \rangle$ and $g(x, y) = x^2 - y^2$.
2. Write $(g \circ \mathbf{f})_{rt}$ in matrix form if $\mathbf{f}(r, s, t) = \langle r - s, t^2, r + 2t \rangle$ and $g(x, y, z) = x^2 y^3 z$.

Proofs

1. Prove Proposition 1.2 for C^2 functions $\mathbf{f}(r, s)$ and $g(x, y, z)$.
2. Given $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ with $f(0, 0) = 0$, show $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Why does this not contradict the theorem on commutativity of partial differentiation?

2. Taylor Polynomial of Degree 2

Given $f(x)$, if d^2f/dx^2 exists, then the Taylor polynomial of degree 2 of $f(x)$ about $x = 0$ is

$$P_2(x) = f(0) + \frac{df}{dx}(0)x + \frac{d^2f}{dx^2}(0)\frac{x^2}{2!}.$$

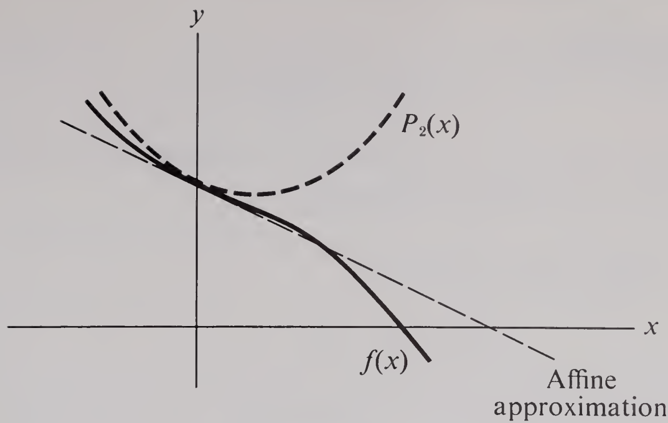


Figure 12.1

The function $P_2(x)$ approximates $f(x)$ near $x=0$; in fact it is a refinement of the affine approximation

$$f(0)+\frac{df}{dx}(0)x.$$

Geometrically, the affine approximation and $P_2(x)$ respectively give the straight line and parabola which best fit the curve of $f(x)$ near $x=0$ (see Figure 12.1).

We now seek a corresponding polynomial in several variables which approximates $f(x_1, \ldots, x_n)$ near a domain vector $\mathbf{u}_o$. It will be obtained by composing $P_2(x)$ with an identification function. A formula will first be obtained for a class C^2 function $f(x, y)$ about $\mathbf{u}_o = \langle x_o, y_o \rangle$. Letting $\mathbf{u} = \langle x, y \rangle$ be arbitrary near $\mathbf{u}_o$, we form the *identification function*

$$\lambda(t) = \mathbf{u}_o + t(\mathbf{u} - \mathbf{u}_o)$$

and obtain the Taylor polynomial of degree 2 of $f \circ \lambda$ about 0 (see Figure 12.2).

$$P_2(t) = (f \circ \lambda)(0) + \frac{d(f \circ \lambda)}{dt}(0)t + \frac{d^2(f \circ \lambda)}{dt^2}(0)\frac{t^2}{2!}.$$

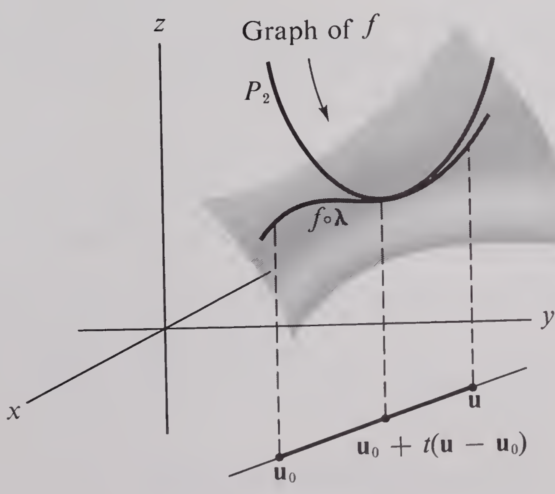


Figure 12.2

It is first noted that

$$(f \circ \lambda)(0) = f(\lambda(0)) = f(\mathbf{u}_o).$$

The other two terms in $P_2(t)$ will now be investigated. From Proposition 1.2 in Chapter X and the gradient formula we have,

$$\begin{aligned}\frac{d(f \circ \lambda)}{dt}(0) &= f_{\mathbf{u}-\mathbf{u}_o}(\mathbf{u}_o)\\ &= \nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o)\\ &= \langle f_x(x_o, y_o), f_y(x_o, y_o)\rangle \cdot \langle x - x_o, y - y_o\rangle.\end{aligned}$$

From

$$\frac{d\lambda}{dt} = \frac{d(\mathbf{u}_o + t(\mathbf{u}_1 - \mathbf{u}_o))}{dt} = \mathbf{u}_1 - \mathbf{u}_o = \langle x - x_o, y - y_o\rangle$$

and

$$\frac{d^2\lambda}{dt^2} = \mathbf{0},$$

we obtain from Proposition 1.2 of this chapter, replacing **f** by λ and g by f,

$$\begin{aligned}\frac{d^2(f \circ \lambda)}{dt^2}(0) &= (\nabla f \circ \lambda) \cdot \mathbf{0} + \langle \mathbf{u}_1 - \mathbf{u}_o\rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u}_1 - \mathbf{u}_o\rangle\\ &= \langle \mathbf{u}_1 - \mathbf{u}_o\rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u}_1 - \mathbf{u}_o\rangle\\ &= [x - x_o \quad y - y_o]\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix}_{\langle x_o, y_o\rangle}\begin{bmatrix} x - x_o \\ y - y_o\end{bmatrix}.\end{aligned}$$

Since $P_2(1) = f \circ \lambda(1) = f(\lambda(1)) = f(\mathbf{u}_o)$, we get the Taylor polynomial of degree 2 of $f(x, y)$ about $\langle x_o, y_o\rangle$ by setting $t = 1$ and substituting for the other expressions in the formula for $P_2(t)$. It is

$$\begin{aligned}&f(x_o, y_o) + \langle f_x(x_o, y_o), f_y(x_o, y_o)\rangle \cdot \langle x - x_o, y - y_o\rangle\\ &\qquad + \tfrac{1}{2}[x - x_o \quad y - y_o]\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix}_{\langle x_o, y_o\rangle}\begin{bmatrix} x - x_o \\ y - y_o\end{bmatrix}\\ &= f(x_o, y_o) + f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)\\ &\qquad + \tfrac{1}{2}f_{xx}(x_o, y_o)(x - x_o)^2 + f_{xy}(x_o, y_o)(x - x_o)(y - y_o)\\ &\qquad + \tfrac{1}{2}f_{yy}(x_o, y_o)(y - y_o)^2.\end{aligned}$$

The property $f_{xy} = f_{yx}$ was used in the last equality.

Example 2.1 Given $f(x, y) = xy^2$ and $\mathbf{u}_o = \langle 1, 3 \rangle$, then from $f_x = y^2$, $f_y = 2xy$, $f_{xx} = 0$, $f_{xy} = f_{yx} = 2y$, $f_{yy} = 2x$, the Taylor polynomial of degree 2 of f about $\mathbf{u}_o$ is

$$\begin{aligned}
&1(3)^2 + \langle 3^2, 2(1)(3) \rangle \cdot \langle x - 1, y - 3 \rangle \\
&\quad + \tfrac{1}{2}[x - 1 \quad y - 3]\begin{bmatrix} 0 & 2(3) \\ 2(3) & 2(1) \end{bmatrix}\begin{bmatrix} x - 1 \\ y - 3 \end{bmatrix} \\
&= 9 + 9(x - 1) + 6(y - 3) + 0(x - 1)^2 + 6(x - 1)(y - 3) + (y - 3)^2.
\end{aligned}$$

From the development of the Taylor polynomial of degree 2 for $f(x, y)$, we observe the following evident extension for a class C^2 function $f(\mathbf{u})$, where $\mathbf{u} = \langle x_1, \ldots, x_n \rangle$ is a variable element in $\mathbf{R}^n$.

Definition of Taylor Polynomial of Degree 2

The Taylor polynomial *of degree 2 of f about* $\mathbf{u}_o$ *is*

$$P(\mathbf{u}) = f(\mathbf{u}_o) + \nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o) + \tfrac{1}{2}\langle \mathbf{u} - \mathbf{u}_o \rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u} - \mathbf{u}_o \rangle.$$

Example 2.2 Given $f(x, y, z) = xy^2z^3$ and $\mathbf{u}_o = \langle 2, 1, 1 \rangle$, then $P(\mathbf{u})$ is the sum of the three terms shown below.

$$\begin{aligned}
f(\mathbf{u}_o) &= 2(1)^2(1)^3 = 2, \\
\nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o) &= \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle_{\langle 2, 1, 1 \rangle} \\
&\quad \cdot \langle x - 2, y - 1, z - 1 \rangle \\
&= (x - 2) + 4(y - 1) + 6(z - 1),
\end{aligned}$$

$$\tfrac{1}{2}\langle \mathbf{u} - \mathbf{u}_o \rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u} - \mathbf{u}_o \rangle =$$

$$\begin{aligned}
&= \tfrac{1}{2}[x - 2 \quad y - 1 \quad z - 1]\begin{bmatrix} 0 & 2yz^3 & 3y^2z^2 \\ 2yz^3 & 2xz^3 & 6xyz^2 \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{bmatrix}_{\langle 2, 1, 1 \rangle}\begin{bmatrix} x - 2 \\ y - 1 \\ z - 1 \end{bmatrix} \\
&= 2(y - 1)^2 + 6(z - 1)^2 + 2(x - 2)(y - 1) \\
&\qquad + 3(x - 2)(z - 1) + 12(y - 1)(z - 1).
\end{aligned}$$

The usefulness of any approximation requires some kind of knowledge about the approximation error. From

$$\nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o) = f_{\mathbf{u} - \mathbf{u}_o}(\mathbf{u}_o) = \mathrm{D}_{\mathbf{u}_o} f(\mathbf{u} - \mathbf{u}_o),$$

it is seen that

$$P(\mathbf{u}) = A(\mathbf{u}) + \tfrac{1}{2}\langle \mathbf{u} - \mathbf{u}_o \rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u} - \mathbf{u}_o \rangle,$$

where $A(\mathbf{u}) = f(\mathbf{u}_o) + D_{\mathbf{u}_o} f(\mathbf{u} - \mathbf{u}_o)$ is the affine approximation of f. The approximation theorem says that

$$\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{|f(\mathbf{u}) - A(\mathbf{u})|}{|\mathbf{u} - \mathbf{u}_o|} = 0.$$

Since $P(\mathbf{u})$ is a second-degree approximation extension of the first-degree approximation $A(\mathbf{u})$ of f, it should be expected that a stronger conclusion could be made about the absolute difference $|f(\mathbf{u}) - P(\mathbf{u})|$.

Second-Degree Approximation Theorem

If f is of class C^3, then

$$\lim_{\mathbf{u} \to \mathbf{u}_o} \frac{|f(\mathbf{u}) - P(\mathbf{u})|}{|\mathbf{u} - \mathbf{u}_o|^2} = 0.$$

The idea of the proof for $f(x, y)$ will now be considered. Taylor's theorem for a function of one variable $g(t)$ asserts that

$$g(t) = g(0) + \frac{dg}{dt}(0)t + \frac{d^2g}{dt^2}(0)\frac{t^2}{2!} + \frac{d^3g}{dt^3}(c)\frac{t^3}{3!},$$

where c lies between 0 and t. Substituting $g(t) = f \circ \lambda(t)$ and setting $t = 1$ gives

$$f(\mathbf{u}) = P(\mathbf{u}) + \frac{1}{3!}\frac{d^3(f \circ \lambda)}{dt^3}(c).$$

Letting $\mathbf{u}_o^* = \lambda(c) = \mathbf{u}_o + c(\mathbf{u} - \mathbf{u}_o)$, it can be shown by three successive applications of the chain rule formula that

$$\begin{aligned}\frac{d^3(f \circ \lambda)}{dt^3}(c) &= f_{xxx}(\mathbf{u}_o^*)(x - x_o)^3 + f_{yyy}(\mathbf{u}_o^*)(y - y_o)^3 \\ &\quad + 3f_{xxy}(\mathbf{u}_o^*)(x - x_o)^2(y - y_o) + 3f_{xyy}(\mathbf{u}_o^*)(x - x_o)(y - y_o)^2.\end{aligned}$$

By the continuity of f_{xxx} there is a $\delta_{xxx} > 0$ such that

$$|f_{xxx}(\mathbf{u})| \leq |f_{xxx}(\mathbf{u}_o)| + 1$$

for each $\mathbf{u}$ in $\sigma[\mathbf{u}_o\colon \delta_{xxx}]$. Similar conclusions can be obtained for the other third-order partial derivatives of f to give $\delta > 0$ and $M > 0$ such that each third-order partial derivative has absolute value $< M$ everywhere on $\sigma[\mathbf{u}_o\colon \delta]$. Thus, if $\mathbf{u}$ is in $\sigma[\mathbf{u}_o\colon \delta]$, then the absolute value of the sum expression for

$$\frac{d^3(f \circ \lambda)}{dt^3}(c)$$

is less than or equal to

$$\begin{aligned}&M|\mathbf{u} - \mathbf{u}_o|^3 + M|\mathbf{u} - \mathbf{u}_o|^3 + 3M|\mathbf{u} - \mathbf{u}_o|^2|\mathbf{u} - \mathbf{u}_o| + 3M|\mathbf{u} - \mathbf{u}_o|\,|\mathbf{u} - \mathbf{u}_o|^2 \\ &= 8M|\mathbf{u} - \mathbf{u}_o|^3.\end{aligned}$$

Hence, for $|\mathbf{u}-\mathbf{u}_o|<\delta$,

$$|f(\mathbf{u})-P(\mathbf{u})| = \left|\frac{1}{3!}\frac{d^3(f\circ\lambda)}{dt^3}(c)\right| \leq \frac{4M|\mathbf{u}-\mathbf{u}_o|^3}{3},$$

from which the desired conclusion is implied by

$$\lim_{\mathbf{u}\to\mathbf{u}_o}\frac{|f(\mathbf{u})-P(\mathbf{u})|}{|\mathbf{u}-\mathbf{u}_o|^2} \leq \lim_{\mathbf{u}\to\mathbf{u}_o}\frac{4}{3}M|\mathbf{u}-\mathbf{u}_o| = 0.$$

Example 2.3 Given $f(x, y) = x^3y$ and $\mathbf{u}_o = \langle 2, 1\rangle$, then $f_{xxx} = 6y$, $f_{xxy} = 6x$, and $f_{xyy} = f_{yyy} = 0$. If $\mathbf{u}$ is in $\sigma[\mathbf{u}_o: .01]$, then $|f_{xxx}(\mathbf{u})| \leq 6(1.01) = 6.06$ and $|f_{xxy}(\mathbf{u})| \leq 6(2.01) = 12.06$. Therefore, on $\sigma[\mathbf{u}_o: .01]$ we have

$$|f(\mathbf{u})-P(\mathbf{u})| \leq (4/3)(12.06)(.01)^3 < .00002.$$

Questions

1. The Taylor polynomial of degree 2 improves the ________ approximation.
2. The first-degree term of the Taylor polynomial $P(\mathbf{u})$ of f about $\mathbf{u}_o$ is the dot product of $\mathbf{u}-\mathbf{u}_o$ and ________.
3. The second-degree term of the Taylor polynomial $P(\mathbf{u})$ of f about $\mathbf{u}_o$ is a quadratic form of the matrix ________.
4. The second-degree approximation theorem says that the Taylor polynomial $P(\mathbf{u})$ of degree 2 of f about $\mathbf{u}_o$ satisfies the equality $\lim_{\mathbf{u}\to\mathbf{u}_o}$ ________ $= 0$.

Problems

1. Do Problem Set B at the end of the chapter.

Exercises

1. Given $f(x, y) = e^x \cos y$, $\mathbf{u}_o = \langle 0, \pi\rangle$:
   (a) find the Taylor polynomial $P(x, y)$ of degree 2 of f about $\mathbf{u}_o$, and
   (b) find an upper bound for $|f(\mathbf{u}) - P(\mathbf{u})|$ if $|\mathbf{u}-\mathbf{u}_o| < .1$.
2. Repeat 1 given $f(x, y) = e^{xy^2}$ and $\mathbf{u}_o = \langle 1, 0\rangle$.

3. Local Maxima and Minima on Open Sets

In this section we consider local maxima and minima of a class C^3 function f having as its domain an open set in $\mathbf{R}^n$. The results of the previous section apply to such functions.

Definition of Local Maxima and Minima

(I) *f has a* local maximum *at* $\mathbf{u}_o$ *provided that there exists* $\delta > 0$ *such that if* $|\mathbf{u} - \mathbf{u}_o| < \delta$, *then* $f(\mathbf{u}_o) \geq f(\mathbf{u})$;

(II) *f has a* local minimum at $\mathbf{u}_o$ *provided that there exists* $\delta > 0$ *such that if* $|\mathbf{u} - \mathbf{u}_o| < \delta$, *then* $f(\mathbf{u}_o) \leq f(\mathbf{u})$.

Example 3.1 Given $f(x, y)$, then f has a local maximum at $\mathbf{u}_o = \langle x_o, y_o \rangle$ if the surface graph $\mathscr{S}$ of f has a *peak* at (x_o, y_o). A local minimum point on $\mathscr{S}$ is called a *pit* (see Figure 12.3). Let f have a local maximum at $\mathbf{u}_o$ and $\mathbf{P}$ be a plane perpendicular to the xy plane and containing the line $y = y_o$ in the xy plane (see Figure 12.4). The intersection of $\mathbf{P}$ and $\mathscr{S}$ is a curve $\mathscr{C}$ which has a local maximum at $x = x_o$. From elementary calculus the tangent line to $\mathscr{C}$ at $(x_o, y_o, f(x_o, y_o))$ is horizontal, and this implies that $f_x(x_o, y_o) = 0$. Using a similar plane through the line $x = x_o$ gives $f_y(x_o, y_o) = 0$; therefore, $\nabla f(\mathbf{u}_o) = \langle f_x(\mathbf{u}_o), f_y(\mathbf{u}_o) \rangle = \langle 0, 0 \rangle$.

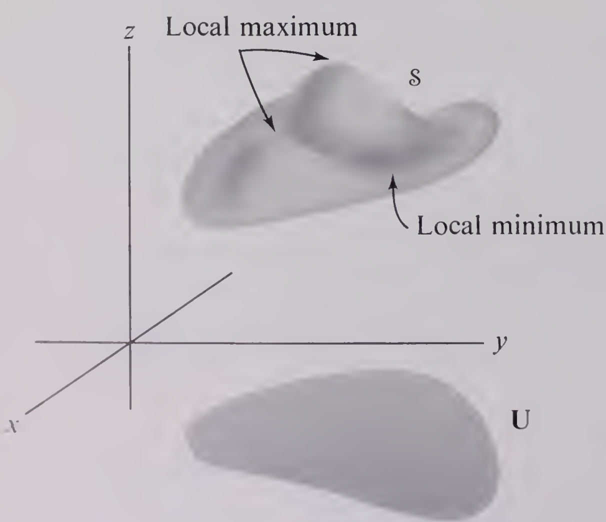


Figure 12.3

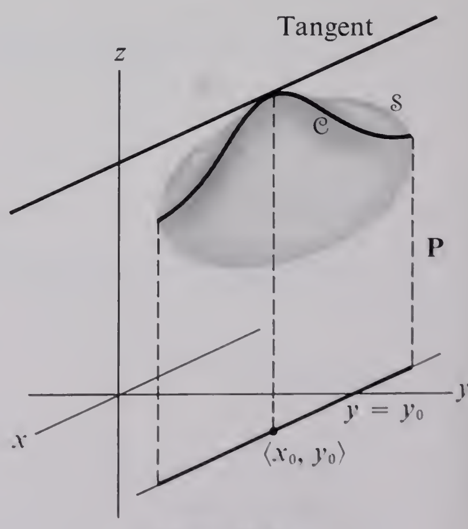


Figure 12.4

From our study of elementary calculus we know that c is a *critical value* of $f(x)$ if $df/dx(c) = 0$. If f is differentiable on an open interval, then its local maxima and minima occur at critical values. The previous example suggests the following extension to functions of several variables.

Definition of Critical Vector

A vector $\mathbf{u}_o$ *is a* critical vector *of f provided* $\nabla f(\mathbf{u}_o) = \mathbf{0}$.

By using the corresponding result from beginning calculus, together with an identification function, we now show the next property.

Proposition 3.1 If f has a local maximum or minimum at $\mathbf{u}_o$, then $\mathbf{u}_o$ is a critical vector of f.

A proof will be made for the local maximum case. Let λ be the identification function $\lambda(t) = \mathbf{u}_o + t\mathbf{v}$, $\mathbf{v}$ (arbitrary) $\neq \mathbf{0}$. There exists a $\delta > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta$, then $f(\mathbf{u}_o) \geq f(\mathbf{u})$. Let $|t_1| < \delta/|\mathbf{v}|$; then $|\lambda(t_1) - \mathbf{u}_o| = |t_1\mathbf{v}| < \delta$ and hence,

$$f \circ \lambda(0) = f(\mathbf{u}_o) \geq f(\lambda(t_1)) = f \circ \lambda(t_1).$$

Therefore, $f \circ \lambda$ has a local maximum at 0 and by elementary calculus and Proposition 1.2 in Chapter X,

$$0 = \frac{d(f \circ \lambda)}{dt}(0) = f_{\mathbf{v}}(\mathbf{u}_o).$$

Thus all vector derivatives vanish at $\mathbf{u}_o$, and this implies $\nabla f(\mathbf{u}_o) = \mathbf{0}$.

Example 3.2 Given $f(x, y) = x^3 + y^3 - xy$, then

$$\nabla f = \langle 3x^2 - y, 3y^2 - x \rangle,$$

and the critical vectors of f are found as simultaneous solutions of $3x^2 - y = 0$ and $3y^2 - x = 0$. It may be verified that $x = y = 0$ and $x = y = \frac{1}{3}$ are the solutions, and hence, $\langle 0, 0 \rangle$ and $\langle \frac{1}{3}, \frac{1}{3} \rangle$ are the critical vectors of f.

We next seek to classify the critical vectors. A critical value of $f(x)$ gives either (I) a local maximum, (II) a local minimum, or (III) an inflection point. An example of an inflection point occurs for $f(x) = x^3$ at $x = 0$. The corresponding term for functions of several variables is *saddle point*. Thus f has a saddle point at $\mathbf{u}_o$ if $\mathbf{u}_o$ is a critical vector of f which gives neither a local maximum nor minimum. The surface graph of $f(x, y) = y^3$ has a saddle point at each point on the x axis

(see Figure 12.5). A more interesting example, and one which illustrates the use of the term "saddle" is given next.

Example 3.3 The hyperbolic paraboloid $f(x, y) = y^2 - x^2$ has $\langle 0, 0\rangle$ for its only critical vector, and we shall now see that this gives a saddle point. An analysis of the *level curves* of f is helpful here. For each number c, the *c-level curve* is the xy-plane hyperbola $y^2 - x^2 = c$. Sketching the curves for various values of c gives a *contour graph* (see Figure 12.6). Since $f(0, 0) = 0$ and every disk centered at the origin intersects c-level curves for both positive and negative c, it follows that the origin is a saddle point.

We now seek an analytical method for classifying critical vectors. If $\mathbf{u}_o = \langle x_o, y_o\rangle$ is a critical vector of a class C^3 function $f(x, y)$, then $\nabla f(\mathbf{u}_o) = \mathbf{0}$. From the previous section the Taylor polynomial of degree 2 of f about $\mathbf{u}_o$ is then

$$P(\mathbf{u}) = f(\mathbf{u}_o) + \tfrac{1}{2}Q_H,$$

where

$$Q_H = [x - x_o \quad y - y_o]\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix}_{\mathbf{u}_o}\begin{bmatrix} x - x_o \\ y - y_o\end{bmatrix}$$

is the quadratic form of $H_f(\mathbf{u}_o)$ in the variables $x - x_o$ and $y - y_o$. Then $\mathbf{u}$ near $\mathbf{u}_o$ implies $f(\mathbf{u}) \approx P(\mathbf{u})$, and hence,

$$f(\mathbf{u}) - f(\mathbf{u}_o) \approx \tfrac{1}{2}Q_H.$$

We now assume that this approximation is sufficiently good to imply that $f(\mathbf{u}) - f(\mathbf{u}_o)$ and $\frac{1}{2}Q_H$ have the same sign when $\mathbf{u}$ is near $\mathbf{u}_o$. With this assumption, if $H_f(\mathbf{u}_o)$ is negative-definite, then $f(\mathbf{u}) - f(\mathbf{u}_o) < 0$ for $\mathbf{u}$ near $\mathbf{u}_o$, $\mathbf{u} \neq \mathbf{u}_o$, and therefore, f has a local maximum at $\mathbf{u}_o$. A similar argument for $H_f(\mathbf{u}_o)$ positive-definite and indefinite suggests the next result; a proof is in the appendix. (For the definition of the various definite properties, see Section 3 in Chapter VIII.)

Theorem on Critical Vectors

Let $\mathbf{u}_o$ be a critical vector of a class C^3 function f having as its domain an open set in $\mathbf{R}^n$;

(a) if $H_f(\mathbf{u}_o)$ is negative-definite, then f has a local maximum at $\mathbf{u}_o$,
(b) if $H_f(\mathbf{u}_o)$ is positive-definite, then f has a local minimum at $\mathbf{u}_o$, and
(c) if $H_f(\mathbf{u}_o)$ is indefinite, then f has a saddle point at $\mathbf{u}_o$.

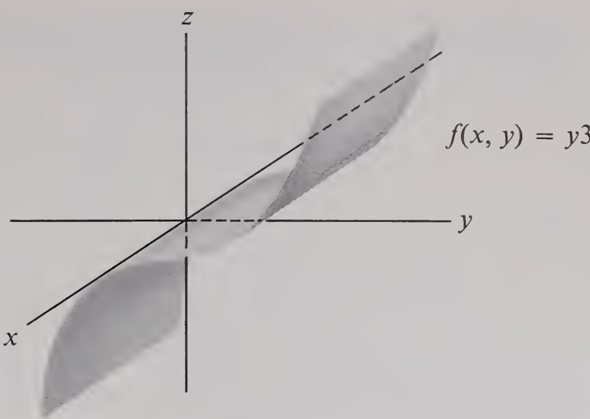


Figure 12.5

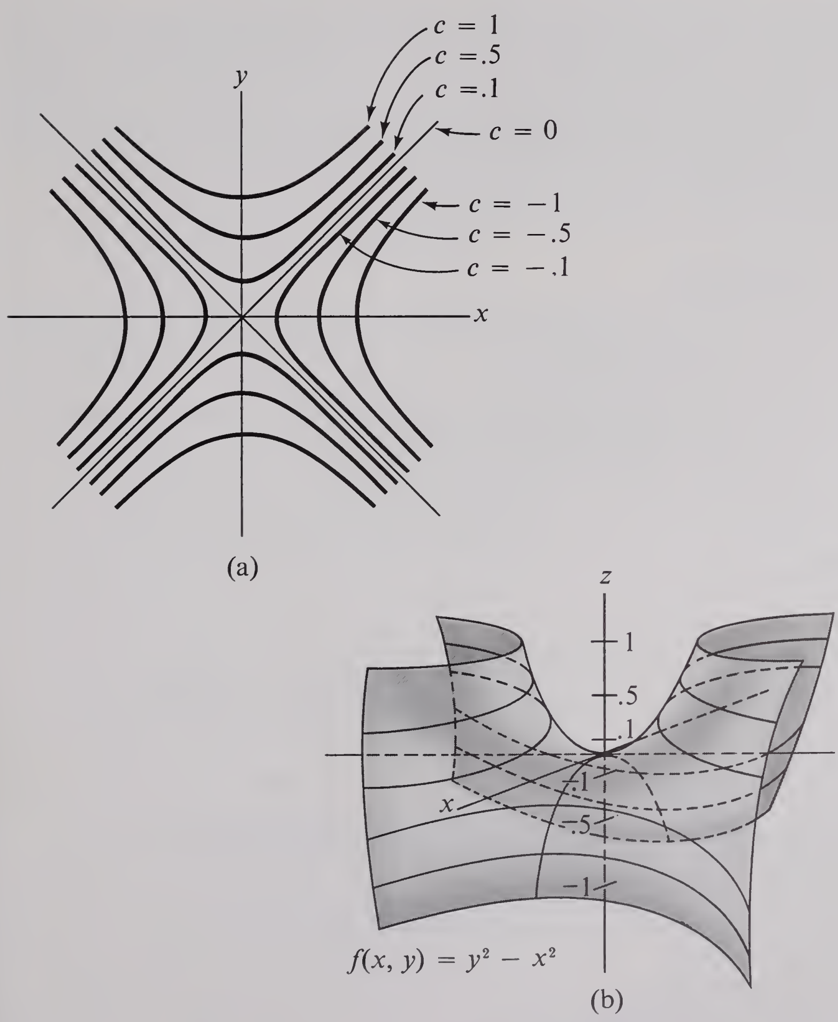


Figure 12.6

Example 3.4 The critical vectors of $f(x, y) = x^3 + y^3 - xy$ are $\langle 0, 0\rangle$ and $\langle \frac{1}{3}, \frac{1}{3}\rangle$ (see Example 3.2). From

$$H_f(\langle 0, 0\rangle) = \begin{bmatrix} 6x & -1 \\ -1 & 6y \end{bmatrix}_{\langle 0,0\rangle} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$H_f(\langle \tfrac{1}{3}, \tfrac{1}{3}\rangle) = \begin{bmatrix} 6x & -1 \\ -1 & 6y \end{bmatrix}_{\langle 1/3,\, 1/3\rangle} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

it may be verified that $H_f(\langle 0, 0\rangle)$ is indefinite and $H_f(\langle \frac{1}{3}, \frac{1}{3}\rangle)$ is positive-definite. Therefore, f has a saddle point at $\langle 0, 0\rangle$ and a local minimum at $\langle \frac{1}{3}, \frac{1}{3}\rangle$.

The theorem on critical vectors does not give information when $H_f(\mathbf{u}_0)$ is semidefinite, but not definite. This corresponds to the one-variable case where $d^2f/dx^2(c) = 0$ at a critical value c, in which case other methods must be used to classify c. The following is a partial result for the semidefinite properties; a proof is in the appendix.

Proposition 3.2

(a) If $\mathbf{u}_0$ is a local maximum of f, then $H_f(\mathbf{u}_0)$ is negative-semidefinite (or negative-definite).

(b) If $\mathbf{u}_0$ is a local minimum of f, then $H_f(\mathbf{u}_0)$ is positive-semidefinite (or positive-definite).

Questions

1. If $\mathbf{u}_0$ is a critical vector of f, then ______ $= 0$.

2. A critical vector of f ______ give a local maximum or minimum.
   (a) must,
   (b) cannot,
   (c) may or may not.

3. If $\nabla f(\mathbf{u}_0) = \mathbf{0}$ and f has neither a local maximum nor minimum at $\mathbf{u}_0$, then f has a ______ at $\mathbf{u}_0$.

4. If f has a local maximum at $\mathbf{u}_0$, then $H_f(\mathbf{u}_0)$ cannot be ______.
   (a) positive-definite,
   (b) negative-definite,
   (c) negative-semidefinite.

5. If $H_f(\mathbf{u}_o)$ is indefinite, then f ______ have a saddle point at $\mathbf{u}_o$.
   (a) must,
   (b) cannot,
   (c) may or may not.

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Determine those values of c for which $f(x, y) = x^2 + cxy + 2y^2 + 4x - 3$ has a local minimum.
2. Given $f(x, y) = xy$:
   (a) Sketch the c-level curves for various c near 0.
   (b) From (a) sketch the graph of f near (0, 0, 0).
   (c) Determine geometrically the nature of the critical vector $\langle 0, 0 \rangle$ of f.
3. Repeat 2 for $f(x, y) = x^2 + 2y^2$.

Proofs

1. Given $\mathbf{u}_o$ as a critical vector of $f(x, y)$, prove that
   (a) $\mathbf{u}_o$ gives a local maximum if $f_{xx}(\mathbf{u}_o) < 0$ and $\det H_f(\mathbf{u}_o) > 0$;
   (b) $\mathbf{u}_o$ gives a local minimum if $f_{xx}(\mathbf{u}_o) > 0$ and $\det H_f(\mathbf{u}_o) > 0$;
   (c) $\mathbf{u}_o$ gives a saddle point if $\det H_f(\mathbf{u}_o) < 0$.

4. Local Maxima and Minima on Curves and Surfaces

In this section we consider techniques of finding local maxima and minima on sets whose dimension is less than that of their embedding space. Instances of such sets are curves and surfaces. All given functions will be assumed to be of class C^3. The approach will be informal; a careful analysis of all factors involved is not appropriate here.

Let $\mathscr{C}$ be a curve in the Cartesian plane described by an injective function $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$ which has an open interval domain. If $g(x, y)$ is a scalar

function having as its domain an open set which includes $\mathscr{C}$, and $\mathbf{u}_o$ is in $\mathscr{C}$, then $g(x, y)$ assumes a local maximum on $\mathscr{C}$ at $\mathbf{u}_o = \langle x_o, y_o \rangle$ if $g(\mathbf{u}_o) \geq g(\mathbf{u})$ for all $\mathbf{u}$ which are on $\mathscr{C}$ near $\mathbf{u}_o$. A similar description holds for local minima. More precisely (see Figure 12.7),

---

$g(x, y)$ has a local maximum *on $\mathscr{C}$ at $\mathbf{u}_o$ provided there exists $\delta > 0$ such that if $\mathbf{u}$ is in both $\mathscr{C}$ and $\sigma[\mathbf{u}_o : \delta]$, then $g(\mathbf{u}_o) \geq g(\mathbf{u})$.*

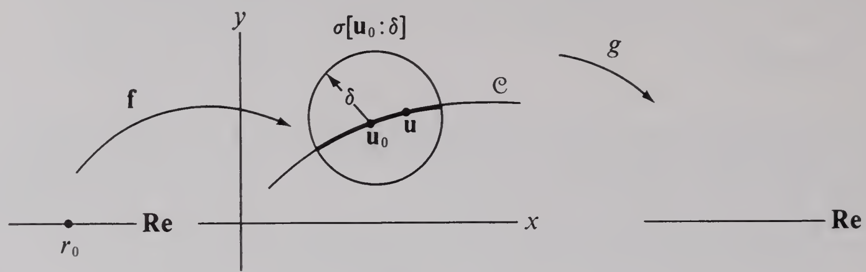


Figure 12.7

The definition for local minimum is obtained from this by replacing "$\geq$" by "$\leq$." In applying these definitions it will be assumed that if $\mathbf{f}(r_o) = \mathbf{u}_o$, then values on $\mathscr{C}$ near $\mathbf{u}_o$ are given by values of r near r_o. We shall not attempt to give precision to this assumed property. Geometrically it means that the curve $\mathscr{C}$ cannot return arbitrarily close to $\mathbf{u}_o$ (see Figure 12.8). It is then evident that g

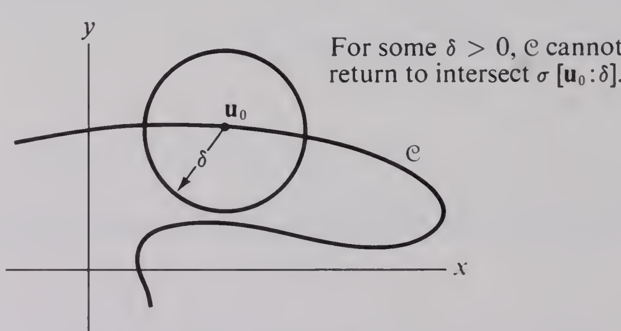


Figure 12.8

has a local maximum or minimum at $\mathbf{u}_o = \mathbf{f}(r_o)$ according to whether or not $g \circ \mathbf{f}$ has a local maximum or minimum at r_o. Therefore, we may conclude from elementary calculus the following statements.

(I) *If* $g(x, y)$ *has a local maximum or minimum at* $\mathbf{f}(r_o) = \mathbf{u}_o$, *then*

$$\frac{d(g \circ \mathbf{f})}{dr}(r_o) = 0.$$

(II) *Given*

$$\frac{d(g \circ \mathbf{f})}{dr}(r_o) = 0,$$

if

$$\frac{d^2(g \circ \mathbf{f})}{dr^2}(r_o) < 0,$$

then g has a local maximum at $\mathbf{f}(r_o)$; *if*

$$\frac{d^2(g \circ \mathbf{f})}{dr^2}(r_o) > 0,$$

then g has a local minimum at $\mathbf{f}(r_o)$.

If

$$\frac{d(g \circ \mathbf{f})}{dr}(r_o) = \frac{d^2(g \circ \mathbf{f})}{dr^2}(r_o) = 0,$$

then other methods from elementary calculus must be used to classify $\mathbf{f}(r_o)$.

Example 4.1 Let $g(x, y) = x - y$ on the curve $\mathscr{C}$ explicitly described by $y = x^2$. Then $\mathscr{C}$ has vector equation $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j}$, and this gives $g \circ \mathbf{f} = r - r^2$. From $d(g \circ \mathbf{f})/dr = 1 - 2r = 0$ we obtain $r = \frac{1}{2}$; therefore, $\mathbf{f}(\frac{1}{2}) = \langle \frac{1}{2}, \frac{1}{4} \rangle$ is the only candidate for a local maximum or minimum. Since $d^2(g \circ \mathbf{f})/dr^2 = -2 < 0$, it follows that g has a local maximum at $\langle \frac{1}{2}, \frac{1}{4} \rangle$.

The analysis of local maxima and minima on curves in Cartesian space is essentially the same as for plane curves. The only change is the addition of a third coordinate.

Example 4.2 Let $g(x, y, z) = x^2 + y^2 + z^2 - 6z$ on the helix

$$\mathbf{f}(r) = 2 \cos r\mathbf{i} + 2 \sin r\mathbf{j} + r\mathbf{k}.$$

Then $g \circ \mathbf{f} = 4 + r^2 - 6r$ and $(d(g \circ \mathbf{f}))/dr = 2r - 6 = 0$ gives $r = 3$.

Hence, $\mathbf{u}_o = \langle 2\cos 3, 2\sin 3, 3\rangle$ is the only possibility for a local maximum or minimum. Since $(d^2(g \circ \mathbf{f}))/dr^2 = 2 > 0$, it is seen that g has a local minimum at $\mathbf{u}_o$.

We next consider a surface $\mathcal{S}$ which is described by an injective vector function $\mathbf{f}(r, s)$ having an open set domain in $\mathbf{R}^2$. If $g(x, y, z)$ has for its domain an open set containing $\mathcal{S}$, then for $\mathbf{u}_o = \langle x_o, y_o, z_o\rangle$ in $\mathcal{S}$ (see Figure 12.9),

---

$g(x, y, z)$ has a local maximum on $\mathcal{S}$ *at* $\mathbf{u}_o$ *provided that there exists $\delta > 0$ such that if $\mathbf{u}$ is in both $\mathcal{S}$ and $\sigma[\mathbf{u}_o : \delta]$, then $g(\mathbf{u}_o) \geq g(\mathbf{u})$.*

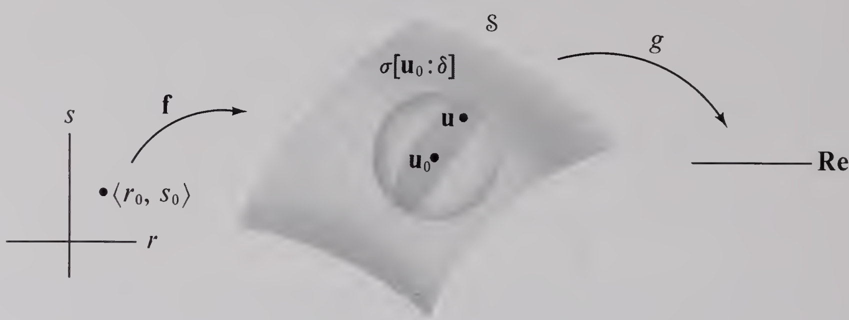


Figure 12.9

A dual definition holds for local minima. As with curves, it will be assumed that if $\mathbf{f}(r_o, s_o) = \mathbf{u}_o$, then vectors in $\mathcal{S}$ near $\mathbf{u}_o$ are images of vectors near $\langle r_o, s_o\rangle$. Then $g(x, y, z)$ has a local maximum or minimum at $\mathbf{u}_o = \mathbf{f}(r_o, s_o)$ according to whether or not $g \circ \mathbf{f}$ has a local maximum or minimum at $\langle r_o, s_o\rangle$. Since the domain of $g \circ \mathbf{f}$ is an open set in $\mathbf{R}^2$, the technique of the previous section may be employed. Hence,

---

(I) *if $g(x, y, z)$ has a local maximum or minimum at $\mathbf{u}_o = \mathbf{f}(r_o, s_o)$, then $\nabla(g \circ \mathbf{f})(r_o, s_o) = \mathbf{0}$; and*

(II) *if $\nabla(g \circ \mathbf{f})(r_o, s_o) = \mathbf{0}$, then g has a local maximum or minimum at $\mathbf{f}(r_o, s_o)$ according to whether $H_{g \circ \mathbf{f}}(r_o, s_o)$ is negative-definite or positive-definite.*

If $H_{g \circ \mathbf{f}}(r_o, s_o)$ is indefinite, then g has neither a local maximum nor minimum at $\mathbf{f}(r_o, s_o)$. If it is semidefinite, then another classification technique must be used.

Example 4.3 The surface of revolution $\mathscr{S}$ obtained by revolving $y = x^2$, $x > 0$ about the x axis has a vector equation

$$\mathbf{f}(x, \theta) = x\mathbf{i} + x^2 \cos \theta\mathbf{j} + x^2 \sin \theta\mathbf{k}.$$

If $g(x, y, z) = y^2 + z^2 - 4x + y/x^2$, then $g \circ \mathbf{f} = x^4 - 4x + \cos \theta$, and hence,

$$\nabla(g \circ \mathbf{f}) = \langle 4x^3 - 4, -\sin \theta\rangle.$$

From $4x^3 - 4 = 0$, $-\sin \theta = 0$, the critical vectors of $g \circ \mathbf{f}$ are $\langle 1, 0\rangle$ and $\langle 1, \pi\rangle$. The Hessian matrix of $g \circ \mathbf{f}$ is

$$H_{g \circ \mathbf{f}} = \begin{bmatrix} 12x^2 & 0 \\ 0 & -\cos \theta \end{bmatrix}.$$

At $\langle 1, 0\rangle$ this gives

$$\begin{bmatrix} 12 & 0 \\ 0 & -1 \end{bmatrix}$$

which is indefinite. At $\langle 1, \pi\rangle$ it gives

$$\begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}$$

which is positive-definite. Therefore, g has a local minimum at $\mathbf{f}(1, \pi) = \langle 1, -1, 0\rangle$, but no local maximum.

Questions

1. In order to find local maxima and minima of $g(x, y)$ on a curve described by $\mathbf{f}(r)$, we investigate critical values of the function ________
2. If $g(x, y, z)$ has a local maximum at $\mathbf{f}(r_o, s_o)$ on a surface described by $\mathbf{f}(r, s)$, then the matrix ________ is negative-definite (or negative-semi-definite).

Problems

1. Do Problem Set D at the end of the chapter.

Proofs

1. Assuming suitable differentiability conditions, prove that if $g(x, y)$ is defined on the graph of $y = f(x)$, then local maxima and minima of g occur at points (x, y) which simultaneously satisfy $g_y = -df/dx(g_x)$ and $y = f(x)$. (*Hint*: Let $\mathbf{f}(x) = x\mathbf{i} + f(x)\mathbf{j}$ and apply the chain rule to $g \circ \mathbf{f}$.)

2. Assuming suitable differentiability conditions, prove that if $g(x, y, z)$ is defined on the surface of revolution of $y = f(x) > 0$ about the x axis, then the local maxima and minima of g occur at points (x, y, z) which satisfy simultaneously

$$y^2 + z^2 = f^2, \quad yg_z = zg_y, \quad \text{and} \quad fg_x + \frac{df}{dx}(yg_y + zg_z) = 0.$$

5. Absolute Maxima and Minima

If $g(x)$ is continuous on a closed interval $[a, b]$, then from first-year calculus g necessarily attains a maximum at some value in $[a, b]$. Furthermore, the maximum value of g can be found by comparing values of g at the critical vectors of g and the end points a and b of $[a, b]$. The minimum value of g also exists and is found in the same way. In this section we extend this technique to finding maxima and minima of functions of several variables. For the purpose of ensuring existence of maxima and minima, we introduce the set properties *closed* and *bounded.* A set in $\mathbf{R}^n$ is *closed* if its complement (consisting of all vectors not in the set) is open. A set in $\mathbf{R}^n$ is *bounded* if there exists $\delta > 0$ such that the set is contained in $\sigma[\mathbf{0}: \delta]$. In $\mathbf{R}^2$ a set is closed provided it contains all its boundary points. A set is bounded if a sufficiently large disk centered at the origin can be chosen to encompass the set. We shall assume without proof the following statements.

(I) If f is continuous on a closed, bounded set in $\mathbf{R}^n$, then f attains a maximum and minimum.

(II) If f is continuous, then the f-image of a closed, bounded set is a closed bounded set.

It is easily seen that $[a, b]$ is a closed, bounded set in $\mathbf{Re}$. Hence, by (II) a curve $\mathscr{C}$ described by a continuous vector function $\mathbf{f}(r)$, $a \leq r \leq b$, is again a closed, bounded set. Let g be continuous on an open set containing $\mathscr{C}$; then the g-image of $\mathscr{C}$ and the $g \circ \mathbf{f}$-image of $[a, b]$ are equal. Hence, the maximum and minimum of g on $\mathscr{C}$ are the same as the maximum and minimum of $g \circ \mathbf{f}$ on $[a, b]$. From the previous section, we may use the following procedure to find the maximum and minimum of g on $\mathscr{C}$ whenever $\mathbf{f}$ and g are of class C^3 and $\mathbf{f}$ is injective on (a, b).

(I) *Find all solutions of* $d(g \circ \mathbf{f})/dr = 0$.

(II) *Compare* $g \circ \mathbf{f}$ *at the values in* (I) *and the end points* a *and* b.

Example 5.1 Given $g(x, y) = x - y$, we seek the maximum and minimum of g on the portion $\mathscr{C}$ of the parabola $y = x^2$ from (0, 0) to (1, 1). Then $\mathscr{C}$ is described by $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j}$, $0 \le r \le 1$, and hence, $g \circ \mathbf{f} = r - r^2$. From

$$\frac{d(g \circ \mathbf{f})}{dr} = 1 - 2r = 0$$

we obtain $r = \frac{1}{2}$. From the values

$$g \circ \mathbf{f}(0) = 0, \quad g \circ \mathbf{f}(\tfrac{1}{2}) = \tfrac{1}{4}, \quad \text{and} \quad g \circ \mathbf{f}(1) = 0,$$

it is seen that $\frac{1}{4}$ is the maximum and 0 the minimum of g on $\mathscr{C}$.

The foregoing technique applies equally well to curves in Cartesian space. We next consider the maximum and minimum of a function g on a closed, bounded set $\mathscr{R}$ in the Cartesian plane which consists of a simple closed curve $\mathscr{C}$ together with all points interior to $\mathscr{C}$. In this case the maximum and minimum of g must either be given by a point on $\mathscr{C}$, or by a local maximum or minimum point in the open set interior to $\mathscr{C}$. Therefore, the following procedure may be used to find the maximum and minimum of g on $\mathscr{R}$.

(I) *Find the maximum and minimum of g on $\mathscr{C}$.*
(II) *Find the critical vectors of g interior to $\mathscr{C}$.*
(III) *Compare the values in* (I) *and the values of g at the critical vectors in* (II).

Example 5.2 Let $g(x, y) = x^2 - 2y^2$ on the closed disk bounded by the circle $x^2 + y^2 = 1$. Then

$$\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad 0 \le \theta \le 2\pi,$$

describes the circle, and from $g \circ \mathbf{f} = \cos^2\theta - 2\sin^2\theta$ we have $d(g \circ \mathbf{f})/d\theta = 0$. This gives a maximum of 1 and a minimum of -2 on $\mathscr{C}$. From $g_x = 2x = 0$ and $g_y = -4y = 0$, the only critical vector of g is $\langle 0, 0\rangle$. Since $g(\langle 0, 0\rangle) = 0$, it follows that the maximum of g on the disk is 1 and the minimum is -2.

The foregoing techniques may be expanded to other types of sets. For instance, let $\mathscr{C}$ be a curve in the first quadrant of the xy plane described by $\mathbf{h}(x) = x\mathbf{i} + f(x)\mathbf{j}$, $a \le x \le b$. If $\mathscr{S}$ is the surface obtained by revolving $\mathscr{C}$ about the x axis, then $\mathscr{C}$ is described by

$$\mathbf{f}(x, \phi) = x\mathbf{i} + f(x)\cos\phi\mathbf{j} + f(x)\sin\phi\mathbf{k},$$

where $a \le x \le b$ and $0 \le \phi \le 2\pi$. The domain of **f** consists of an open rectangle and its boundary which consists of four closed line segments. The maximum and minimum of a function $g(x, y, z)$ on $\mathscr{S}$ can be found by comparing maximum and minimum values of $g \circ \mathbf{f}$ on each of the line segments with the values of $g \circ \mathbf{f}$ at its critical vectors on the open rectangle. In this case two of the line segments have the same **f**-image, and hence, attention can be restricted to three of the boundary segments (see Figure 12.10).

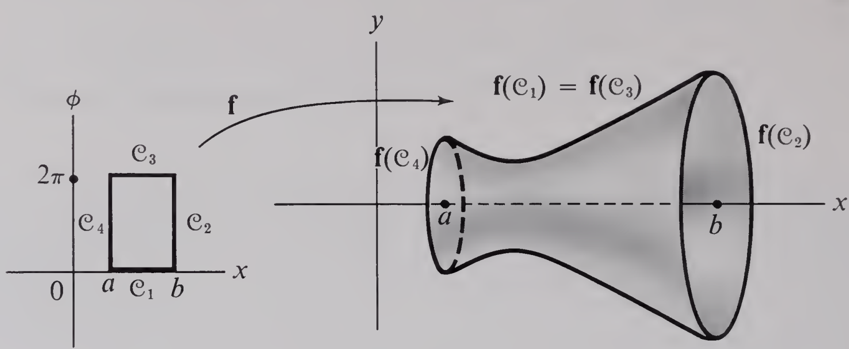


Figure 12.10

Example 5.3 Let

$$g(x, y, z) = 2y^2 + 2z^2 - 27x + \frac{y}{x^2}$$

on the surface $\mathscr{S}$ obtained by revolving $y = x^2$, $1 \le x \le 2$, about the x axis. Then $\mathscr{S}$ is described by

$$\mathbf{f}(x, \phi) = x\mathbf{i} + x^2 \cos \phi\mathbf{j} + x^2 \sin \phi\mathbf{k}, \qquad 1 \le x \le 2, \quad 0 \le \phi \le 2\pi.$$

We find the maximum and minimum of g on $\mathscr{S}$ from the following chart, which may be easily verified.

Set	$g \circ \mathbf{f}$	Max	Min
$\{\langle x, 0\rangle : 1 \le x \le 2\}$	$2x^4 - 27x + 1$	-21	$-235/8$
$\{\langle 1, \phi\rangle : 0 \le \phi \le 2\pi\}$	$-25 + \cos\phi$	-24	-26
$\{\langle 2, \phi\rangle : 0 \le \phi \le 2\pi\}$	$-22 + \cos\phi$	-21	-23
$\{\langle x, \phi\rangle : 1 < x < 2, 0 < \phi < 2\pi\}$	$2x^4 - 27x - \cos\phi$	None (local)	$-251/8$ (local)

It follows that on $\mathscr{S}$ the maximum of g is -21 and the minimum is $-251/8$.

Questions

1. A continuous function assumes a maximum and minimum if its domain is ______ and ______.
2. A set is closed if its complement is ______.

3. A set is bounded if it ______ a sufficiently large sphere.
   (a) includes,
   (b) is included in,
   (c) intersects.
4. The maximum of a continuous function $g(x, y)$ on a simple smooth curve $\mathscr{C}$ described by $\mathbf{f}(r)$, $a \le r \le b$, ______ occur at a critical value of $g \circ \mathbf{f}$.
   (a) must,
   (b) cannot,
   (c) may or may not.

Problems

1. Do Problem Set E at the end of the chapter.

Exercises

1. Find the maximum and minimum of $g(x, y) = 4x^2 + 5y^2 - 4x - 4y$ on the closed region bounded by the curves $y = x$ and $y = x^2$.
2. Find the distance from $(1, 0, 6)$ to the surface $z = y^2 + x$. (*Hint*: Find a vector equation $\mathbf{f}(r, s)$ of the surface and minimize $g \circ \mathbf{f}$ where
$$g(x, y, z) = (x - 1)^2 + y^2 + (z - 6)^2.)$$

Proofs

1. Let (x_1, y_1) be the point on $y = f(x)$ nearest (x_o, y_o). Show the equation
$$\frac{df}{dx}(x_1) = -\frac{x_1 - x_o}{y_1 - y_o}$$
is satisfied.
2. Let (x_o, y_o) be the point on the circle $(x - c)^2 + (y - d)^2 = a^2$ nearest the origin. Prove the line through the origin and (x_o, y_o) is perpendicular to the tangent line to the circle at (x_o, y_o).

6. *Lagrange's Multipliers*

Thus far we have considered maxima and minima on sets described by vector equations. In this section we shall obtain a technique for finding

maxima and minima on implicitly described sets. The procedure studied here is called the *method of Lagrange* (French: 1736–1813). In order to simplify the discussion it will always be assumed that conditions prevail to ensure that the required maximum and minimum necessarily exist. Suitable differentiability conditions on given functions will also be assumed.

Let $\mathscr{C}$ be a curve in the Cartesian plane implicitly described by $h(x, y) = 0$, and let $g(x, y)$ be a scalar function whose maximum and minimum on $\mathscr{C}$ are to be determined. Since a maximum or minimum is also a local maximum or minimum, we first seek conditions on $\mathbf{u}_o = \langle x_o, y_o \rangle$ for it to give a local maximum or minimum of g on $\mathscr{C}$. The curve $\mathscr{C}$ may be described near $\mathbf{u}_o$ by some vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, where $\mathbf{f}(r_o) = \mathbf{u}_o$. Then $g \circ \mathbf{f}$ has a local maximum or minimum at r_o, and hence, by the chain-rule formula,

$$0 = \frac{d(g \circ \mathbf{f})}{dr}(r_o) = g_x(\mathbf{u}_o)\frac{d\bar{x}}{dr}(r_o) + g_y(\mathbf{u}_o)\frac{d\bar{y}}{dr}(r_o).$$

Furthermore, since $h \circ \mathbf{f}(r) = h(\bar{x}(r), \bar{y}(r)) = 0$ for all r near r_o, we have, again by the chain-rule formula,

$$0 = \frac{d(h \circ \mathbf{f})}{dr}(r_o) = h_x(\mathbf{u}_o)\frac{d\bar{x}}{dr}(r_o) + g_y(\mathbf{u}_o)\frac{d\bar{y}}{dr}(r_o).$$

These two equations may be written in matrix form,

$$\begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix}_{\mathbf{u}_o} \begin{bmatrix} \dfrac{d\bar{x}}{dr} \\ \dfrac{d\bar{y}}{dr} \end{bmatrix}_{r_o} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We now assume that $d\bar{x}/dr(r_o)$ and $d\bar{y}/dr(r_o)$ are not both zero. It may be recalled from the study of tangents that this condition is also needed for $\mathbf{f}$ to assign a tangent to $\mathscr{C}$ at $\mathbf{u}_o$. Hence, this assumption is justified if some vector function assigns a tangent to $\mathscr{C}$ at $\mathbf{u}_o$. With this assumption, it follows from vector algebra (see Proposition 4.5 in Chapter VI) that the coefficient matrix has rank 0 or 1, and therefore the row vectors

$$\langle g_x(\mathbf{u}_o), g_y(\mathbf{u}_o) \rangle \quad \text{and} \quad \langle h_x(\mathbf{u}_o), h_y(\mathbf{u}_o) \rangle$$

are dependent. If $h_x(\mathbf{u}_o)$ and $h_y(\mathbf{u}_o)$ are not both zero, then there exists a real number λ such that

$$\langle g_x(\mathbf{u}_o), g_y(\mathbf{u}_o) \rangle + \lambda \langle h_x(\mathbf{u}_o), h_y(\mathbf{u}_o) \rangle = \langle 0, 0 \rangle.$$

This equation may be written

$$\nabla(g + \lambda h)(\mathbf{u}_o) = \mathbf{0},$$

meaning that $\mathbf{u}_o$ is a critical vector of $g + \lambda h$. We now summarize our informal development.

Let $\mathscr{C}$ be implicitly described by $h(x, y) = 0$ where h_x and h_y do not vanish simultaneously at any point. If $g(x, y)$ has a maximum or minimum on $\mathscr{C}$ at $\mathbf{u}_o$, then there exists a real number λ such that $\mathbf{u}_o$ is a critical vector of $g + \lambda h$.

The values for λ are called *Lagrange's multipliers.* The procedure for finding the maximum and minimum of g on $\mathscr{C}$ now follows from the foregoing discussion. Equating the coordinates of $\nabla(g + \lambda h) = \mathbf{0}$ together with $h = 0$ gives three equations,

$$\begin{aligned} g_x + \lambda h_x &= 0, \\ g_y + \lambda h_y &= 0, \\ h &= 0, \end{aligned}$$

in three unknowns x, y, and λ. Comparison of all $g(x_o, y_o)$ among solution tuples $\langle x_o, y_o, \lambda_o\rangle$ gives the maximum and minimum of g. The parameter λ plays only an intermediary role in finding the maximum and minimum. An alternative procedure is to find the simultaneous solutions of

$$\det\begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix} = 0 \quad \text{and} \quad h = 0.$$

Example 6.1 Let $g(x, y) = x^2 + y^2$ on the ellipse $5x^2 + 6xy + 5y^2 - 8 = 0$. The three equations

$$\begin{aligned} 0 &= g_x + \lambda h_x = 2x + \lambda(10x + 6y), \\ 0 &= g_y + \lambda h_y = 2y + \lambda(6x + 10y), \\ 0 &= h = 5x^2 + 6xy + 5y^2 - 8 \end{aligned}$$

give

$$\langle x, y\rangle = \langle\sqrt{2}, -\sqrt{2}\rangle, \langle-\sqrt{2}, \sqrt{2}\rangle, \left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle, \left\langle-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$$

as possible maximum or minimum points of g. Comparison of g values at these points shows the maximum of g is 4 and the minimum is 1.

Finding the maximum and minimum of $g(x, y, z)$ on an implicitly described surface $h(x, y, z) = 0$ follows essentially the same pattern. It is assumed that the surface is everywhere described locally by some $\mathbf{f}(r, s)$, where at each point $\mathbf{f}_r$ and $\mathbf{f}_s$ cannot both give the zero vector. In this case comparison of $g(x_o, y_o, z_o)$ is made among solutions $\langle x_o, y_o, z_o, \lambda_o\rangle$ of the four equations

$$\begin{aligned} g_x + \lambda h_x &= 0, \\ g_y + \lambda h_y &= 0, \\ g_z + \lambda h_z &= 0, \\ h &= 0. \end{aligned}$$

An alternative procedure is to find solutions of $\nabla g \times \nabla h = \mathbf{0}$ (see Proofs, exercise 1).

Example 6.2 Let

$$g(x, y, z) = x + y - \frac{35}{4} z^2$$

on the ellipsoidal surface

$$h(x, y, z) = x^2 + 2xy + 4y^2 + \frac{35}{4} z^2 - 9 = 0.$$

The four equations

$$\begin{aligned} 0 &= g_x + \lambda h_x = 1 + \lambda(2x + 2y), \\ 0 &= g_y + \lambda h_y = 1 + \lambda(2x + 8y), \\ 0 &= g_z + \lambda h_z = -\frac{70}{4} z + \lambda\left(\frac{70}{4}\right) z, \\ 0 &= h = x^2 + 2xy + 4y^2 + \frac{35}{4} z^2 - 9 \end{aligned}$$

have solutions $\langle x, y, z\rangle = \langle -\frac{1}{2}, 0, 1\rangle$, $\langle -\frac{1}{2}, 0, -1\rangle$, $\langle 3, 0, 0\rangle$, and $\langle -3, 0, 0\rangle$. Comparison of g at these values shows the maximum of g is 3 and the minimum is $-37/4$.

We next consider maximum and minimum values of $g(x, y, z)$ on the curve $\mathscr{C}$ in Cartesian space described implicitly by $h_1(x, y, z) = 0$ and $h_2(x, y, z) = 0$. If g has a maximum or minimum at $\mathbf{u}_o = \langle x_o, y_o, z_o\rangle$, and $\mathscr{C}$ is described near $\mathbf{u}_o$ by

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k},$$

where $\mathbf{f}(r_o) = \mathbf{u}_o$, then we may apply the chain rule to the equations

$$\frac{d(g \circ \mathbf{f})}{dr}(r_o) = 0,$$

$$\frac{d(h_1 \circ \mathbf{f})}{dr}(r_o) = 0,$$

$$\frac{d(h_2 \circ \mathbf{f})}{dr}(r_o) = 0$$

to obtain in matrix form

$$\begin{bmatrix} g_x & g_y & g_z \\ (h_1)_x & (h_1)_y & (h_1)_z \\ (h_2)_x & (h_2)_y & (h_2)_z \end{bmatrix}_{\mathbf{u}_o} \begin{bmatrix} \dfrac{d\bar{x}}{dr} \\ \dfrac{d\bar{y}}{dr} \\ \dfrac{d\bar{z}}{dr} \end{bmatrix}_{r_o} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $d\bar{x}/dr$, $d\bar{y}/dr$, and $d\bar{z}/dr$ do not vanish simultaneously at r_o, then from vector algebra, the rows $\nabla g(\mathbf{u}_o)$, $\nabla h_1(\mathbf{u}_o)$, and $\nabla h_2(\mathbf{u}_o)$ are dependent. If

$$\{\nabla h_1(\mathbf{u}_o), \nabla h_2(\mathbf{u}_o)\}$$

is independent, then there exist Lagrange multiplier numbers λ_1 and λ_2 such that

$$\nabla g(\mathbf{u}_o) + \lambda_1 \nabla h_1(\mathbf{u}_o) + \lambda_2 \nabla h_2(\mathbf{u}_o) = \mathbf{0}.$$

Thus maximum and minimum values of g are found among the critical vectors of $g + \lambda_1 h_1 + \lambda_2 h_2$ for some λ_1 and λ_2. We may therefore find the maximum and minimum of g by comparing values of $g(x_o, y_o, z_o)$ among solution tuples of $\langle \lambda_{1_o}, \lambda_{2_o}, x_o, y_o, z_o \rangle$ of the equations

$$\begin{aligned} g_x + \lambda_1 (h_1)_x + \lambda_2 (h_2)_x &= 0, \\ g_y + \lambda_1 (h_1)_y + \lambda_2 (h_2)_y &= 0, \\ g_z + \lambda_1 (h_1)_z + \lambda_2 (h_2)_z &= 0, \\ h_1 &= 0, \\ h_2 &= 0. \end{aligned}$$

An alternative procedure is to solve $h_1 = h_2 = 0$ simultaneously with the equation

$$\det \begin{bmatrix} g_x & g_y & g_z \\ (h_1)_x & (h_1)_y & (h_1)_z \\ (h_2)_x & (h_2)_y & (h_2)_z \end{bmatrix} = 0.$$

Example 6.3 We seek the maximum and minimum of $g(x, y, z) = 3x - y + 2z$ on the curve $\mathscr{C}$ of intersection of the sphere $x^2 + y^2 + z^2 - 105 = 0$ and the plane $x + 2y - 4z = 0$. The simultaneous solutions of

$$x^2 + y^2 + z^2 - 105 = 0,$$
$$x + 2y - 4z = 0,$$

and

$$\det\begin{bmatrix} 3 & -1 & 2 \\ 2x & 2y & 2z \\ 1 & 2 & -4 \end{bmatrix} = -28y - 14z = 0$$

are $\langle x, y, z\rangle = \langle 10, -1, 2\rangle$ and $\langle -10, 1, -2\rangle$. The maximum of g on $\mathscr{C}$ is $g(10, -1, 2) = 35$; the minimum is $g(-10, 1, -2) = -35$.

Questions

1. Lagrange's method is used to find the maximum and minimum on sets which are described ______.
   (a) explicitly,
   (b) implicitly,
   (c) by a vector equation.
2. The derivation of Lagrange's method for a curve $\mathscr{C}$ described by $h(x, y) = 0$ uses the assumption that ______ and ______ do not vanish simultaneously.
3. Lagrange's multipliers are ______.
   (a) functions,
   (b) vectors,
   (c) auxiliary numbers.

Problems

1. Do Problem Set F at the end of the chapter.

Proofs

1. Prove that the maximum and minimum of $g(x, y, z)$ on the surface $h(x, y, z) = 0$ occur at points (x, y, z) which satisfy $\nabla g \times \nabla h = \mathbf{0}$. (*Hint*: First show $\det\begin{bmatrix} g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = 0$.)
2. Using Lagrange's method prove that the distance from (x_0, y_0, z_0) to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Problems

A. *Second-Order Partial Derivatives*

The four *second-order partial derivatives of* $g(x, y)$ are

A.1 (a) $g_{xx} = (g_x)_x$, (b) $g_{xy} = (g_x)_y$,
(c) $g_{yx} = (g_y)_x$, (d) $g_{yy} = (g_y)_y$.

The following formula holds for functions to be considered here.

A.2 $g_{yx} = g_{xy}$.

1. Find all second-order partial derivatives of the given functions.

   (a) $g(x, y) = x^2y^3$, (b) $g(x, y) = e^{xy^2}$.

Similar formulas hold for $g(x, y, z)$. There are in this case 9 second-order partial derivatives with $g_{xy} = g_{yx}$, $g_{xz} = g_{zx}$, and $g_{yz} = g_{zy}$.

2. Find all second-order partial derivatives of $g(x, y, z) = xy^2z^3$.

Given $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ and $g(x, y)$, by the chain rule we have

$$(g \circ \mathbf{f})_r = (g_x \circ \mathbf{f})\bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_r .$$

The chain rule may again be applied to find $(g \circ \mathbf{f})_{rs}$. First applying the product rule of differentiation gives the equation

$$(g \circ \mathbf{f})_{rs} = (g_x \circ \mathbf{f})\bar{x}_{rs} + (g_x \circ \mathbf{f})_s \bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_{rs} + (g_y \circ \mathbf{f})_s \bar{y}_r .$$

Using the chain rule again gives

$$(g_x \circ \mathbf{f})_s = (g_{xx} \circ \mathbf{f})\bar{x}_s + (g_{xy} \circ \mathbf{f})\bar{y}_s ,$$

$$(g_y \circ \mathbf{f})_s = (g_{yx} \circ \mathbf{f})\bar{x}_s + (g_{yy} \circ \mathbf{f})\bar{y}_s .$$

Substitution into the equation for $(g \circ \mathbf{f})_{rs}$ gives the desired result. It is the foregoing procedure, rather than any new formulas, which should be learned.

3. Given $\mathbf{f}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$ and $g(x, y)$, find

   (a) $(g \circ \mathbf{f})_r$, (b) $(g_x \circ \mathbf{f})_\theta$, (c) $(g_y \circ \mathbf{f})_\theta$,
   (d) $(g \circ \mathbf{f})_{r\theta}$, (e) $(g_x \circ \mathbf{f})_r$, (f) $(g_y \circ \mathbf{f})_r$,
   (g) $(g \circ \mathbf{f})_{rr}$, (h) $(g \circ \mathbf{f})_{\theta\theta}$.

4. Given $\mathbf{f}(r, s) = (r^2 - s^2)\mathbf{i} + 2rs\mathbf{j}$ and $g(x, y)$, find

   (a) $(g \circ \mathbf{f})_{rr}$, (b) $(g \circ \mathbf{f})_{rs}$, (c) $(g \circ \mathbf{f})_{ss}$.

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5. Find all second-order partial derivatives of the following functions.

   (a) x^4y^5, (b) $\ln (x + y^2)$, (c) xze^{xy}.

6. Given $\mathbf{f}(r, s) = rs^2\mathbf{i} + (4r^3 - s)\mathbf{j}$ and $g(x, y)$, find all second-order partial derivatives of $g \circ \mathbf{f}$.

B. *Taylor Polynomials of Degree 2*

Given $f(x, y)$, the Taylor polynomial of degree 2 about $\mathbf{u}_0 = \langle x_0, y_0 \rangle$ is

B.1
$$f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \tfrac{1}{2}f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \tfrac{1}{2}f_{yy}(x_0, y_0)(y - y_0)^2.$$

1. Find the Taylor polynomial of degree 2 for each given function.

   (a) $f(x, y) = x^3y$ about $\langle 2, -1 \rangle$,
   (b) $f(x, y) = e^{xy^2}$ about $\langle 1, 0 \rangle$.

In finding Taylor polynomials of degree 2 for functions having more than two variables, it is convenient to use the *Hessian matrix*. For $f(x, y)$ and $f(x, y, z)$ the Hessian matrix H_f is, respectively,

B.2
$$\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}.$$

2. Find the Hessian matrix of

   (a) $f(x, y) = xy^3$, (b) $f(x, y, z) = x^2yz^3$.

The Taylor polynomial of degree 2 of f about $\mathbf{u}_o$ is in general the sum of three expressions,

(I) constant: $f(\mathbf{u}_o)$,
(II) linear: $\nabla f(\mathbf{u}_o) \cdot (\mathbf{u} - \mathbf{u}_o)$, and
(III) quadratic: $\frac{1}{2}\langle \mathbf{u} - \mathbf{u}_o\rangle^* H_f(\mathbf{u}_o)\langle \mathbf{u} - \mathbf{u}_o\rangle$.

For $f(x, y, z)$, $\mathbf{u}_o = \langle x_o, y_o, z_o\rangle$, and $\mathbf{u} = \langle x, y, z\rangle$, this becomes

B.3 (I) $f(x_o, y_o, z_o)$,

(II) $\langle f_x, f_y, f_z\rangle|_{\langle x_o, y_o, z_o\rangle} \cdot \langle x - x_o, y - y_o, z - z_o\rangle$

(III) $$\tfrac{1}{2}[x - x_o \quad y - y_o \quad z - z_o]\begin{bmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz}\end{bmatrix}_{\langle x_o, y_o, z_o\rangle}\begin{bmatrix} x - x_o \\ y - y \\ z - z_o\end{bmatrix}$$

3. Find the constant, linear, and quadratic terms of the Taylor polynomial of f about $\mathbf{u}_o$ for each of the situations below.

(a) $f(x, y, z) = xy^2z^3$ and $\mathbf{u}_o = \langle 1, 1, 2\rangle$,
(b) $f(x, y, z) = x^2e^{yz}$ and $\mathbf{u}_o = \langle 1, 0, 2\rangle$.

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4. Find the Taylor polynomial of degree 2 for the following functions and points.

(a) $f(x, y) = x^2y^2$ about $\langle 3, 2\rangle$,
(b) $f(x, y) = \cos x^2y$ about $\langle 1, \pi\rangle$.

5. Find the constant, linear, and quadratic terms of the Taylor polynomial of $f(x, y, z) = xy - ze^x$ about $\langle 0, 1, 3\rangle$.

C. *Local Maxima and Minima on Regions, Solids*

Given $g(x, y)$ with domain a region in the Cartesian plane, then $\mathbf{u}_o$ is a *critical vector* of g if $\nabla g(\mathbf{u}_o) = \mathbf{0}$. Thus, critical vectors of $g(x, y)$ are found among solutions of the two equations

C.1 $g_x(\mathbf{u}_o) = 0$,
$g_y(\mathbf{u}_o) = 0$.

1. Find the critical vectors of $g(x, y) =$

(a) $x^2 + y^3 - 3xy$, (b) $x^2 + y^2 - 4xy$,
(c) $x^2 + y^2 - 2xy$, (d) $x^3 + y^3 + x$.

A corresponding definition holds for critical vectors of a function $g(x, y, z)$ having for its domain a solid in Cartesian space. Thus critical vectors satisfy the criteria

C.2 $$g_x(\mathbf{u}_o) = 0,$$
$$g_y(\mathbf{u}_o) = 0,$$
$$g_z(\mathbf{u}_o) = 0.$$

2. Find the critical vectors of $g(x, y, z) =$

   (a) $xy - xz^2 + 2z - y$, (b) $x^2z - y^2z + 6y - x^2$,
   (c) $2xy + yz - xz + x - 4y$.

Critical vectors give either local maxima, local minima, or saddle points. If $H_g(\mathbf{u}_o)$ is the Hessian matrix of g evaluated at a critical vector $\mathbf{u}_o$ then

C.3 (a) g has a local maximum at $\mathbf{u}_o$ if $H_g(\mathbf{u}_o)$ is negative-definite,
(b) g has a local minimum at $\mathbf{u}_o$ if $H_g(\mathbf{u}_o)$ is positive-definite, and
(c) g has a saddle point at $\mathbf{u}_o$ if $H_g(\mathbf{u}_o)$ is indefinite.

If $H_g(\mathbf{u}_o)$ is semidefinite, but not definite, then other means must be devised; this case will not be considered here.

3. For each critical vector $\mathbf{u}_o$ in 1(a) follow the procedure outlined below.

   (a) Find $H_g(\mathbf{u}_o)$.
   (b) Determine the definite nature of $H_g(\mathbf{u}_o)$ using the techniques in Problem Set B of Chapter VIII.
   (c) Specify from (b), if possible, whether g has a local maximum, local minimum, or saddle point at $\mathbf{u}_o$.

4. Repeat Problem 3 for the critical vectors in 1(b), (c), (d) and 2(a), (b), (c).

5. Find the local maxima, local minima, and saddle points for the following functions.

   (a) $g(x, y) = x^3 + y^3 - 6xy$,
   (b) $g(x, y, z) = x^2 + y^2 + z^3 - xy - 6x - 3z$.

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6. Find the local maxima, local minima, and saddle points.

   (a) $x^3 + y^2 - 6x^2$, (b) $x^3 + y^3 - 3xy$,
   (c) $x^2 + 2x + y^3 - 3y + 5$, (d) $-x^2 - 2x - y^3 + 12y + 8$,
   (e) $xz + y^2z - x + 4y$, (f) $x^2 + y^2 + z^2 - xy - x - y$,
   (g) $x^2 + 5y^2 + 2z^2 + 4y + 8z$.

D. *Local Maxima and Minima on Curves, Surfaces*

Given $g(x, y)$ with its domain a curve $\mathscr{C}$ in the Cartesian plane described by $\mathbf{f}(r)$, the local maxima and local minima points of g on $\mathscr{C}$ may be obtained as follows.

D.1 (I) Set $\dfrac{d(g \circ \mathbf{f})}{dr} = 0$ and solve for r.

(II) If r_o is a solution of (I), then g has a local maximum at r_o if $d^2(g \circ \mathbf{f})/dr^2(r_o) < 0$ and a local minimum if $d^2(g \circ \mathbf{f})/dr^2(r_o) > 0$.

If $d^2(g \circ \mathbf{f})/dr^2(r_o) = 0$, then other methods from first-year calculus, such as the comparison of values of $g \circ \mathbf{f}$ at and near r_o, may be used to determine the nature of $\mathbf{f}(r_o)$.

1. Find the local maxima and local minima points of

(a) $g(x, y) = x^2 + y$ on the circle $\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$,
(b) $g(x, y) = 24x - 2y$ on the parabola $y = 3x^2$,
(c) $g(x, y) = x + y^2$ on the ellipse $x^2/4 + y^2/9 = 1$.

The technique of D.1 also applies to curves in space.

2. Find the local maxima and local minima points of $g(x, y, z) = x - y + 4z$ on the curve $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j} - r\mathbf{k}$.

Given $g(x, y, z)$ on a surface $\mathscr{S}$ represented by $\mathbf{f}(r, s)$, the local maxima and local minima points may be determined according to the following procedure.

D.2 (I) Set $(g \circ \mathbf{f})_r = (g \circ \mathbf{f})_s = 0$ and solve for r and s.
(II) If (r_o, s_o) is a solution of (I), then g has a local maximum at $\mathbf{f}(r_o, s_o)$ if $H_g \circ \mathbf{f}(r_o, s_o)$ is negative-definite and a local minimum if $H_g \circ \mathbf{f}(r_o, s_o)$ is positive-definite.

If $H_g \circ \mathbf{f}(r_o, s_o)$ is indefinite, then $\langle r_o, s_o \rangle$ gives neither a local maximum nor minimum. If $H_g \circ \mathbf{f}(r_o, s_o)$ is semidefinite, then the method of D.2 fails.

3. Find the local maxima and local minima points of

(a) $g(x, y, z) = x^3 + y^2 - 6z$ on the surface $z = xy$,
(b) $g(x, y, z) = xy + z$ on the surface

$$\mathbf{f}(r, s) = r\mathbf{i} + (r^2 - 12)\mathbf{j} + (s^3 - 6s^2)\mathbf{k},$$

(c) $g(x, y, z) = x(y^2 + z^2) - 3z$ on the conical surface obtained by revolving $y = x$ about the x axis.

4. Find the local maxima and local minima points of

   (a) $g(x, y) = x^3 - 6y$ on the parabola $x = y^2$,
   (b) $g(x, y) = 12x - y$ on $y = x^3$,
   (c) $g(x, y, z) = x^2 + y - 2z$ on $\mathbf{f}(r) = r\mathbf{i} + 4r\mathbf{j} + r^3\mathbf{k}$.

5. Find the local maxima and local minima points of

   (a) $g(x, y, z) = x^2 + y(z - 3)$ on the surface
   $$\mathbf{f}(r, s) = (r + 1)\mathbf{i} + s\mathbf{j} + s^2\mathbf{k},$$
   (b) $g(x, y, z) = x(4y + z) + 2y^2$ on the surface $z = x^2$,
   (c) $g(x, y, z) = x + y$ on the surface of revolution of $y = x^2$ about the y axis.

E. *Absolute Maxima and Minima*

In elementary calculus the (absolute) maximum and minimum of a function $g(x)$, $a \leq x \leq b$, was found by comparison of values of g at zero derivative points with values of g at the end points a and b. A similar procedure applies to finding the maximum and minimum of a function on a curve in the Cartesian plane or space.

1. Find the maximum and minimum of $g(x, y) = x^2 + y$ on the semicircle $x^2 + y^2 = 1$, $y \geq 0$ as follows.

   (a) Represent the semicircle by $\mathbf{f}(\theta)$, $a \leq \theta \leq b$.
   (b) Compare values of g where $d(g \circ \mathbf{f})/d\theta = 0$ and at end points $\mathbf{f}(a)$, $\mathbf{f}(b)$.

2. Find the maximum and minimum of

   (a) $g(x, y) = x - y$ on the parabolic arc $y = x^2$, $0 \leq x \leq 1$,
   (b) $g(x, y) = x - 2y$ on the ellipse $x^2/4 + y^2/9 = 1$. (*Note*: The ellipse has no end point but the foregoing technique is nevertheless valid. The ellipse is described by $\mathbf{f}(\theta)$, $a \leq \theta \leq b$, where $\mathbf{f}(a) = \mathbf{f}(b)$.)

The distance from a point (c, d) to a curve $\mathbf{f}(r)$ may be obtained as follows in E.1.

E.1 (I) Let $g(x, y) = (x - c)^2 + (y - d)^2$.
(II) Find the value $r = r_0$ which minimizes $g \circ \mathbf{f}$.
(III) The desired distance is $\sqrt{g \circ \mathbf{f}(r_0)}$.

This procedure works, since the functions $g \circ \mathbf{f}$ and $\sqrt{g \circ \mathbf{f}}$ attain a minimum at the same point.

3. Find the distance

   (a) from the origin to $\mathbf{f}(r) = (r^2 - 8)\mathbf{i} + (4r + 3)\mathbf{j}$,
   (b) from the origin to the circle $(x - 3)^2 + (y - 4)^2 = 1$,
   (c) from $(-1, 1)$ to the parabola $f(x) = x^2 + x$.

A simple extension of E.1 gives the distance from a point to a curve in space.

4. Find the distance from the origin to

$$\mathbf{f}(r) = (2r + 1)\mathbf{i} + r^2\mathbf{j} + (r^2 - 2)\mathbf{k}.$$

Next let $g(x, y)$ have as its domain a region in the Cartesian plane consisting of a simple closed curve, such as a circle, ellipse, triangle, rectangle, together with its interior. The general procedure for finding the absolute maximum and minimum of g is described in E.2.

E.2 (I) Find the critical vectors of g in the interior of the domain.
(II) Find the maximum and minimum of g on the boundary curve of the domain.
(III) Compare values in (II) with values of g at the critical vectors in (I).

5. Find the maximum and minimum of $g(x, y) = x^2 - 2y^2$ on the closed disc bounded by the circle $x^2 + y^2 = 1$.

6. Find the maximum and minimum of $g(x, y) = x^2 - 4xy + 2y$ on the closed rectangular region with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. (*Hint*: Find maximum and minimum values on each side of the rectangle and compare with values at critical vectors in the interior of the rectangle.)

7. Find the maximum and minimum of $g(x, y) = 5x^2 + 5y^2 - 4x - 4y$ on the triangular region with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

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8. Find the maximum and minimum of

   (a) $g(x, y) = x - 24y$ on $x - 1 = 3y^2$, $0 \le y \le 5$,
   (b) $g(x, y) = 2x - y$ on $(x - 2)^2 + y^2 = 1$.

9. Find the distance

   (a) from $(-15, 5)$ to $f(x) = x^2 + 3x$,
   (b) from $(2, 13/2, -1)$ to $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j} + 2r\mathbf{k}$.

10. Find the maximum and minimum of

    (a) $g(x, y) = 4x^2 + 4y^2 + 6x - 7y$ on the disc $x^2 + y^2 \le 1$,
    (b) $g(x, y) = 5x^2 + 2y^2 - 9x - 3y + 4$ on the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

F. *Lagrange's Method*

Given $g(x, y)$ on a curve implicitly described by $h(x, y) = 0$, the maximum and minimum values of g (if they exist) can often be found by the following procedure.

F.1 (I) Solve simultaneously $h = 0$ and

$$\det \begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix} = 0.$$

(II) Compare values of g at solution values found in (I). The maximum and minimum of g on the curve $h = 0$ occur among these.

1. Find the maximum and minimum of

   (a) $g(x, y) = x^2 + y$ on the circle $x^2 + y^2 = 1$,
   (b) $g(x, y) = x^2 + y^2$ on the ellipse $5x^2 + 6xy + 5y^2 - 8 = 0$.

2. Find the distance from the origin to the ellipse $x^2 + xy + y^2 = 1$. (*Hint*: Minimize $g(x, y) = x^2 + y^2$.)

Given $g(x, y, z)$ on a curve in space implicitly described by $h_1(x, y, z) = 0$ and $h_2(x, y, z) = 0$, the maximum and minimum can be found as follows.

F.2 (I) Solve simultaneously $h_1 = 0$, $h_2 = 0$ and

$$\det \begin{bmatrix} g_x & g_y & g_z \\ (h_1)_x & (h_1)_y & (h_1)_z \\ (h_2)_x & (h_2)_y & (h_2)_z \end{bmatrix} = 0.$$

(II) Compare values of g at solution values found in (I). The maximum and minimum of g on the curve $h_1 = h_2 = 0$ occur among these.

3. Find the

   (a) maximum and minimum of $g(x, y, z) = y - z$ on the intersection of the cylinder $x^2 + y^2 = 40$ and plane $x + 2y + z = 8$,
   (b) minimum of $g(x, y, z) = x^2 + 2y^2 + 3z^2$ on the line which is the intersection of the planes $x + y + z = 3$ and $x + 2y + z = 2$.

4. Find the distance from the origin to the curve of intersection of $x^2 + 2y^2 + z^2 = 5$ and $x + 3y + 2z = 0$.

Given $g(x, y, z)$ on the surface implicitly described by $h(x, y, z) = 0$, then the maximum and minimum of g, if they exist, may often be found as follows:

F.3 (I) Solve $\nabla g \times \nabla h = \mathbf{0}$ (this gives three equations) and $h = 0$ simultaneously.
(II) Compare values of g at solution values found in (I). The maximum and minimum of g on the surface $h = 0$ occur among these.

5. Find the minimum of $g(x, y, z) = x^2 + 2y^2 + 3z^2$ on the plane $x + 4y + 3z = 24$.

6. Find the distance from the origin to the surface $xyz^2 = 32$.

The foregoing methods are also frequently used when the equations $h = 0$, $h_1 = 0$, and $h_2 = 0$ are interpreted as *constraints* or *side conditions*, rather than equations of curves and surfaces.

7. A rectangle is constructed inside the ellipse $x^2/9 + y^2/16 = 1$ with sides parallel to the coordinate axes. Find the largest possible area. (Maximize the area using the ellipse equation as a constraint.)

8. A right triangle is constructed in the right half-plane region bound by the y axis and the ellipse $x^2/9 + y^2/16 = 1$. If the right-angle vertex and one side of the right triangle lie on the y axis, find the maximum area.

9. A rectangular parallelepiped is constructed inside the ellipsoid
$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1$$
with its faces parallel to the coordinate planes. Find the maximum volume of the parallelepiped.

10. A 500 cubic inch box is to be constructed without a top. Find the least possible surface area of the box.

11. A 12 cubic inch box is to be constructed. If the material for the top and bottom costs 3 cents per square inch and the material for the sides costs 2 cents per square inch, find the minimum cost of the box.

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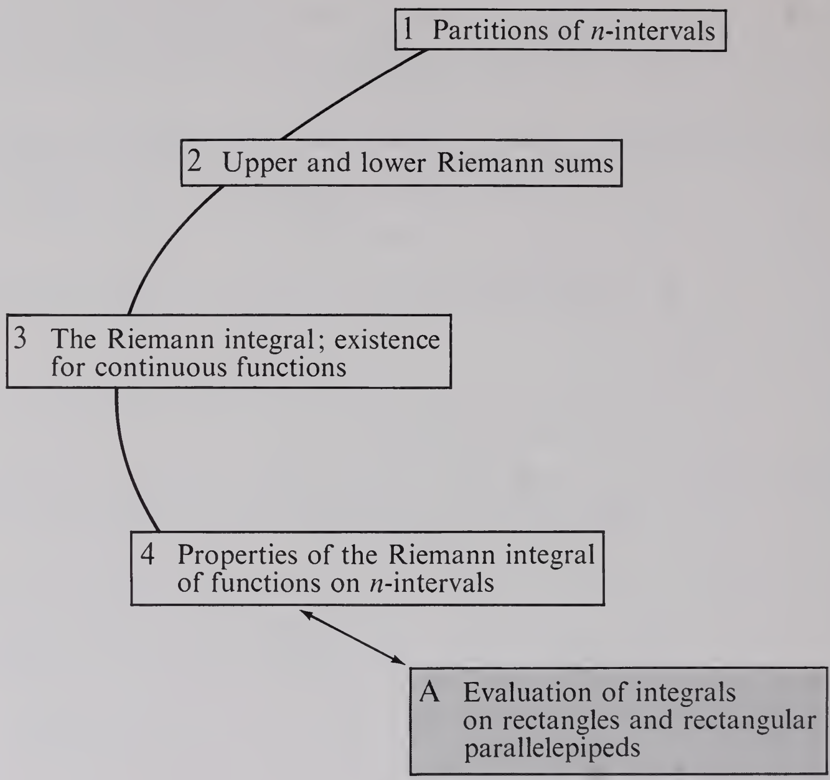
12. Find the maximum and minimum of $g(x, y) = x + y$ on the ellipse $2x^2 + xy + 2y^2 + 5x + 2y = 0$.

13. Find the distance from the origin to the curve of intersection of $x^2 + y^2 = 20$ and $x + 2y + 4z = 6$.

14. Find the maximum and minimum of $g(x, y, z) = x + 6y + 2z$ on the sphere $x^2 + y^2 + z^2 = 41$.

15. A rectangle lies within the ellipse $x^2/4 + y^2/9 = 1$ and has sides parallel to the coordinate axes. Find the largest possible area.

16. A box, without top and having 300 cubic inches, is to be constructed. If the material for the bottom is 3 cents per square inch and the material for the sides is 5 cents per square inch, find the minimum cost.


1 Partitions of n-intervals
2 Upper and lower Riemann sums
3 The Riemann integral; existence for continuous functions
4 Properties of the Riemann integral of functions on n-intervals
A Evaluation of integrals on rectangles and rectangular parallelepipeds

XIII

Integrals on n-Intervals

Thus far our development has involved extensions of the derivative; the remainder of the text will be primarily concerned with extensions of the integral $\int_a^b f(x)\,dx$. Integration is rooted in the problem of computing areas, volumes, and centers of mass which dates back to the works of the Greek scholar Archimedes [287–212 B.C.]. The integral was conceived by Newton as an inverse of the derivative, and by Leibniz as a limit of sums; both understood the connection between these two viewpoints. Current mathematics usually regards the integral as a limit of sums or a common least upper bound and greatest lower bound of sums. The antiderivative relationship, which is established by the equality

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a),$$

introduces a fundamental property that provides a useful technique for computing the integral.

In 1823 Cauchy produced a rigorous development of the integral of a continuous function $f(x)$ on an interval $[a, b]$, and this we now informally sketch. Consider a partition of $[a, b]$ into subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$, where $a = x_0$, and $b = x_n$, together with arbitrary intermediate points $\tilde{x}_j$ in $[x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$. We form the sum

$$R = f(\tilde{x}_1)(x_1 - x_0) + f(\tilde{x}_2)(x_2 - x_1) + \cdots + f(\tilde{x}_n)(x_n - x_{n-1}).$$

If $f(x) \geq 0$ for all x in $[a, b]$, then this sum may be interpreted geometrically as a sum of areas of rectangles which approximates the area of the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ (see Figure 13.1). The approximation improves as the number n of subintervals increases in such a way that the length of the longest subinterval approaches zero. Let a sequence R_k of sums be formed corresponding to partitions p_k such that if $|p_k|$ is the length of the longest subinterval of p_k, then $\lim_{k\to\infty} |p_k| = 0$; then the integral of f is

$$\int_a^b f\,dx = \lim_{k\to\infty} R_k.$$

Although the choice of the sequence R_k of sums has infinitely many possibilities, each has the same limit, and therefore, the integral is well defined.

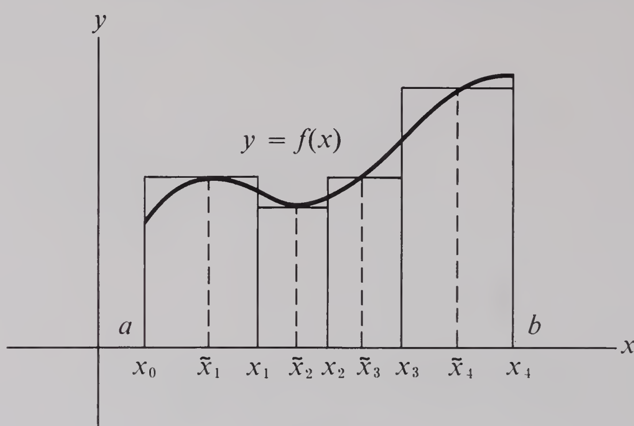


Figure 13.1

In 1854 the German mathematician G. F. B. Riemann [1826–1866] generalized the integral of Cauchy, and in his honor we call our integral the *Riemann integral* and the sum R a *Riemann sum*. In this extension we begin with a bounded function $f(x)$ on $[a, b]$ and make correspond to each partition p of $[a, b]$ the least upper bound $U_f(p)$ and the greatest lower bound $L_f(p)$ of all Riemann sums of f associated with p, using the various possibilities for intermediate points. The set of all $U_f(p)$ has a greatest lower bound $\overline{\int_a^b} f\,dx$ called the *upper Riemann integral* of f, and the set of all $L_f(p)$ has a least upper bound $\underline{\int_a^b} f\,dx$ called the *lower Riemann integral* of f. It is always true that

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx.$$

If equality holds, then f is said to be *integrable*, and the common value is denoted $\int_a^b f\,dx$ and called the *Riemann integral* of f. In the case in which f is continuous, the Riemann integral exists and equals the integral of Cauchy.

The extension of the integral in this chapter will involve scalar functions on sets in $\mathbf{R}^n$ called *n-intervals*. The development here closely parallels the Riemann construction of $\int_a^b f\,dx$; much of the terminology is the same and proofs carry over with only slight modification. For this reason many details will be omitted from the construction of the theory.

1. Partitions of n-Intervals

The simplest type of domain for the development of the integral of a function $g(x_1, \ldots, x_n)$ is an *n-interval*. The n-intervals may be described as a product of intervals of real numbers. The *product* of the closed intervals $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ of real numbers is defined by

$$\mathbf{I}_1 \times \mathbf{I}_2 \times \cdots \times \mathbf{I}_n = \{\langle x_1, x_2, \ldots, x_n\rangle : x_1 \text{ in } \mathbf{I}_1,\ x_2 \text{ in } \mathbf{I}_2, \ldots, x_n \text{ in } \mathbf{I}_n\}.$$

Example 1.1 Given $\mathbf{I}_1 = [a_1, b_1]$ and $\mathbf{I}_2 = [a_2, b_2]$, then the graph of

$$\mathbf{I}_1 \times \mathbf{I}_2 = \{\langle x, y\rangle : a_1 \le x \le b_1,\ a_2 \le y \le b_2\}$$

is a rectangle. If also $\mathbf{I}_3 = [a_3, b_3]$, then the graph of $\mathbf{I}_1 \times \mathbf{I}_2 \times \mathbf{I}_3$ is a rectangular parallelepiped (see Figure 13.2).

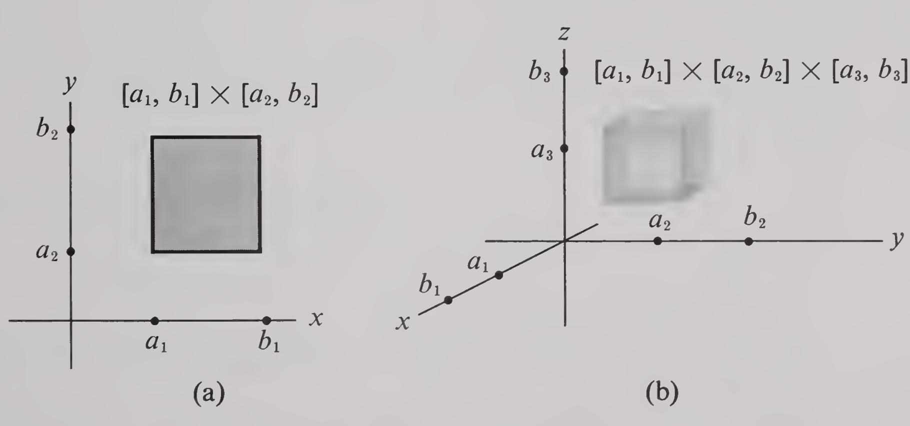


Figure 13.2

A 1–interval is simply a closed interval of real numbers. From vector algebra the n-interval $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is the n-parallelotope

$$\mathbf{u}_0 + [r_1\mathbf{u}_1 + \cdots + r_n\mathbf{u}_n],$$

where $\mathbf{u}_o = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{u}_1 = (b_1 - a_1)\mathbf{e}_1, \ldots, \mathbf{u}_n = (b_n - a_n)\mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are standard basis vectors. The volume of this parallelotope is the absolute determinant.

$$\begin{aligned}\mathrm{Vol}([a_1, b_1] \times \cdots \times [a_n, b_n]) &= |\langle (b_1 - a_1)\mathbf{e}_1, \ldots, (b_n - a_n)\mathbf{e}_n \rangle| \\ &= (b_1 - a_1) \cdots (b_n - a_n).\end{aligned}$$

Special cases are

(a) the length of $[a, b]$ is $b - a$,
(b) the area of $[a_1, b_1] \times [a_2, b_2]$ is $(b_1 - a_1)(b_2 - a_2)$, and
(c) the volume of $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is $(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$.

Example 1.2 The length of $[2, 6]$ is $6 - 2 = 4$; the area of $[2, 5] \times [5, 7]$ is $(5 - 2)(7 - 5) = 6$; the volume of $[-1, 3] \times [2, 5] \times [7, 8]$ is $(4)(3)(1) = 12$.

The development of our integral involves sums associated with *partitions.* A *partition* of a 1-interval $[a, b]$ may be defined as a finite collection of subintervals (closed interval subsets) of $[a, b]$ satisfying the following conditions.

(a) Every number in $[a, b]$ is in at least one subinterval of the collection.
(b) Any two subintervals of the collection have at most one number in their intersection.

Thus, a partition of $[a, b]$ consists of a chain of closed subintervals which make up all of $[a, b]$ and have only end points in common. Such a partition p of $[a, b]$ may be described by the symbol $p = \{[x_0, x_1, \ldots, x_k]\}$, where $a = x_0 < x_1 < \cdots < x_k = b$ are end points of the subintervals of p. The extension of our definition replaces (b) by the equivalent property that no two subintervals have an interior vector in common.

Definition of Partition of n-Interval

A partition p of an n-interval $\mathbf{I}^n$ *is a collection of n-subintervals of* $\mathbf{I}^n$ *which satisfies the requirements that*

(a) *every element of* $\mathbf{I}^n$ *is in at least one n-subinterval of the collection, and*
(b) *no element of* $\mathbf{I}^n$ *is in the interior of two n-subintervals of the collection.*

The graph of the interior of $\mathbf{I}^2 = [a_1, b_1] \times [a_2, b_2]$ consists of the open rectangle enclosed within the perimeter, which is composed of four line segments

called the *edges* of $\mathbf{I}^2$. The *vertices* of $\mathbf{I}^2$ are $\langle a_1, a_2\rangle$, $\langle b_1, a_2\rangle$, $\langle a_1, b_2\rangle$, and $\langle b_1, b_2\rangle$. This terminology can be extended to the case of n-intervals. A partition of $\mathbf{I}^2$ may be described by giving its subintervals, or by a listing of its vertices; it may also be indicated by a sketch.

Example 1.3 If $\mathbf{I}^2 = [0, 3] \times [1, 3]$, then

$$p = \{[0, 1] \times [1, 3], [1, 3] \times [1, 2], [1, 2] \times [2, 3], [2, 3] \times [2, 3]\}$$

is the partition having vertices $\langle 0,1\rangle, \langle 1,1\rangle, \langle 3,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3, 2\rangle$, $\langle 0, 3\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$, and $\langle 3, 3\rangle$ (see Figure 13.3).

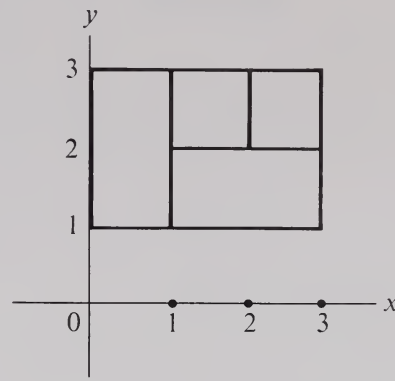


Figure 13.3

Among the simplest and most useful partitions of $\mathbf{I}_1 \times \cdots \times \mathbf{I}_n$ are those which can be represented by partitions of $\mathbf{I}_1, \ldots, \mathbf{I}_n$. If $p_1, \ldots, p_n$ are respective partitions of $\mathbf{I}_1, \ldots, \mathbf{I}_n$, then their *product partition* $p_1 \times \cdots \times p_n$ is the partition of $\mathbf{I}_1 \times \cdots \times \mathbf{I}_n$ consisting of all subintervals of the form $\mathbf{I}_{11} \times \cdots \times \mathbf{I}_{n1}$, where $\mathbf{I}_{11}$ is in p_1, $\mathbf{I}_{21}$, is in p_2, and, in general, $\mathbf{I}_{j1}$ is in p_j for $j = 1, 2, \ldots, n$. For the case $n = 2$ a product partition may be pictured as a grid.

Example 1.4 Let $p_1 = \{[1, 2, 4, 5]\}$ and $p_2 = \{[0, 1, 2, 3, 6]\}$ be respective partitions of $[1, 5]$ and $[0, 6]$. The sketch of $p_1 \times p_2$ is obtained by drawing vertical segments at $x = 2$, 4 and horizontal segments at $y = 1$, 2, 3 within the graph of $[1, 5] \times [0, 6]$ (see Figure 13.4).

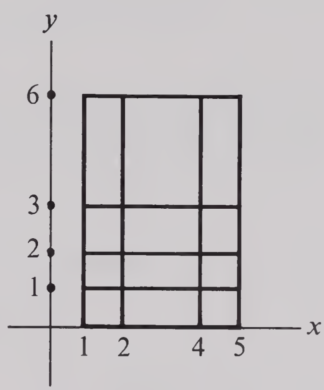


Figure 13.4

It is next desired to obtain a meaning for "fineness," with opposite meaning "coarseness," of partitions. Two definitions, one an absolute and the other comparative, are needed to achieve the results of this chapter. The absolute meaning measures fineness in terms of the largest subinterval of a partition. We first define the *diameter* of $\mathbf{I}^n$ to be the maximum $|\mathbf{u} - \mathbf{v}|$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{I}^n$. Since $\mathbf{I}^n$ is a closed bounded set and the norm is a continuous function, this maximum norm necessarily exists. It is geometrically evident that the diameter of a 2-interval or 3-interval is the length of a diagonal.

Definition of Norm of a Partition

The norm $|p|$ of a partition p *is the maximum of the diameters of the subintervals of* p.

Thus, the norm of a partition of a 2-interval or 3-interval is the maximum length of the diagonal segments of its subintervals.

Example 1.5 (a) In Example 1.3 the subintervals have diameters $\sqrt{5}$, $\sqrt{5}$, $\sqrt{2}$, $\sqrt{2}$; thus, the norm is $\sqrt{5}$.

(b) Given $\mathbf{I}^2 = [0, 12] \times [0, 3]$, let p_1 and p_2 be respective partitions of [0, 12], [0, 3], each having 6 subintervals of equal length. Then each subinterval of $p = p_1 \times p_2$ has length 2 and width $\frac{1}{2}$; therefore,

$$|p| = \sqrt{2^2 + (\tfrac{1}{2})^2} = \sqrt{\frac{17}{2}}.$$

The next result says that it is possible to select arbitrarily fine product partitions.

Proposition 1.1 Given an n-interval $\mathbf{I}^n$ and $\varepsilon > 0$, there exists a product partition p of $\mathbf{I}^n$ such that $|p| < \varepsilon$.

A proof is obtained for $\mathbf{I}^2 = [a_1, b_1] \times [a_2, b_2]$ by letting $p = p_1 \times p_2$, where p_1 and p_2 are respective partitions of $[a_1, b_1]$ and $[a_2, b_2]$, each containing

$$k > \frac{1}{\varepsilon}\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

subintervals of equal length. The diameter of p is

$$\frac{1}{k}\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} < \varepsilon$$

(see Figure 13.5).

Example 1.6 Let $\varepsilon = .001$ and $\mathbf{I}^2 = [0, 10] \times [1, 6]$. If p_1 and p_2 are respective partitions of [0, 10] and [1, 6] each containing k subintervals of equal length, then

$$|p_1 \times p_2| = \frac{1}{k}\sqrt{(10 - 0)^2 + (6 - 1)^2} = \frac{5\sqrt{5}}{k}.$$

Thus $|p_1 \times p_2| < \varepsilon$ provided $5\sqrt{5}/k < .001$, or, equivalently, $k > 5000\sqrt{5}$.

The second meaning for fineness compares two partitions.

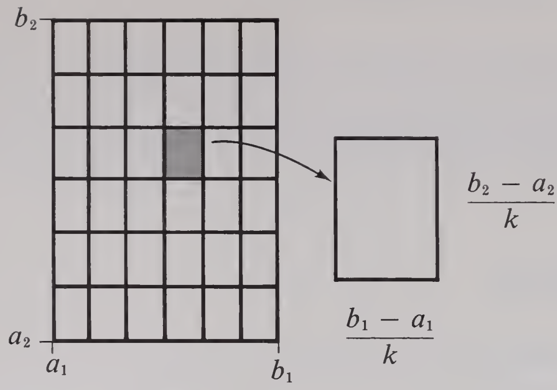


Figure 13.5

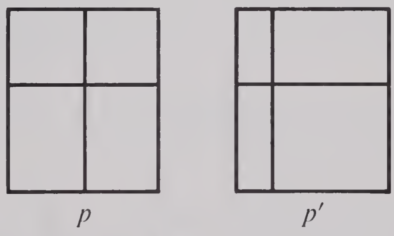
Definition of Refines

If p and p′ are partitions of $\mathbf{I}^n$, *then p* refines *p′, written* $p \succ p'$, *provided each subinterval of p is contained in some subinterval of p′.*

An alternative definition states $p \succ p'$ if each vertex of p' is also a vertex of p. A relation between refinement and the partition norm is given in the next result.

Proposition 1.2 If $p \succ p'$, then $|p| \leq |p'|$.

The proof follows from the definition of norm. The converse is false (see Figure 13.6); in fact, given two partitions of $\mathbf{I}^n$ it is not necessarily true that either refines the other, even for the case $n = 1$. However, refinement does obey the following properties, which are useful in the development of the integral.



$|p| \leq |p'|: p \not\succ p'$

Figure 13.6

Proposition 1.3
(a) If $p \succ p'$ and $p' \succ p''$, then $p \succ p''$ (*transitive property*).
(b) Given p and p', then there exists p'' such that $p'' \succ p$ and $p'' \succ p'$ (*endless* or *directed property*).

A proof of (a) follows from the definition of refine and the transitive property of set inclusion. For the proof of (b), the desired p'' may be obtained by forming a product partition that uses all vertices of subintervals of p and p' as vertices of subintervals of p''.

Questions

1. The graph of a 2-interval is a ________.
2. The graph of a 3-interval is a ________.
3. Any two sets in a partition of an n-interval ________.
   (a) have an empty intersection,
   (b) have no common interior points,
   (c) have boundary points in common.
4. The maximum distance between any two points of an n-interval is called its ________.
5. If p refines p', then ________.
   (a) p' contains more subintervals than p,
   (b) the norm of p is not larger than the norm of p',
   (c) p' does not refine p.

Exercises

1. Find the length, area, or volume for each set below.
   (a) $[2, 7]$,
   (b) $[-1, 3] \times [4, 6]$,
   (c) $[-1, 2] \times [3, 5] \times [-3, 8]$,
   (d) $[0, 1] \times [0, 2] \times [0, 3] \times [0, 4]$.
2. Given the partitions $p_1 = \{[0, .1, .2, .5, .8, 1]\}$ and $p_2 = \{[0, .1, .3, 1]\}$ of $[0, 1]$,
   (a) sketch $p_1 \times p_2$,
   (b) give the diameter of each subinterval of $p_1 \times p_2$, and
   (c) give the norm of $p_1 \times p_2$.
3. Give the vertices of $\mathbf{I}^2 = [0, 2] \times [1, 4]$.
4. Sketch and find the norm of $p = \{[0, 2] \times [0, 1], [2, 3] \times [0, 1], [0, 1] \times [1, 2], [1, 3] \times [1, 2]\}$.
5. Given that p_1 and p_2 are respective partitions of $[1, 9]$ and $[0, 6]$, each having k subintervals of equal length, find a value of k so that
$$|p_1 \times p_2| < .000001.$$
6. Give the (a) vertices and (b) diameter of $\mathbf{I}^3 = [0, 2] \times [1, 3] \times [0, 4]$.

Proofs

1. Obtain and justify a formula for the number of vertices of an n-interval (n an arbitrary positive integer).

2. Riemann Sums

In this section we study Riemann sums. These lead to the Riemann integral of the next section. Let g denote a bounded, scalar function having as its domain $\mathbf{I}^n$, and let $p = \{\mathbf{I}_1{}^n, \ldots, \mathbf{I}_k{}^n\}$ be a partition of $\mathbf{I}^n$. Since g is bounded, the values of g on each subinterval $\mathbf{I}_j{}^n$ have a least upper bound M_j and a greatest lower bound m_j. In some cases, as when g is continuous, M_j is the maximum and m_j the minimum value of g on $\mathbf{I}_j{}^n$.

Definition of Upper and Lower Riemann Sums

The upper and lower Riemann sums *of g associated with p are*, *respectively*,

$$U_g(p) = M_1 \operatorname{Vol} \mathbf{I}_1{}^n + \cdots + M_k \operatorname{Vol} \mathbf{I}_k{}^n,$$
$$L_g(p) = m_1 \operatorname{Vol} \mathbf{I}_1{}^n + \cdots + m_k \operatorname{Vol} \mathbf{I}_k{}^n.$$

Example 2.1 If $g(x, y)$ is continuous and $g_x \geq 0$, $g_y \geq 0$ on $\mathbf{I}^2$, then the maximum value of g on each subinterval occurs at the upper right-hand corner of the graph and the minimum occurs at the lower left corner. Thus, if

$$g = xy^2,$$
$$\mathbf{I}^2 = [0, 4] \times [0, 4],$$

and

$$p = \{[0, 3] \times [0, 2], [0, 3] \times [2, 4], [3, 4] \times [0, 2], [3, 4] \times [2, 4]\},$$

then

$$U_g(p) = 3(2)^2 6 + 3(4)^2 6 + 4(2)^2 2 + 4(4)^2 2 = 520,$$
$$L_g(p) = 0(0)^2 6 + 0(2)^2 6 + 3(0)^2 2 + 3(2)^2 2 = 24.$$

If $g_x \leq 0$ and $g_y \leq 0$, then the maximum and minimum positions are reversed. In other cases more sophisticated techniques would be required to determine upper and lower Riemann sums.

Example 2.2 If $g(x, y) \geq 0$ on $\mathbf{I}^2$ and g is continuous, then the "geometric volume" V of the solid between the graph of g and graph of $\mathbf{I}^2$ in the xy plane satisfies the inequality chain

$$L_g(p) \leq V \leq U_g(p)$$

for each partition p of $\mathbf{I}^2$ (see Figure 13.7). In order to see this, let $p = \{\mathbf{I}_1{}^2, \ldots, \mathbf{I}_k{}^2\}$ and M_j and m_j be respectively the maximum and minimum values of g on $\mathbf{I}_j{}^2$. Also let V_j be the portion of V above $\mathbf{I}_j{}^2$. If a rectangular parallelepiped of height M_j is constructed on the base $\mathbf{I}_j{}^2$, then its volume is $M_j \operatorname{Vol} \mathbf{I}_j{}^2$ and by geometric reasoning $V_j \leq M_j \operatorname{Vol} \mathbf{I}_j{}^2$. Therefore,

$$V = V_1 + \cdots + V_k \leq M_1 \operatorname{Vol} \mathbf{I}_1{}^2 + \cdots + M_k \operatorname{Vol} \mathbf{I}_k{}^2 = U_g(p).$$

In a similar manner the inequality $L_g(p) \leq V$ can be obtained. It may be observed that if $U_g(p) - L_g(p)$ is small, then either is a good approximation to V.

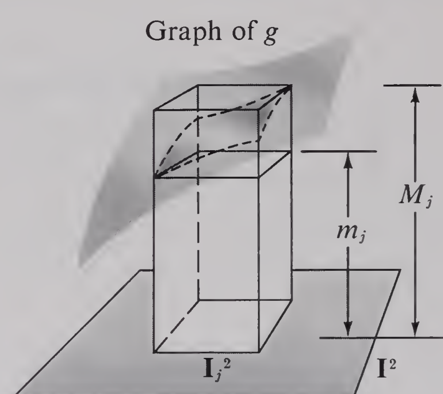


Figure 13.7

We next consider results which describe the type of change produced in upper and lower Riemann sums by refinement of a partition.

Proposition 2.1 If $p \succ p'$, then

(a) $U_g(p) \leq U_g(p')$, and
(b) $L_g(p) \geq L_g(p')$.

Thus, refinement of a partition tends to decrease the upper Riemann sum and increase the lower Riemann sum. The idea of the proof of (a) can be conveyed by the case where $p' = \{\mathbf{I}^n\}$, $p = \{\mathbf{I}_1{}^n, \ldots, \mathbf{I}_k{}^n\}$, and g is continuous. If M,

$M_1, \ldots, M_k$ are respectively the maximum of g on $\mathbf{I}^n, \mathbf{I}_1{}^n, \ldots, \mathbf{I}_k{}^n$, then $M \geq M_j$ for each $j = 1, \ldots, k$, and hence,

$$\begin{aligned} U_g(p') &= M \operatorname{Vol} \mathbf{I}^n \\ &= M(\operatorname{Vol} \mathbf{I}_1{}^n + \cdots + \operatorname{Vol} \mathbf{I}_k{}^n) \\ &\geq M_1 \operatorname{Vol} \mathbf{I}_1{}^n + \cdots + M_k \operatorname{Vol} \mathbf{I}_k{}^n \\ &= U_g(p). \end{aligned}$$

Application of this argument to each subinterval of an arbitrary partition p' gives the general result.

As a corollary of Proposition 2.1, we see that if p and p' are any partitions of $\mathbf{I}^n$, then we can state a further result.

Proposition 2.2 $\quad L_g(p') \leq U_g(p).$

For the proof let p'' be chosen by the endless property to satisfy $p'' \succ p$ and $p'' \succ p'$. Then $L_g(p'') \leq U_g(p'')$ is evident from the definition of upper and lower Riemann sums; and hence by Proposition 2.1,

$$L_g(p') \leq L_g(p'') \leq U_g(p'') \leq U_g(p).$$

We may therefore conclude that refinement of partitions brings L_g and U_g closer together, each U_g remaining to the right of each L_g on the real number line.

Questions

1. Refinement of a partition ______ an upper Riemann sum.
   (a) cannot increase,
   (b) cannot decrease,
   (c) might either increase or decrease.

2. The least upper bound of g on an n-interval is the maximum value of g provided that ______.
   (a) $n = 1$,
   (b) g is continuous,
   (c) the partial derivatives of g are positive.

3. By a suitably chosen partition, a lower Riemann sum can ______ be made more than an upper Riemann sum.
   (a) always,
   (b) sometimes,
   (c) never.

Exercises

1. Find $L_g(p)$ and $U_g(p)$ for each given function and partition;
   (a) $g(x, y) = xy, p = \{[0, 1, 3, 5]\} \times \{[1, 2, 4]\}$,
   (b) $g(x, y) = 1/(xy), p = \{[1, 2, 4]\} \times \{[1, 2, 3]\}$,
   (c) $g(x, y) = x - y, p = \{[0, 1, 2, 3]\} \times \{[0, 1, 2]\}$.
2. Given $g(x, y) = x^2y^2$ on $[0, 1] \times [0, 2]$, by using suitable Riemann sums find to an accuracy of .1 the volume of the solid between the graph of g and the xy plane.

Proofs

1. Prove that $L_g(p) \geq L_g(p')$ if $p = \{\mathbf{I}_1{}^2, \mathbf{I}_2{}^2\}$ is a partition of $\mathbf{I}^2$ and $p' = \{\mathbf{I}^2\}$.

3. The Riemann Integral

In this section we define the Riemann integral for certain bounded functions on n-intervals, and prove that for uniformly continuous functions it always exists and can be described as a limit of Riemann sums. For g, a bounded function on $\mathbf{I}^n$, we set

$$U_g = \{U_g(p) : p \text{ a partition of } \mathbf{I}^n\},$$

and

$$L_g = \{L_g(p) : p \text{ a partition of } \mathbf{I}^n\}.$$

Thus U_g and L_g are sets of numbers which respectively represent the set of all upper and lower Riemann sums of g. By the previous section, the set U_g is bounded below by each number in L_g; similarly L_g is bounded above by each number in U_g. The completeness property of the real number system justifies the next definition.

Definition of Upper and Lower Riemann Integrals

(a) The upper Riemann integral $\overline{\int}_{\mathbf{I}^n} g\, dV$ *is the greatest lower bound of* U_g.
(b) The lower Riemann integral $\underline{\int}_{\mathbf{I}^n} g\, dV$ *is the least upper bound of* L_g.

By elementary properties of the real numbers it is easily seen (see Proofs, exercise 1) that the following result holds.

Proposition 3.1 $\overline{\int}_{\mathbf{I}^n} g\,dV \geq \underline{\int}_{\mathbf{I}^n} g\,dV.$

We shall be interested in those functions g for which the upper and lower Riemann integrals are equal. Such functions are said to be (Riemann) *integrable*. In this case the distinction between upper and lower integrals is unnecessary, and their common value is called the *Riemann integral*. Thus we have arrived at the following definition.

Definition of Riemann Integral

If g is integrable, then the Riemann integral *of g is*

$$\int_{\mathbf{I}^n} g\,dV = \overline{\int}_{\mathbf{I}^n} g\,dV = \underline{\int}_{\mathbf{I}^n} g\,dV.$$

The following criterion for integrability is a consequence of the definitions of the upper and lower Riemann integrals and of the elementary properties of the real numbers.

Proposition 3.2 The function g is integrable if and only if for each $\varepsilon > 0$ there exists a partition p such that $U_g(p) - L_g(p) < \varepsilon$.

It will suffice for our purposes to know that continuous functions on $\mathbf{I}^n$ are integrable. Unfortunately, a careful proof of this cannot be made without introducing some ideas that would divert us from our main goals. Instead, we shall prove integrability for a class of functions called *uniformly continuous* functions.

Definition of Uniform Continuity

A function g is uniformly continuous *on $\mathbf{I}^n$ provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{I}^n$ and $|\mathbf{u} - \mathbf{v}| < \delta$, then $|g(\mathbf{u}) - g(\mathbf{v})| < \varepsilon$.*

Uniform continuity implies continuity, a relationship which follows from a comparison of their definitions; for each $\mathbf{u}_0$ in $\mathbf{I}^n$ and $\varepsilon > 0$, the value of δ from

the definition of uniform continuity serves to prove the continuity of g at $\mathbf{u}_0$. It is the converse of this implication, however, which is needed here; it will be assumed without proof.

Proposition 3.3 A function g on $\mathbf{I}^n$ is continuous if and only if it is uniformly continuous.

We shall now prove that a uniformly continuous function g is integrable. By Proposition 3.2 it suffices to find for $\varepsilon > 0$ a single partition p such that $U_g(p) - L_g(p) < \varepsilon$. In order to establish a later result, it will be convenient to obtain a stronger conclusion as stated in Proposition 3.4.

Proposition 3.4 If g is uniformly continuous, then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|p| < \delta$, then $U_g(p) - L_g(p) < \varepsilon$.

For the proof we choose δ to satisfy the definition of uniform continuity for g and $\varepsilon/\text{Vol } \mathbf{I}^n$. Thus, if $|\mathbf{u} - \mathbf{v}| < \delta$, then

$$|g(\mathbf{u}) - g(\mathbf{v})| < \frac{\varepsilon}{\text{Vol } \mathbf{I}^n}.$$

Let p be a partition of $\mathbf{I}^n$ such that $|p| < \delta$; for simplicity of notation we assume that $p = \{\mathbf{I}_1{}^n, \mathbf{I}_2{}^n, \mathbf{I}_3{}^n\}$. Since g is also continuous, we let $M_j = g(\mathbf{u}_j)$ and $m_j = g(\mathbf{v}_j)$ be the respective maximum and minimum values of g on $\mathbf{I}_j{}^n$, where $j = 1, 2, 3$. Then, for $j = 1, 2, 3$,

$$(M_j - m_j) < \frac{\varepsilon}{\text{Vol } \mathbf{I}^n},$$

since $|\mathbf{u}_j - \mathbf{v}_j| \leq |p| < \delta$. Therefore,

$$\begin{aligned} U_g(p) - L_g(p) &= (M_1 \text{Vol } \mathbf{I}_1{}^n + M_2 \text{Vol } \mathbf{I}_2{}^n + M_3 \text{Vol } \mathbf{I}_3{}^n) \\ &\quad - (m_1 \text{Vol } \mathbf{I}_1{}^n + m_2 \text{Vol } \mathbf{I}_2{}^n + m_3 \text{Vol } \mathbf{I}_3{}^n) \\ &= (M_1 - m_1)\text{Vol } \mathbf{I}_1{}^n + (M_2 - m_2)\text{Vol } \mathbf{I}_2{}^n \\ &\quad + (M_3 - m_3)\text{Vol } \mathbf{I}_3{}^n \\ &< \frac{\varepsilon}{\text{Vol } \mathbf{I}^n} (\text{Vol } \mathbf{I}_1{}^n + \text{Vol } \mathbf{I}_2{}^n + \text{Vol } \mathbf{I}_3{}^n) \\ &= \varepsilon. \end{aligned}$$

From Propositions 3.2, 3.3, and 3.4 we now conclude the following result.

Proposition 3.5 If g is continuous on $\mathbf{I}^n$, then g is integrable.

The Riemann integral has been defined as a common least upper bound and greatest lower bound. We now show that it can also be described as a limit, in the tradition of Cauchy as expressed in the chapter introduction. A *Riemann sum of g associated with p* is any sum of the form

$$R_g(p) = g(\mathbf{u}_1)\text{Vol } \mathbf{I}_1{}^n + \cdots + g(\mathbf{u}_k)\text{Vol } \mathbf{I}_k{}^n,$$

where $p = \{\mathbf{I}_1{}^n, \ldots, \mathbf{I}_k{}^n\}$ is a partition of $\mathbf{I}^n$ and $\mathbf{u}_j$ is in $\mathbf{I}_j{}^n$ for each $j = 1, \ldots, k$. It is clear that

$$L_g(p) \leq R_g(p) \leq U_g(p)$$

for each $R_g(p)$. Using this we now show the first step in our limit description.

> **Proposition 3.6** If g is continuous, then for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|p| < \delta$ and $R_g(p)$ is a Riemann sum of g associated with p, then
> $$\left| \int_{\mathbf{I}^n} g \, dV - R_g(p) \right| < \varepsilon.$$

The proof follows from Proposition 3.4, since both $\int_{\mathbf{I}^n} g \, dV$ and $R_g(p)$ lie between $L_g(p)$ and $U_g(p)$ which differ by less than ε.

From Proposition 3.6 the Riemann integral of a continuous function can be described as the limit of any sequence of Riemann sums associated with partitions whose norms approach zero.

> **Proposition 3.7** Let g be continuous and p_k a sequence of partitions of $\mathbf{I}^n$ such that $\lim_{k\to\infty} |p_k| = 0$. If R_k is a Riemann sum of g associated with p_k, then
> $$\lim_{k\to\infty} R_k = \int_{\mathbf{I}^n} g \, dV.$$

Proposition 3.7 provides a method for computing integrals of certain simple functions. A far superior technique will be found in the next section. It is customary to denote the integral of a function $g(x, y)$ of two variables by $\int_{\mathbf{I}^n} g \, dA$ rather than $\int_{\mathbf{I}^n} g \, dV$.

> **Example 3.1** Let $g(x, y) = x^2y$ on $\mathbf{I}^2 = [0, 1] \times [0, 2]$. Also let p_{1k} be the partition of $[0, 1]$ containing k equal length subintervals, and p_{2k} the partition of $[0, 2]$ containing k equal length subintervals (see Figure 13.8). If $p_k = p_{1k} \times p_{2k}$, then for an arbitrary subinterval
> $$\left[\frac{i-1}{k}, \frac{i}{k}\right] \times \left[\frac{2(j-1)}{k}, \frac{2j}{k}\right]$$

of p_k the maximum value of g occurs at $\langle i/k, 2j/k \rangle$. Thus,

$$U_g(p_k) = \sum_{i=1}^{k} \sum_{j=1}^{k} \left[\left(\frac{i}{k}\right)^2 \left(\frac{2j}{k}\right)\right]\left[\left(\frac{1}{k}\right)\left(\frac{2}{k}\right)\right]$$
$$= \frac{4}{k^5}\left(\sum_{i=1}^{k} i^2 \sum_{j=1}^{k} j\right).$$

Using the identities

$$\sum_{i=1}^{k} i^2 = \tfrac{1}{6}k(k+1)(2k+1) \quad \text{and} \quad \sum_{j=1}^{k} j = \tfrac{1}{2}k(k+1),$$

it may be seen after algebraic simplification that

$$U_g(p_k) = \frac{(k+1)^2(2k+1)}{3k^3}.$$

Hence by Proposition 3.7,

$$\int_{I^2} g\, dA = \lim_{k \to \infty} U_g(p_k) = \tfrac{2}{3}.$$

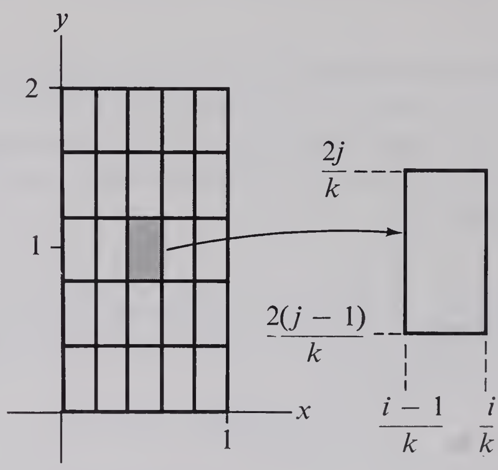


Figure 13.8

Questions

1. The upper Riemann integral is defined as ______.
   (a) a least upper bound,
   (b) a greatest lower bound,
   (c) a common least upper bound and greatest lower bound.
2. A function on an n-interval is integrable if it ______.
   (a) is bounded,
   (b) has a maximum,
   (c) is continuous.

3. The integral of a continuous function can be described as a limit of Riemann sums associated with a sequence of partitions p_k, where $\lim_{k\to\infty}$ ______ $= 0$.

4. A uniformly continuous function ______ continuous.
   (a) is always,
   (b) may or may not be,
   (c) cannot be.

Exercises

1. Given $g(x, y) = xy$ and $p = \{[0, 1, 2]\} \times \{[0, 1, 2]\}$, find $R_g(p)$ using (a) the midpoints (b) the upper left corner points of the subrectangles as the evaluation points for g.

2. In Example 3.1 show that $\lim\limits_{k\to\infty} L_g(p_k) = 2/3$.

3. Let $g(x, y) = xy^2$ on $\mathbf{I}^2 = [0, 1] \times [0, 1]$ and p_k be the partition of $\mathbf{I}^2$ containing k^2 equal size subintervals. In terms of k, find an upper bound for the error in approximating the volume between the graph of g and the xy plane by $U_g(p_k)$.

Proofs

1. Prove that $\overline{\int}_{\mathbf{I}^n} g\, dV \geq \underline{\int}_{\mathbf{I}^n} g\, dV$. (*Hint*: Assume not, and show this implies $L_g(p) \geq U_g(p)$ for some p.)

4. Properties of the Riemann Integral

The properties of $\int_{\mathbf{I}^n} g\, dV$ are mostly direct extensions of the integral properties of the one-variable function $g(x)$ on $[a, b]$, and are proved in a similar way. In order to keep things simple we shall assume all given functions in this section are continuous. Then $\int_{\mathbf{I}^n} (\quad)\, dV$ may be regarded as a real function having for its domain the vector space of continuous functions on $\mathbf{I}^n$. Our first result says that this function is linear.

Proposition 4.1

(a) $\int_{\mathbf{I}^n} (g_1 + g_2)\, dV = \int_{\mathbf{I}^n} g_1\, dV + \int_{\mathbf{I}^n} g_2\, dV$, and
(b) $\int_{\mathbf{I}^n} cg\, dV = c \int_{\mathbf{I}^n} g\, dV$.

The proof, like the proofs of certain other properties in this section, follows a pattern. We first determine relationships among upper and lower Riemann sums. Then, using previously established properties of Riemann sums, it is shown that each side of the desired integral equality is less than the other side plus an arbitrary preassigned $\varepsilon > 0$. A proof of (b) will be indicated for the case $c > 0$. If p is a partition of $\mathbf{I}^n$, then $U_{cg}(p) = cU_g(p)$, for if M_j is the maximum value of g on an n-subinterval $\mathbf{I}_j{}^n$, then cM_j is the maximum value of cg on $\mathbf{I}_j{}^n$; similarly $L_{cg}(p) = cL_g(p)$. Next, letting $\varepsilon > 0$, there exist partitions p' and p'' of $\mathbf{I}^n$ such that

$$U_g(p') - \int_{\mathbf{I}^n} g_n \, dV < \frac{\varepsilon}{c},$$

$$U_{cg}(p'') - \int_{\mathbf{I}^n} cg \, dV < \varepsilon.$$

The desired conclusion is implied by the following chains of inequalities.

$$\begin{aligned} \int_{\mathbf{I}^n} cg \, dV &\le U_{cg}(p') = cU_g(p') \\ &< c\left(\int_{\mathbf{I}^n} g \, dV + \frac{\varepsilon}{c}\right) = c\int_{\mathbf{I}^n} g \, dV + \varepsilon, \\ c\int_{\mathbf{I}^n} g \, dV &\le cU_g(p'') = U_{cg}(p'') \\ &< \int_{\mathbf{I}^n} cg \, dV + \varepsilon. \end{aligned}$$

If g is the constant function $g(\mathbf{u}) = c$ for all $\mathbf{u}$ in $\mathbf{I}^n$, then $L_g(p) = U_g(p) = c \operatorname{Vol} \mathbf{I}^n$ for each partition p. This implies our next result.

Proposition 4.2 $\int_{\mathbf{I}^n} c \, dV = c \operatorname{Vol} \mathbf{I}^n$.

Useful special cases are obtained by setting $c = 0$ and $c = 1$.

Proposition 4.3 (a) $\int_{\mathbf{I}^n} 1 \, dV = \operatorname{Vol} \mathbf{I}^n$,
(b) $\int_{\mathbf{I}^n} 0 \, dV = 0$.

We define the inequality $g \ge f$ to mean that $g(\mathbf{u}) \ge f(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbf{I}^n$. The next result says that larger functions give larger integrals.

Proposition 4.4

(a) If $g_2 \ge g_1$, then

$$\int_{\mathbf{I}^n} g_2 \, dV \ge \int_{\mathbf{I}^n} g_1 \, dV, \text{ and}$$

(b) if $g_2 \ge g_1$ and $g_2 \ne g_1$, then

$$\int_{\mathbf{I}^n} g_2 \, dV > \int_{\mathbf{I}^n} g_1 \, dV.$$

The proof of (a) follows from $U_{g_2}(p) \geq U_{g_1}(p)$ for every p. For the proof of (b), there exists $\mathbf{u}_o$ in $\mathbf{I}^n$ such that

$$g_2(\mathbf{u}_o) - g_1(\mathbf{u}_o) = c > 0.$$

By the continuity of g there exists an n–subinterval $\mathbf{I}_1{}^n$ of $\mathbf{I}^n$, where $\mathbf{u}_o$ is in $\mathbf{I}^n$, such that

$$g_2(\mathbf{u}) - g_1(\mathbf{u}) \geq \frac{c}{2}$$

for each $\mathbf{u}$ in $\mathbf{I}^n$. If p is any partition of $\mathbf{I}^n$ such that $\mathbf{I}_1{}^n$ is a union of n-subintervals of p, then the contribution of this union to $U_{g_2}(p)$ is at least $(c/2)\text{Vol}\,\mathbf{I}_1{}^n$ more than its contribution to $U_{g_1}(p)$. Therefore,

$$U_{g_2}(p) \geq U_{g_1}(p) + \frac{c}{2}\text{Vol}\,\mathbf{I}_1{}^n.$$

From this it is easily seen that

$$\begin{aligned}\int_{\mathbf{I}^n} g_2\, dV &\geq \int_{\mathbf{I}^n} g_1\, dV + \frac{c}{2}\text{Vol}\,\mathbf{I}_1{}^n \\ &> \int_{\mathbf{I}^n} g_1\, dV.\end{aligned}$$

Useful special cases of Proposition 4.4 are found by setting $g_1 = 0$.

Proposition 4.5

(a) If $g \geq 0$, then $\int_{\mathbf{I}^n} g\, dV \geq 0$, and
(b) if $g \geq 0$ and $g \neq 0$, then $\int_{\mathbf{I}^n} g\, dV > 0$.

The foregoing properties enable us to define an important inner product on the vector space of continuous functions on $\mathbf{I}^n$. It is given by the rule

$$g_1 \cdot g_2 = \int_{\mathbf{I}^n} g_1 g_2\, dV.$$

The proof that this rule defines an inner product is a consequence of Propositions 4.1, 4.5, and properties of functions.

The *absolute value function* $|g|$ of g is defined by $|g|(\mathbf{u}) = |g(\mathbf{u})|$ for each $\mathbf{u}$. It satisfies the relationship expressed in the next result.

Proposition 4.6 $|\int_{\mathbf{I}^n} g\, dV| \leq \int_{\mathbf{I}^n} |g|\, dV.$

For the proof we apply Propositions 4.4 and 4.1(b) to the inequalities $|g| \geq g$ and $|g| \geq (-1)g$. The desired conclusion is implied by the inequalities,

$$\int_{\mathbf{I}^n} |g|\, dV \geq \int_{\mathbf{I}^n} g\, dV$$

and

$$\int_{\mathbf{I}^n} |g|\, dV \geq \int_{\mathbf{I}^n} (-1) g\, dV = (-1) \int_{\mathbf{I}^n} g\, dV.$$

Our final property extends the fundamental theorem of calculus equality,

$$\int_a^b \frac{dg}{dx}\, dx = g(b) - g(a).$$

It provides an effective method for computing $\int_{\mathbf{I}^n} g\, dV$ provided certain antiderivatives can be found. We shall first observe it for the case where $n = 2$.

Iterated Integral Theorem

Let $g(x, y)$ be continuous on $\mathbf{I}^2 = [a_1, b_1] \times [a_2, b_2]$, and for each x in $[a_1, b_1]$ let $g^x(y)$ be the function on $[a_2, b_2]$ defined by $g^x(y) = g(x, y)$. If $G(x)$ is the function on $[a_1, b_1]$ defined by $G(x) = \int_{a_2}^{b_2} g^x\, dy$, then

$$\int_{\mathbf{I}^2} g\, dA = \int_{a_1}^{b_1} G\, dx.$$

A proof is in the appendix. The function $G(x)$ may be described by

$$G(x) = \int_{a_2}^{b_2} g(x, y)\, dy,$$

with x being held constant in solving the right-side integral. Hence, the conclusion of the iterated integral theorem may be written in the more conventional form

$$\int_{\mathbf{I}^n} g\, dA = \int_{a_1}^{b_1} dx \int_{a_2}^{b_2} g\, dy.$$

Example 4.1 Given $g(x, y) = x^2 y$ on $\mathbf{I}^2 = [0, 1] \times [0, 2]$, then

$$G(x) = \int_0^2 x^2 y\, dy = \left.\frac{x^2 y^2}{2}\right|_{y=0}^{y=2} = 2x^2,$$

$$\int_{\mathbf{I}^2} g\, dA = \int_0^1 2x^2\, dx = \tfrac{2}{3}.$$

From the proof of the iterated integral theorem it can be seen that the order of the coordinate variables is immaterial in the evaluation of $\int_{\mathbf{I}^2} g\, dA$. Hence, we also have the equality

$$\int_{\mathbf{I}^2} g\, dA = \int_{a_2}^{b_2} dy \int_{a_1}^{b_1} g\, dx.$$

Example 4.2 Given $g(x, y) = x^2 y$ on $\mathbf{I}^2 = [0, 1] \times [0, 2]$ as in Example 4.1, then

$$\int_{\mathbf{I}^2} g\, dA = \int_0^2 dy \int_0^1 x^2 y\, dx = \int_0^2 \frac{x^3 y}{3}\Big|_{x=0}^{x=1} dy$$
$$= \int_0^2 \frac{y}{3}\, dy = \frac{2}{3}.$$

The iterated integral expression for $g(x_1, \ldots, x_n)$ on $\mathbf{I}^n = [a_1, b_1] \times \cdots \times [a_n, b_n]$ may be written

$$\int_{\mathbf{I}^n} g\, dV = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_{n-1}}^{b_{n-1}} dx_{n-1} \int_{a_n}^{b_n} g\, dx_n.$$

The value obtained by the iteration process is independent of the order in which the variables are used.

Example 4.3 Let $g(x, y, z) = xy^2 z$ on $\mathbf{I}^3 = [0, 1] \times [0, 1] \times [0, 2]$. There are six orderings of the variables; one such choice gives

$$\int_{\mathbf{I}^3} g\, dV = \int_0^1 dx \int_0^1 dy \int_0^2 xy^2 z\, dz = \int_0^1 dx \int_0^1 \frac{xy^2 z^2}{2}\Big|_{z=0}^{z=2} dy$$
$$= \int_0^1 dx \int_0^1 2xy^2\, dy = \int_0^1 \frac{2x}{3}\, dx = \frac{1}{3}.$$

There is a simple geometric interpretation to the iterated integral theorem for the case $g(x, y) \geq 0$ on $\mathbf{I}^2$. The function $g^x(y)$ describes the curve of intersection of the surface $z = g(x, y)$ and a plane perpendicular to the xy plane (see Figure 13.9). Then $G(x)$ is the area under the graph of $g^x(y)$, and $\int_{\mathbf{I}^2} g\, dA$ is the

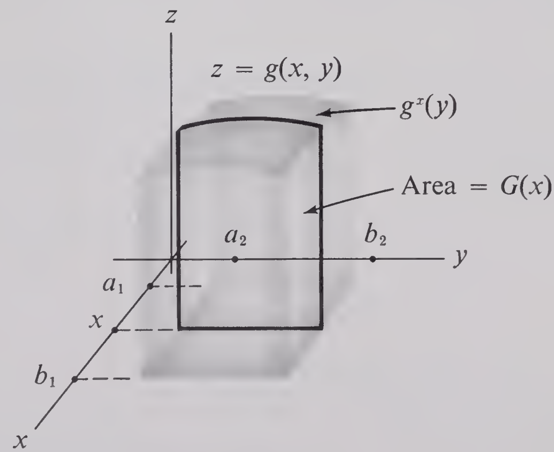


Figure 13.9

volume of the solid under the graph of $g(x, y)$. Thus, the iterated integral theorem says that the volume under $g(x, y)$ can be obtained by integrating the area function $G(x)$ on the interval $[a_1, b_1]$.

Questions

1. The volume of $\mathbf{I}^n$ is the integral of $f(x) =$ ______ on $\mathbf{I}^n$.
2. If $U_g(p) = c$, then $U_{2g}(p) =$ ______.
   (a) $2c$,
   (b) $c/2$,
   (c) c^2.
3. If $\int_a^b |g| \, dV = c$ and $|\int_a^b g \, dV| = d$, then ______.
   (a) $0 \leq c \leq d$,
   (b) $0 \leq d \leq c$,
   (c) $d \leq 0 \leq c$.
4. The iterated integral theorem extends the ______.
   (a) mean value theorem,
   (b) fundamental theorem,
   (c) chain rule of elementary calculus.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. State the following inequalities for the inner product $g_1 \cdot g_2$ of continuous functions on $\mathbf{I}^n$.
   (a) Cauchy-Schwarz,
   (b) Minkowski.

Proofs

1. For $c < 0$ establish a relationship between
   (a) $U_{cg}(p)$ and $L_g(p)$, and
   (b) $L_{cg}(p)$ and $U_g(p)$.
   (*Hint*: If m is the minimum of g on an interval, then cm is the maximum of cg.)

Problems

A. *Integrals on Rectangles and Rectangular Parallelepipeds*

The *product* of the real intervals $[a_1, b_1]$ and $[a_2, b_2]$ is

A.1 $\quad [a_1, b_1] \times [a_2, b_2] = \{\langle x, y\rangle : a_1 \leq x \leq b_1 \text{ and } a_2 \leq y \leq b_2\}.$

Its graph is the rectangular region with vertices (a_1, a_2), (a_1, b_2), (b_1, a_2), and (b_1, b_2). Given $g(x, y)$ with domain $\mathbf{I} = [a_1, b_1] \times [a_2, b_2]$, the *integral* $\int_{\mathbf{I}} g\, dA$ may be determined by the following procedure.

A.2 (I) Find a function $h(x, y)$ such that $h_y = g$.
(II) Set $G(x) = h(x, b_2) - h(x, a_2)$.
(III) Evaluate $\int_{a_1}^{b_1} G(x)\, dx = \int_{\mathbf{I}} g\, dA$.

The steps in A.2 may be symbolized as shown below.

$$\begin{aligned}
\int_{\mathbf{I}} g\, dA &= \int_{a_1}^{b_1} dx \left(\int_{a_2}^{b_2} g\, dy\right) \\
&= \int_{a_1}^{b_1} dx \left(h(x, y)\Big|_{y=a_2}^{y=b_2}\right) \\
&= \int_{a_1}^{b_1} G(x)\, dx.
\end{aligned}$$

The same value for $\int_{\mathbf{I}} g\, dA$ can also be obtained by reversing the roles of x and y. Thus, letting $h'_x = g$ and $G'(y) = h'(b_1, y) - h'(a_1, y)$ gives

$$\int_{\mathbf{I}} g\, dA = \int_{a_2}^{b_2} dy \left(\int_{a_1}^{b_1} g\, dx \right)$$
$$= \int_{a_2}^{b_2} dy \left(h'(x, y) \Big|_{x=a_1}^{x=b_1} \right)$$
$$= \int_{a_2}^{b_2} G'(y)\, dy.$$

1. Given $g(x, y) = x^2 y^3$ on $\mathbf{I} = [0, 1] \times [0, 2]$,

   (a) find $h(x, y)$ so that $h_y = g$,
   (b) find $g(x) = h(x, 2) - h(x, 1)$,
   (c) evaluate $\int_0^1 G(x)\, dx$,
   (d) find $h'(x, y)$ so that $h'_x = g$,
   (e) find $G'(y) = h'(1, y) - h'(0, y)$, and
   (f) evaluate $\int_1^2 G'(y)\, dy$.

2. Evaluate $\int_{\mathbf{I}} g\, dA$ in two different ways for each set of conditions below.

   (a) $g(x, y) = xy^2$, $\mathbf{I} = [1, 2] \times [0, 3]$,
   (b) $g(x, y) = e^{x+2y}$, $\mathbf{I} = [0, 1] \times [0, 2]$.

If g has the form $g(x, y) = g_1(x) g_2(y)$, then we have

A.3 $$\int_{\mathbf{I}} g\, dA = \left(\int_{a_1}^{b_1} g_1\, dx \right) \left(\int_{a_2}^{b_2} g_2\, dy \right).$$

Otherwise one order of integration may be simpler than the other.

3. Evaluate $\int_{\mathbf{I}} g\, dA$ for each function and rectangle below.

   (a) $g(x, y) = xe^{xy}$, $\mathbf{I} = [0, 1] \times [0, 1]$,
   (b) $g(x, y) = y \cos xy$, $\mathbf{I} = [0, 1] \times [0, \pi/4]$.

The product $\mathbf{I} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ of three intervals is a simple extension of the product of two intervals. The graph of $\mathbf{I}$ is then a rectangular parallelepiped. If $g(x, y, z)$ is a function with domain $\mathbf{I}$, then $\int_{\mathbf{I}} g\, dV$ is evaluated by two applications of the procedure indicated in A.2. There are now 6 possible orderings of integration corresponding to the orderings of x, y, z; each gives the same value. If $g(x, y, z) = g_1(x) g_2(y) g_3(z)$, then we have

A.4 $$\int_{\mathbf{I}} g\, dV = \left(\int_{a_1}^{b_1} g_1\, dx \right) \left(\int_{a_2}^{b_2} g_2\, dy \right) \left(\int_{a_3}^{b_3} g_3\, dz \right).$$

4. Evaluate $\int_{\mathbf{I}} g\, dV$ given the following conditions.

   (a) $g(x, y, z) = xyz^2$, $\mathbf{I} = [0, 2] \times [2, 4] \times [0, 3]$,
   (b) $g(x, y, z) = ze^{xz+y}$, $\mathbf{I} = [0, 1] \times [0, 1] \times [0, 1]$.

Review

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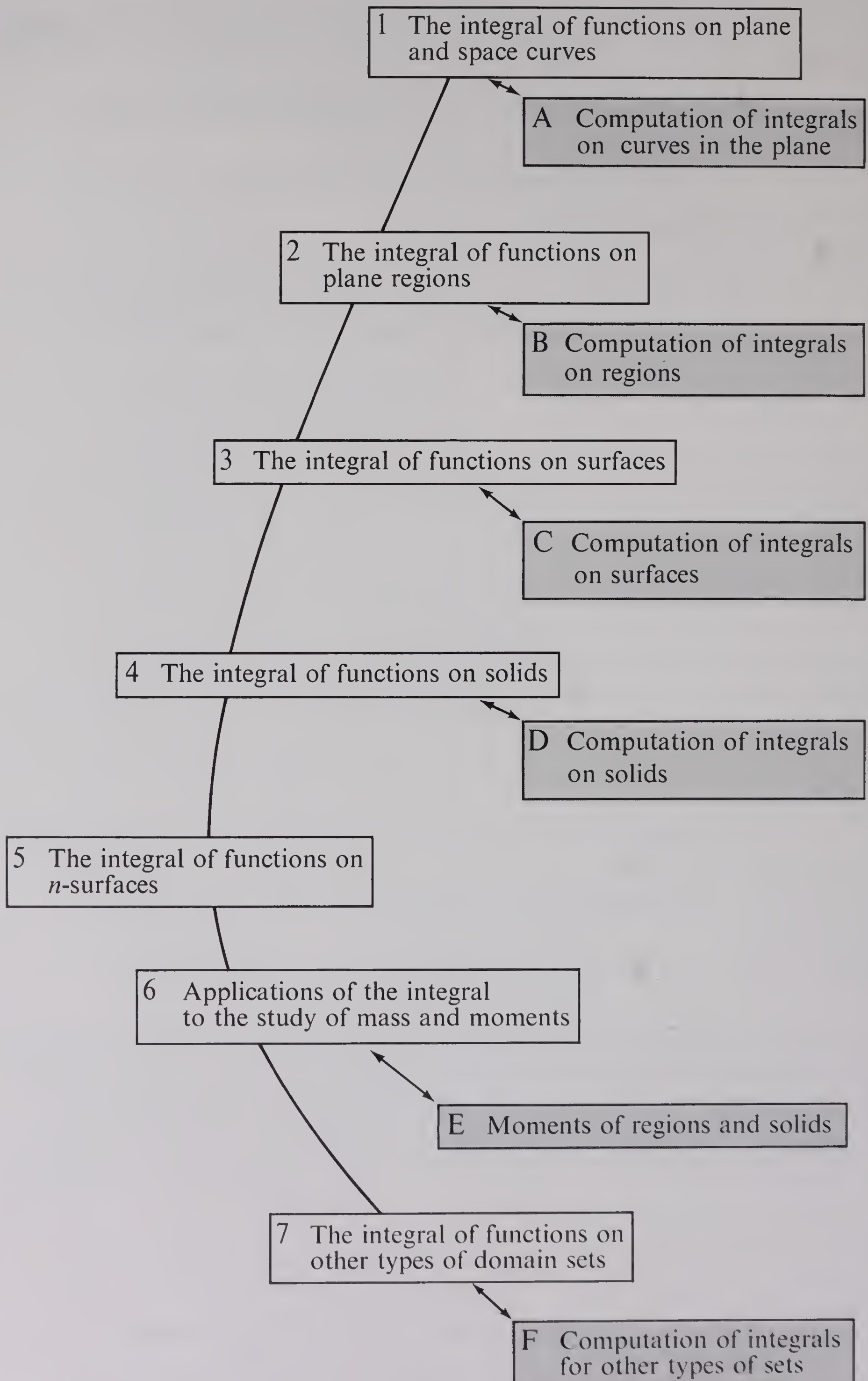
5. Evaluate $\int_{\mathbf{I}} g \, dA$ given the following functions and rectangles.

   (a) $g(x, y) = x^3y, \quad \mathbf{I} = [0, 2] \times [0, 1],$
   (b) $g(x, y) = xy \cos xy^2, \quad \mathbf{I} = [0, \pi/6] \times [0, 1].$

6. Evaluate $\int_{\mathbf{I}} g \, dV$ in each case.

   (a) $g(x, y, z) = x^2yz^3, \quad \mathbf{I} = [0, 3] \times [0, 1] \times [0, 2],$
   (b) $g(x, y, z) = x \cos(xy + z), \quad \mathbf{I} = [0, \pi/2] \times [0, 1] \times [0, \pi],$
   (c) $g(x, y, z) = xze^{yz}, \quad \mathbf{I} = [0, 1] \times [0, 1] \times [0, 1].$


1 The integral of functions on plane and space curves
A Computation of integrals on curves in the plane
2 The integral of functions on plane regions
B Computation of integrals on regions
3 The integral of functions on surfaces
C Computation of integrals on surfaces
4 The integral of functions on solids
D Computation of integrals on solids
5 The integral of functions on n-surfaces
6 Applications of the integral to the study of mass and moments
E Moments of regions and solids
7 The integral of functions on other types of domain sets
F Computation of integrals for other types of sets

XIV

Integrals on n-Surfaces

In the previous chapter we developed the integral for bounded functions on n–intervals. Here we shall study the integral for continuous functions on a broader class of domains called *n–surfaces*. The n–surfaces include various curves, plane regions, surfaces, and solids. Integrals on such sets have numerous applications; in geometry they are applied to finding lengths, areas, and volumes and in physics to problems in mass, force, and moment.

As in previous chapters, our main interest lies with sets that are described by vector functions. In the study of the integral, unlike that of the vector derivative, it is necessary to define precisely the extent of such sets. Special emphasis will be placed on n-dimensional sets which are suitable images of n–intervals; this keeps the theory at a simple, yet reasonably comprehensive level. Such images are called *n–surfaces*, and are described by *representations.* A representation of an n–surface $\mathscr{S}^n$ is a pair $\{\mathbf{f}, \mathbf{I}^n\}$ consisting of a vector function $\mathbf{f}$ and an n–interval $\mathbf{I}^n$, where $\mathbf{I}^n$ is contained in the domain of $\mathbf{f}$, such that $\mathscr{S}^n$ is the $\mathbf{f}$-image of $\mathbf{I}^n$. Suitable conditions, to be specified in this chapter, must be imposed on $\mathbf{f}$. If g is a continuous function on an n–surface $\mathscr{S}^n$ which is the $\mathbf{f}$-image of an n–interval $\mathbf{I}^n$, then the *integral* of g on $\mathscr{S}^n$ is always given by the formula

$$\int_{\mathscr{S}^n} g\, dV = \int_{\mathbf{I}^n} (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dV,$$

where $|J_{\mathbf{f}}|$ is the absolute determinant (see Section 3 of Chapter VII) of the Jacobian matrix of **f**. This formula is obtained with help of the magnification theorem from vector algebra. It is an extension of the change-of-variable formula in first-year calculus, which says that given $f(r)$ on $[a, b]$, with $df/dr > 0$, and $g(x)$ on $[f(a), f(b)]$, then

$$\int_{f(a)}^{f(b)} g\,dx = \int_a^b (g \circ f)\frac{df}{dr}\,dr.$$

In this case

$$|J_{\mathbf{f}}| = \left|\begin{bmatrix}\dfrac{df}{dr}\end{bmatrix}\right| = \frac{df}{dr},$$

and the f-image of $[a, b]$ is $[f(a), f(b)]$. The symbol $[df/dr]$ describes 1×1 matrix whose only entry is df/dr.

1. Integrals on Curves

We first consider the integral of a continuous function $g(x, y)$ on a *simple smooth curve* $\mathscr{C}$ which is the **f**-image of $[a, b]$ where

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}.$$

We say that $\mathbf{f}(r)$, $a \le r \le b$, is a representation of $\mathscr{C}$. The simple smooth property was encountered in the study of tangents. It implies that (a) **f** is of class C^1, (b) **f** is injective, and (c) $d\bar{x}/dr$ and $d\bar{y}/dr$ do not vanish simultaneously. Corresponding to each partition $p = [r_0, r_1, \ldots, r_k]$ of $[a, b]$ is a subdivision $[\mathbf{f}(r_0), \mathbf{f}(r_1), \ldots, \mathbf{f}(r_k)]$ of $\mathscr{C}$. Intermediate values $\tilde{r}_j$ in $[r_{j-1}, r_j]$ give points $\tilde{\mathbf{u}}_j = \mathbf{f}(\tilde{r}_j)$ intermediate to the portion $\mathscr{C}_j = \mathbf{f}([r_{j-1}, r_j])$ of $\mathscr{C}$ from $\mathbf{f}(r_{j-1})$ to $\mathbf{f}(r_j)$ (see Figure 14.1).

The integral of g on $\mathscr{C}$ is to be approximated by the sum

$$R_p = g(\tilde{\mathbf{u}}_1)L(\mathscr{C}_1) + \cdots + g(\tilde{\mathbf{u}}_k)L(\mathscr{C}_k),$$

where $L(\mathscr{C}_j)$ denotes the length of $\mathscr{C}_j$, $j = 1, \ldots, k$. We seek a means of estimating R_p when p is a fine partition. To do this we must find an estimate for each $L(\mathscr{C}_j)$.

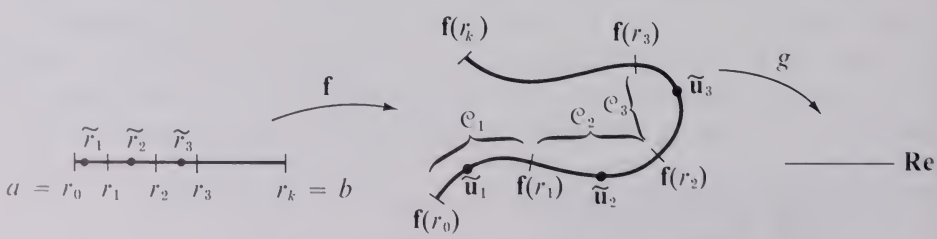


Figure 14.1

The function **f**, since it is of class C^1, may be approximated on the interval $[r_{j-1}, r_j]$ by the affine approximation of **f** at $\tilde{r}_j$ (see Figure 14.2)

$$\mathbf{A}_j(r) = \tilde{\mathbf{u}}_j + r\left(\frac{d\bar{x}}{dr}(\tilde{r}_j)\mathbf{i} + \frac{d\bar{y}}{dr}(\tilde{r}_j)\mathbf{j}\right).$$

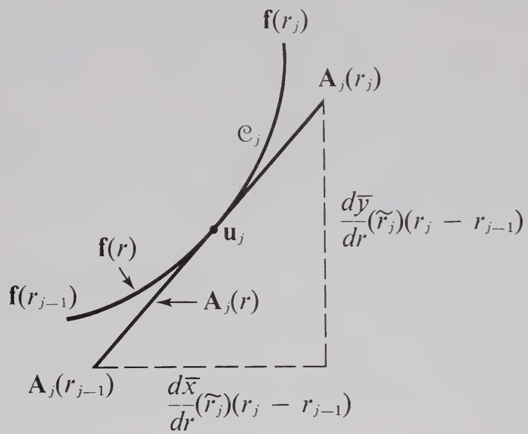


Figure 14.2

The length $L(\mathscr{C}_j)$ of $\mathscr{C}_j$ may be approximated by the length of the $\mathbf{A}_j$-image of $[r_{j-1}, r_j]$. Since $\mathbf{A}_j$ has the linear matrix

$$\begin{bmatrix} \dfrac{d\bar{x}}{dr} \\ \\ \dfrac{d\bar{y}}{dr} \end{bmatrix}_{\tilde{r}_j},$$

it follows from the magnification theorem from vector algebra (Chapter VII) that

$$\begin{aligned} L(\mathscr{C}_j) &\approx \left\| \begin{bmatrix} \dfrac{d\bar{x}}{dr} \\ \\ \dfrac{d\bar{y}}{dr} \end{bmatrix}_{\tilde{r}_j} \right\| (r_j - r_{j-1}) \\ &= \sqrt{\left(\frac{d\bar{x}}{dr}(\tilde{r}_j)\right)^2 + \left(\frac{d\bar{y}}{dr}(\tilde{r}_j)\right)^2}\,(r_j - r_{j-1}). \end{aligned}$$

Substituting, we have the integral of g on $\mathscr{C}$ approximated by the sum

$$\begin{aligned} R_p = g(\tilde{\mathbf{u}}_1) &\sqrt{\left(\frac{d\bar{x}}{dr}(\tilde{r}_1)\right)^2 + \left(\frac{d\bar{y}}{dr}(\tilde{r}_1)\right)^2}\,(r_1 - r_0) + \cdots \\ &+ g(\tilde{\mathbf{u}}_k)\sqrt{\left(\frac{d\bar{x}}{dr}(\tilde{r}_k)\right)^2 + \left(\frac{d\bar{y}}{dr}(\tilde{r}_k)\right)^2}\,(r_k - r_{k-1}). \end{aligned}$$

Replacing $g(\tilde{\mathbf{u}}_j)$ by $g(\bar{x}(\tilde{r}_j), \bar{y}(\tilde{r}_j))$, it is seen that R_p is a Riemann sum of

$$g(\bar{x}, \bar{y}) \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2}$$

It may be observed that

$$g(\bar{x}, \bar{y}) = g \circ \mathbf{f} \quad \text{and} \quad \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2} = |J_{\mathbf{f}}|,$$

where $|J_{\mathbf{f}}|$ is the absolute determinant of the Jacobian matrix of $\mathbf{f}$. We therefore have the following definition of the integral of g on the curve $\mathscr{C}$, which is represented by $\mathbf{f}(r)$, $a \le r \le b$.

Definition of an Integral on a Curve (Plane)

$$\int_{\mathscr{C}} g\, dL = \int_a^b (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dr$$

$$= \int_a^b g(\bar{x}, \bar{y}) \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2}\, dr.$$

Example 1.1 Let $g(x, y) = x^2 y$ and let $\mathscr{C}$ be the curve represented by $\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $0 \le \theta \le \pi$. Then $g \circ \mathbf{f} = \cos^2\theta \sin\theta$ and $|J_{\mathbf{f}}| = 1$. Hence,

$$\int_{\mathscr{C}} g\, dL = \int_0^\pi \cos^2\theta \sin\theta\, d\theta = \frac{-\cos^3\theta}{3}\bigg|_0^\pi = \frac{2}{3}.$$

The formula for $\int_{\mathscr{C}} g\, dL$ is also valid if $\mathscr{C}$ is a *smooth simple closed curve.* In this case the representing function $\mathbf{f}$ satisfies $\mathbf{f}(a) = \mathbf{f}(b)$, but otherwise meets the previous conditions on $\mathbf{f}$. Since a single domain point has a null contribution to our integral, the argument for the formula remains essentially unchanged. Similarly we may permit $d\bar{x}/dr$ and $d\bar{y}/dr$ to vanish simultaneously at the interval end points.

Example 1.2 Let $g(x, y) = xy$ have for its domain the ellipse $\mathscr{C}$ given by $x^2/4 + y^2/9 = 1$. Using the representation $\mathbf{f}(\theta) = 2\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}$, $0 \le \theta \le 2\pi$, for $\mathscr{C}$, we obtain

$$g \circ \mathbf{f} = 6\cos\theta \sin\theta \quad \text{and} \quad |J_{\mathbf{f}}| = (4\sin^2\theta + 9\cos^2\theta)^{1/2}.$$

Thus, $\int_{\mathscr{C}} g\, dL = \int_0^{2\pi} 6(4\sin^2\theta + 9\cos^2\theta)^{1/2} \sin\theta \cos\theta\, d\theta = 0.$

If $\mathscr{C}$ is defined explicitly by $y = f(x)$, $a \leq x \leq b$, then a representation is $\mathbf{f}(x) = x\mathbf{i} + f(x)\mathbf{j}$, $a \leq x \leq b$. In this case

$$|J_{\mathbf{f}}| = \sqrt{1 + \left(\frac{df}{dx}\right)^2}$$

and the integral of $g(x, y)$ on $\mathscr{C}$ takes the form shown below.

Integral Formula For a Curve Graph

$$\int_{\mathscr{C}} g\, dL = \int_a^b g(x, f(x)) \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx.$$

Example 1.3 Let $\mathscr{C}$ be the portion of the parabola $y = f(x) = x^2$ from $\langle 1, 1\rangle$ to $\langle 2, 4\rangle$, and $g(x, y) = y^2/x^3$. Then

$$\frac{df}{dx} = 2x \quad \text{and} \quad g(x, f(x)) = \frac{x^4}{x^3} = x.$$

Hence,

$$\int_{\mathscr{C}} g\, dL = \int_1^2 x\sqrt{1 + 4x^2}\, dx = \frac{1}{12}(17^{3/2} - 5^{3/2}).$$

It may be recalled that the *polar coordinates* (ρ, θ) of a point in the Cartesian plane have the following geometric meaning.

$\rho =$ distance from point to origin,
$\theta =$ counterclockwise angle (in radians) from positive x axis to a ray through the point and emanating from the origin.

Polar coordinates of the origin are $(0, \theta)$, where θ is arbitrary, $0 \leq \theta < 2\pi$. The polar coordinates (ρ, θ) and rectangular coordinates (x, y) of a point satisfy the relationships shown below.

$$x = \rho \cos \theta, \qquad \rho = \sqrt{x^2 + y^2},$$

$$y = \rho \sin \theta, \qquad \theta = \arctan \frac{y}{x}.$$

If $\mathscr{C}$ is a curve described by the explicit polar coordinate equation $\rho = h(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, then the rectangular coordinates of an arbitrary point on $\mathscr{C}$ are

$$x = h(\theta) \cos \theta, \qquad y = h(\theta) \sin \theta.$$

Therefore, $\mathscr{C}$ is represented by

$$\mathbf{f}(\theta) = h(\theta)\cos\theta\mathbf{i} + h(\theta)\sin\theta\mathbf{j}.$$

Then

$$|J_{\mathbf{f}}| = \left[\left(\frac{d}{d\theta}(h\cos\theta)\right)^2 + \left(\frac{d}{d\theta}(h\sin\theta)\right)^2\right]^{1/2} = \sqrt{h^2 + \left(\frac{dh}{d\theta}\right)^2}$$

and it follows that the integral of $g(x, y)$ on $\mathscr{C}$ is given by

$$\int_{\mathscr{C}} g\,dL = \int_{\theta_1}^{\theta_2} g(h\cos\theta, h\sin\theta)\sqrt{h^2 + \left(\frac{dh}{d\theta}\right)^2}\,d\theta.$$

If g is given as a function $g(\rho, \theta)$ of ρ and θ, then the integral may be written

Integral Formula for a Curve Graph (Polar Coordinates)

$$\int_{\mathscr{C}} g\,dL = \int_{\theta_1}^{\theta_2} g(h(\theta), \theta)\sqrt{h^2 + \left(\frac{dh}{d\theta}\right)^2}\,d\theta.$$

Example 1.4 Given $g(x, y) = y$ on the curve $\mathscr{C}$ with polar equation $\rho = \cos\theta$, $0 \leq \theta \leq \pi/2$, then

$$\int_{\mathscr{C}} g\,dL = \int_0^{\pi/2} \cos\theta\sin\theta\sqrt{\cos^2\theta + (-\sin\theta)^2}\,d\theta = \tfrac{1}{2}.$$

If g is the constant function $g = 1$, then the expression for R_p in the derivation of $\int_{\mathscr{C}} g\,dL$ becomes for each p

$$L(\mathscr{C}_1) + \cdots + L(\mathscr{C}_k) = L(\mathscr{C}).$$

From this we observe that the *arc length* of $\mathscr{C}$ may be defined as follows.

Definition of Arc Length (Plane)

$$L(\mathscr{C}) = \int_{\mathscr{C}} 1\,dL$$
$$= \int_a^b |J_{\mathbf{f}}|\,dr,$$

where $\mathscr{C}$ is represented by $\mathbf{f}(r)$, $a \leq r \leq b$.

Special cases are given next.

Arc Length of a Curve Graph

The arc length of $y = f(x)$, $a \le x \le b$, is

$$\int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx.$$

Arc Length of a Curve Graph (Polar Coordinates)

The arc length of $\rho = h(\theta)$, $\theta_1 \le \theta \le \theta_2$, is

$$\int_{\theta_1}^{\theta_2} \sqrt{h^2 + \left(\frac{dh}{d\theta}\right)^2}\, d\theta.$$

Example 1.5 The circle $\mathscr{C}$ of radius a centered at the origin is represented by $\mathbf{f}(\theta) = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j}$, $0 \le \theta \le 2\pi$. Then $|J_\mathbf{f}| = a$ and the length is $\int_\mathscr{C} 1\, dL = \int_0^{2\pi} a\, d\theta = 2\pi a$.

If $g(x, y, z)$ is continuous on a curve $\mathscr{C}$ in Cartesian space, then the development of $\int_\mathscr{C} g\, dL$ is basically the same as the above with the addition of a third coordinate. Thus, if $\mathscr{C}$ is represented by

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}, \qquad a \le r \le b,$$

then there is the definition,

Definition of an Integral on a Curve (Space)

$$\begin{aligned}\int_\mathscr{C} g\, dL &= \int_a^b (g \circ \mathbf{f})|J_\mathbf{f}|\, dr \\ &= \int_a^b g(\bar{x}, \bar{y}, \bar{z}) \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2 + \left(\frac{d\bar{z}}{dr}\right)^2}\, dr.\end{aligned}$$

The *arc length* of $\mathscr{C}$ is given by the formula

Definition of Arc Length (Space)

$$\begin{aligned}L(\mathscr{C}) &= \int_\mathscr{C} 1\, dL \\ &= \int_a^b \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2 + \left(\frac{d\bar{z}}{dr}\right)^2}\, dr.\end{aligned}$$

Example 1.6 Let $\mathscr{C}$ be the helix coil represented by

$$\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \theta\mathbf{k}, \quad 0 \le \theta \le 2\pi.$$

Then

$$|J_{\mathbf{f}}| = \sqrt{\sin^2\theta + \cos^2\theta + 1} = \sqrt{2};$$

hence, the length of the coil is

$$L(\mathscr{C}) = \int_0^{2\pi} \sqrt{2}\, d\theta = 2\sqrt{2}\pi.$$

Questions

1. If $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, then the absolute determinant of the Jacobian matrix of $\mathbf{f}$ is ________.
2. The length of a small arc may be estimated by the length of its ________ approximation.
3. Given $\mathscr{C}_j = \mathbf{f}([r_{j-1}, r_j])$, then in terms of $\tilde{r}_j$, where $r_{j-1} \le \tilde{r}_j \le r_j$, the length of $\mathscr{C}_j$ is approximately the product of $(r_j - r_{j-1})$ and ________.
4. The formula $\int_{\mathscr{C}} g\, dL = \int_{\mathbf{I}} (g \circ \mathbf{f})\, |J_{\mathbf{f}}|\, dr$, where $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$, requires that $d\bar{x}/dr$ and $d\bar{y}/dr$ not vanish simultaneously except possibly at ________.

Problems

1. Do Problem Set A at the end of the chapter.

Proofs

1. Let $\mathscr{C}$ be the graph of $z = f(x, y)$ on the curve in the xy plane represented by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$. Obtain a formula for $\int_{\mathscr{C}} g\, dL$ in terms of derivatives of f, $\bar{x}$, and $\bar{y}$.
2. Obtain a formula for $\int_{\mathscr{C}} g\, dL$ if $\mathscr{C}$ is the line segment with end points (a_1, b_1) and (a_2, b_2).
3. A representation $\mathbf{f}(s)$, $0 \le s \le L(\mathscr{C})$, is called the *arc length representation* of $\mathscr{C}$ provided s is the length of the portion of $\mathscr{C}$ from $\mathbf{f}(0)$ to $\mathbf{f}(s)$. Prove that if $\mathbf{f}(s) = \bar{x}(s)\mathbf{i} + \bar{y}(s)\mathbf{j}$, $0 \le s \le L(\mathscr{C})$, is the arc length representation, then

$$\left(\frac{d\bar{x}}{ds}\right)^2 + \left(\frac{d\bar{y}}{ds}\right)^2 = 1.$$

2. Integrals on Plane Regions

Let $\mathscr{R}$ be a set in the plane which is the **f**-image of a 2-interval

$$\mathbf{I}^2 = [a_1, b_1] \times [a_2, b_2].$$

We say that $\mathbf{f}(r, s)$, $a_1 \leq r \leq b_1$, $a_2 \leq s \leq b_2$, is a representation of $\mathscr{R}$. We shall initially require of $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$ that

(a) **f** is of class C^1,
(b) **f** is injective, and
(c) $\bar{x}_r\bar{y}_s - \bar{x}_s\bar{y}_r \neq 0$ at each $\langle r, s\rangle$ in $\mathbf{I}^2$.

We seek a definition for the integral of a continuous function $g(x, y)$ on $\mathscr{R}$. If $p = \{\mathbf{I}_1{}^2, \ldots, \mathbf{I}_k{}^2\}$ is a partition of $\mathbf{I}^2$ and $\tilde{\mathbf{v}}_j$ is in $\mathbf{I}_j{}^2$, $j = 1, \ldots, k$, then, letting $\mathbf{f}(\mathbf{I}_j) = \mathscr{R}_j$, there is a corresponding subdivision $\{\mathscr{R}_1, \ldots, \mathscr{R}_k\}$ of $\mathscr{R}$ with intermediate vectors $\tilde{\mathbf{u}}_j = \mathbf{f}(\tilde{\mathbf{v}}_j)$ (see Figure 14.3).

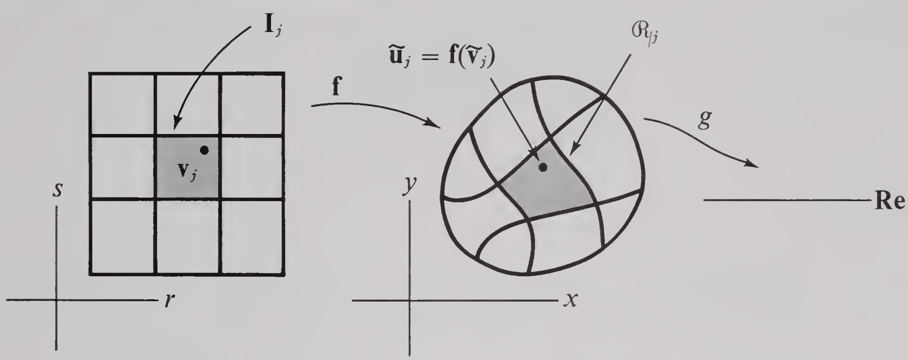


Figure 14.3

The integral of g on $\mathscr{R}$ is to be approximated by the sum

$$R_p = g(\tilde{\mathbf{u}}_1)A(\mathscr{R}_1) + \cdots + g(\tilde{\mathbf{u}}_k)A(\mathscr{R}_k)$$

where $A(\mathscr{R}_j)$ denotes the area of $\mathscr{R}_j$, $j = 1, \ldots, k$. As with the development of $\int_{\mathscr{C}} g\, dL$, we seek to approximate $A(\mathscr{R}_j)$ in order to obtain an estimate of R_p when p is a fine partition. Replacing **f** on $\mathbf{I}_j{}^2$ by its affine approximation $\mathbf{A}_j$ at $\tilde{\mathbf{v}}_j$, we approximate each $\mathscr{R}_j$ by the $\mathbf{A}_j$-image of $\mathbf{I}_j{}^2$. This image is a parallelogram $\mathbf{P}_j$ (see Figure 14.4). Since the linear matrix of $\mathbf{A}_j$ is $J_{\mathbf{f}}(\tilde{\mathbf{v}}_j)$, by the magnification theorem we have

$$A(\mathscr{R}_j) \sim A(\mathbf{P}_j) = |J_{\mathbf{f}}(\tilde{\mathbf{v}}_j)|A(\mathbf{I}_j{}^2),$$

where $A(\mathbf{I}_j{}^2)$ is the area of $\mathbf{I}_j{}^2$. Therefore,

$$R_p \approx g(\tilde{\mathbf{u}}_1)|J_{\mathbf{f}}(\tilde{\mathbf{v}}_1)|A(\mathbf{I}_1{}^2) + \cdots + g(\tilde{\mathbf{u}}_k)|J_{\mathbf{f}}(\tilde{\mathbf{v}}_k)|A(\mathbf{I}_k{}^2).$$

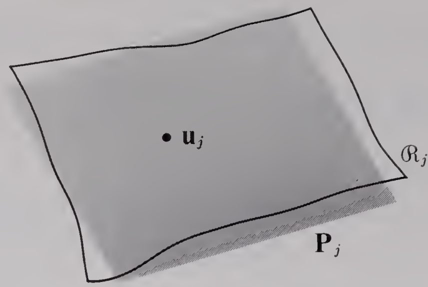


Figure 14.4

Replacing $g(\tilde{\mathbf{u}}_j)$ by $g \circ \mathbf{f}(\tilde{\mathbf{v}}_j)$, R_p is approximated by a Riemann sum of $(g \circ \mathbf{f})|J_{\mathbf{f}}|$ on $\mathbf{I}^2$, where

$$|J_{\mathbf{f}}| = \left| \begin{bmatrix} \bar{x}_r & \bar{x}_s \\ \bar{y}_r & \bar{y}_s \end{bmatrix} \right| = |\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r|.$$

This suggests defining the integral of g on $\mathcal{R}$ by the formula

Definition of an Integral on a Region

$$\int_{\mathcal{R}} g\, dA = \int_{\mathbf{I}^2} (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dA$$
$$= \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} g(\bar{x}, \bar{y})|\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r|\, ds.$$

Example 2.1 Let $g(x, y) = xy$ on the region $\mathcal{R}$, represented by $\mathbf{f}(r, s) = (r + s)\mathbf{i} + r^2\mathbf{j}$, $1 \le r \le 2$, $0 \le s \le 1$. Then

$$g \circ \mathbf{f} = r^2(r + s) \quad \text{and} \quad |J_{\mathbf{f}}| = |1(0) - 1(2r)| = 2r.$$

Hence,

$$\int_{\mathcal{R}} g\, dA = \int_1^2 dr \int_0^1 r^2(r + s)(2r)ds = \frac{323}{20}.$$

Since the boundary of $\mathbf{I}^2$ gives a null contribution to the integral of a continuous function on $\mathbf{I}^2$, we may relax the requirements on $\mathbf{f}$ so that $\mathbf{f}$ need be injective and $\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r \ne 0$ only on the interior of $\mathbf{I}^2$. Thus our integral formula may be applied to a much larger class of domains.

Example 2.2 Let $\mathcal{R}$ be the region consisting of the circle $x^2 + y^2 = a^2$, together with its interior. It has a representation $\mathbf{f}(r, \theta) = r \cos \theta\, \mathbf{i} + r \sin \theta\, \mathbf{j}$, $0 \le r \le a$, $0 \le \theta \le 2\pi$. Then $|J_{\mathbf{f}}| = r$, and, given $g(x, y)$,

$$\int_{\mathcal{R}} g\, dA = \int_0^a dr \int_0^{2\pi} g(r \cos \theta, r \sin \theta) r\, d\theta.$$

Frequently a change of variable will simplify an integral or facilitate its evaluation. Let $\mathscr{R}$ be the region bounded by the graphs of $y = h_1(x)$, $y = h_2(x)$, $x = a$, and $x = b$, where $h_2(x) > h_1(x)$ for $a < x < b$. This region will be symbolized $[a, b] \times [h_1(x), h_2(x)]$. From vector algebra, the segment joining $\langle x, h_1(x)\rangle$ and $\langle x, h_2(x)\rangle$ is

$$\langle x, h_1(x)\rangle + [r\langle 0, h_2(x) - h_1(x)\rangle]$$

(see Figure 14.5). It is easily verified that $\mathscr{R}$ is represented by

$$\mathbf{f}(x, r) = x\mathbf{i} + (h_1(x) + r(h_2(x) - h_1(x)))\mathbf{j}, \qquad a \le x \le b, \quad 0 \le r \le 1.$$

Then

$$|J_{\mathbf{f}}| = \begin{bmatrix} 1 & 0 \\ - & h_2 - h_1 \end{bmatrix},$$

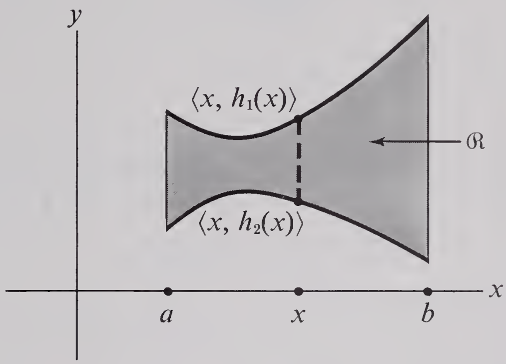


Figure 14.5

where the lower left entry is immaterial, and hence, $|J_{\mathbf{f}}| = h_2 - h_1$. Therefore, given $g(x, y)$, then

$$\int_{\mathscr{R}} g\, dA = \int_a^b dx \int_0^1 g(x, h_1 + r(h_2 - h_1))(h_2 - h_1)\, dr.$$

For a fixed x let $y(x, r) = h_1(x) + r(h_2(x) - h_1(x))$; then $y_r = h_2 - h_1$ and by the change-of-variable formula from beginning calculus, the preceding integral is given by the equality,

Integral Formula for the Region $[a, b] \times [h_1(x), h_2(x)]$

$$\int_{\mathscr{R}} g\, dA = \int_a^b dx \int_{h_1(x)}^{h_2(x)} g(x, y)\, dy.$$

Example 2.3 Let $g(x, y) = xy$ and $\mathscr{R}$ be the region bounded by $y = 2x$, $y = x^2$, $x = 0$, and $x = 1$. Then

$$\int_{\mathscr{R}} g\,dA = \int_0^1 dx \int_{x^2}^{2x} xy\,dy = \int_0^1 \frac{xy^2}{2}\bigg|_{y=x^2}^{y=2x} dx = \int_0^1 \left(2x^3 - \frac{x^5}{2}\right) dx = \frac{5}{12}.$$

Next, let $\mathscr{R}$ be the region bounded by the polar equation curves $\rho = h(\theta)$, $\theta = \theta_1$, and $\theta = \theta_2$, where $0 \le \theta_1 < \theta_2 \le 2\pi$ (see Figure 14.6). This region is

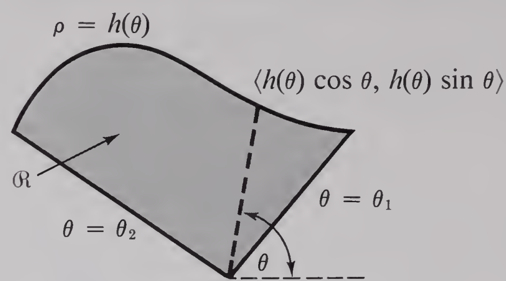


Figure 14.6

symbolized $[0, h(\theta)] \times [\theta_1, \theta_2]$. Since the line segment joining the origin to $\langle h(\theta)\cos\theta, h(\theta)\sin\theta\rangle$ is

$$[r\langle h(\theta)\cos\theta, h(\theta)\sin\theta\rangle],$$

it follows that a representation of $\mathscr{R}$ is

$$\mathbf{f}(r, \theta) = rh(\theta)\cos\theta\,\mathbf{i} + rh(\theta)\sin\theta\,\mathbf{j}, \qquad 0 \le r \le 1, \quad \theta_1 \le \theta \le \theta_2.$$

It may be verified that $|J_{\mathbf{f}}| = rh^2$, and hence, the integral of $g(x, y)$ on $\mathscr{R}$ is

$$\int_{\mathscr{R}} g\,dA = \int_{\theta_1}^{\theta_2} d\theta \int_0^1 g(rh\cos\theta, rh\sin\theta) rh^2\,dr.$$

A substitution of $\rho = rh(\theta)$ for θ fixed converts this to

$$\int_{\mathscr{R}} g\,dA = \int_{\theta_1}^{\theta_2} d\theta \int_0^{h(\theta)} g(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho.$$

If g is given as a function $g(\rho, \theta)$ of ρ and θ on $\mathscr{R}$, then this integral takes the following form.

Integral Formula for the Region $[0, h(\theta)] \times [\theta_1, \theta_2]$ ***(Polar Coordinates)***

$$\int_{\mathscr{R}} g\,dA = \int_{\theta_1}^{\theta_2} d\theta \int_0^{h(\theta)} g(\rho, \theta)\,\rho\,d\rho.$$

Example 2.4 Given $g(\rho, \theta) = \rho\theta$ on the region $\mathscr{R}$ bounded by $\rho(\theta) = \theta^2$, $\theta = 0$, $\theta = \pi$, then

$$\int_{\mathscr{R}} g\, dA = \int_0^{\pi} d\theta \int_0^{\theta^2} (\rho\theta)\rho\, d\rho = \int_0^{\pi} \frac{\rho^3\theta}{3}\bigg|_0^{\theta^2} d\theta = \int_0^{\pi} \frac{\theta^7}{3}\, d\theta = \frac{\pi^8}{24}.$$

Analogous to the arc length of a curve, the *area* of a region $\mathscr{R}$ represented by $\mathbf{f}(r, s)$, with $\langle r, s\rangle$ in $\mathbf{I}^2 = [a_1, b_1] \times [a_2, b_2]$, is given next.

Definition of Area (Region)

$$\begin{aligned} A(\mathscr{R}) &= \int_{\mathscr{R}} 1\, dA \\ &= \int_{\mathbf{I}^2} |J_{\mathbf{f}}|\, dV \\ &= \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} |\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r|\, ds. \end{aligned}$$

A special case is the area of the region bounded by $y = h_1(x)$, $y = h_2(x)$, $x = a$, and $x = b$. It is given by

$$A(\mathscr{R}) = \int_a^b dx \int_{h_1(x)}^{h_2(x)} dy.$$

We see that this gives the formula

Area of $[a, b] \times [h_1(x), h_2(x)]$

$$A(\mathscr{R}) = \int_a^b (h_2 - h_1)\, dx.$$

Similarly the area of the region bounded by $\rho = h(\theta)$, $\theta = \theta_1$, $\theta = \theta_2$ is given by

$$A(\mathscr{R}) = \int_{\theta_1}^{\theta_2} d\theta \int_0^{h(\theta)} \rho\, d\rho.$$

By inspection we have the formula

Area of $[0, h(\theta)] \times [\theta_1, \theta_2]$

$$A(\mathscr{R}) = \frac{1}{2}\int_{\theta_1}^{\theta_2} h^2\, d\theta.$$

Example 2.5 Let $\mathscr{R}$ be the region enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$. A representation of $\mathscr{R}$ is

$$\mathbf{f}(r, \theta) = ar\cos\theta\mathbf{i} + br\sin\theta\mathbf{j}, \qquad 0 \le r \le 1 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

Then,

$$|J_{\mathbf{f}}| = |(a\cos\theta)(br\cos\theta) - (-ar\sin\theta)(b\sin\theta)| = abr.$$

Hence, $A(\mathscr{R}) = \int_0^1 dr \int_0^{2\pi} abr\, d\theta = \pi\, ab.$

Questions

1. If $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, then the absolute determinant of the Jacobian matrix of $\mathbf{f}$ is ______.

2. The absolute determinant of the Jacobian matrix of a representing function $\mathbf{f}(r, s)$ is required to be nonzero everywhere except possibly on the ______ of the representing 2-interval.
   (a) interior,
   (b) boundary,
   (c) vertices.

Problems

1. Do Problem Set B at the end of the chapter.

Exercises

1. Let $\mathscr{R}$ be the region bounded by the quadrilateral with successive vertices $\mathbf{u}_1 = \langle 1, 1\rangle$, $\mathbf{u}_2 = \langle 2, 3\rangle$, $\mathbf{u}_3 = \langle 6, 4\rangle$, and $\mathbf{u}_4 = \langle 7, 2\rangle$.
   (a) Find a representation for $\mathscr{R}$.
   (b) Find $\int_{\mathscr{R}} g\, dA$ if $g(x, y) = xy$.
   (*Hint*: Two sides of the quadrilateral are $\mathbf{u}_1 + [r(\mathbf{u}_2 - \mathbf{u}_1)]$ and $\mathbf{u}_4 + [r(\mathbf{u}_3 - \mathbf{u}_4]$. Find a representation of the segment from $\mathbf{u}_1 + r(\mathbf{u}_2 - \mathbf{u}_1)$ to $\mathbf{u}_4 + r(\mathbf{u}_3 - \mathbf{u}_4)$.)

Proofs

1. Let $P = (a_1, b_1)$, $Q = (a_2, b_2)$, and $R = (a_3, b_3)$.
   (a) Using the vector algebra form, $\mathbf{u}_o + [r\mathbf{u}_1 + s\mathbf{u}_2]$, obtain a representation $\mathbf{f}(r, s)$, $[0, 1] \times [0, 1]$ for the parallelogram region with sides $\overline{PQ}$ and $\overline{QR}$.

(b) Using the vector algebra form, $\mathbf{u}_o + [r\mathbf{u}_1 + rs\mathbf{u}_2]$, obtain a representation $\mathbf{f}'(r, s)$, $[0, 1] \times [0, 1]$ of the triangular region with vertices P, Q, and R.

(c) Show from (a) and (b) that $|J_{\mathbf{f}'}| = r|J_{\mathbf{f}}|$.

(d) Obtain from (a) and (b) a formula for the areas of the parallelogram and triangular regions.

2. Let $\mathscr{R}$ be the region bounded by $\rho = h_1(\theta)$, $\rho = h_2(\theta)$, $\theta = \theta_1$, and $\theta = \theta_2$, where $0 \leq h_1(\theta) < h_2(\theta)$.
   (a) Find a representation for $\mathscr{R}$.
   (b) Given $g(\rho, \theta)$, show $\int_{\mathscr{R}} g\, dA = \int_{\theta_1}^{\theta_2} d\theta \int_{h_1(\theta)}^{h_2(\theta)} g\rho\, d\rho$.
3. Let $\mathscr{R}$ be the region bounded by the quadrilateral with successive vertices $\mathbf{u}_1 = (a_1, b_1)$, $\mathbf{u}_2 = (a_2, b_2)$, $\mathbf{u}_3 = (a_3, b_3)$, and $\mathbf{u}_4 = (a_4, b_4)$.
   (a) Find a representation for $\mathscr{R}$ (see Exercise 1).
   (b) Find $|J_{\mathbf{f}}|$ in (a).
   (c) Obtain a formula for the area of the region in (a).
4. Let $\mathscr{R}$ be the region bounded by the graphs of $x = h_1(y)$, $x = h_2(y)$, $y = a$, and $y = b$, where $h_2(y) > h_1(y)$ for $a < y < b$. Prove $\int_{\mathscr{R}} g\, dA = \int_a^b dy \int_{h_1(y)}^{h_2(y)} g(x, y)\, dx$ by the following two methods:
   (a) Rename the xy coordinate system as the y', $-x'$ coordinate system and apply the integral formula for a region of the form $[a, b] \times [h_1(x'), h_2(x')]$. Then obtain the desired formula by a suitable change of variables.
   (b) Find a representation for $\mathscr{R}$ and proceed as with the derivation of the integral formula for $[a, b] \times [h_1(x), h_2(x)]$.

3. Integrals on Surfaces

In this section we shall study the integral of a continuous function $g(x, y, z)$ on a surface $\mathscr{S}$. The construction is similar to that for integrals on plane regions. Thus we shall assume that $\mathscr{S}$ is the $\mathbf{f}$-image of a 2-interval $\mathbf{I}^2$, where

(a) $\mathbf{f}$ is of class C^1,
(b) $\mathbf{f}$ is injective on the interior of $\mathbf{I}^2$, and
(c) $|J_{\mathbf{f}}(\mathbf{u}_o)| \neq 0$ for each $\mathbf{u}_o$ in the interior of $\mathbf{I}^2$.

The integral of g on $\mathscr{S}$ is then derived in an analogous way to that in the previous sections and is given by the definition

Definition of an Integral on a Surface

$$\int_{\mathscr{S}} g\, dA = \int_{\mathbf{I}^2} (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dA.$$

Example 3.1 Let $g(x, y, z) = x$, where the domain of g is the surface $\mathscr{S}$ represented by

$$\mathbf{f}(r, s) = (r + s)\mathbf{i} + (2r - 3s)\mathbf{j} + (4r + s)\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le s \le 2.$$

Then $g \circ \mathbf{f} = r + s$ and from

$$J_{\mathbf{f}} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

we obtain $|J_{\mathbf{f}}| = \sqrt{230}$. Hence

$$\int_{\mathscr{S}} g\, dA = \int_0^1 dr \int_0^2 (r + s)\sqrt{230}\, ds = 3\sqrt{230}.$$

We shall next obtain special integral formulas for various types of surfaces. Let $\mathscr{S}$ be the surface graph of $z = f(x, y)$ on a region $\mathscr{R}$ represented by $\mathbf{h}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, where $\langle r, s\rangle$ is in $\mathbf{I}^2$ (see Figure 14.7). Then $\mathscr{S}$ is represented by

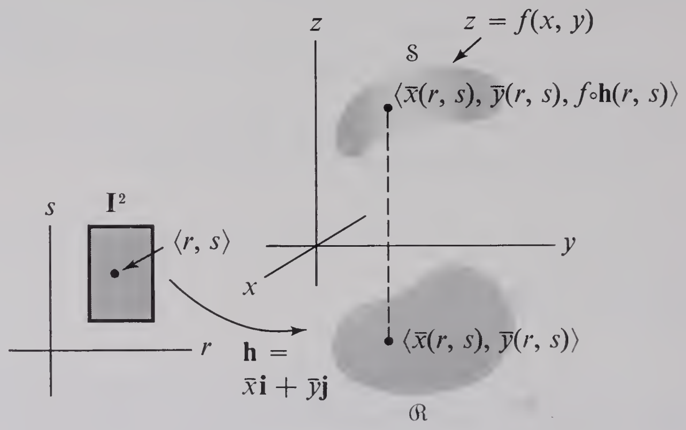


Figure 14.7

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + f \circ \mathbf{h}(r, s)\mathbf{k}, \ \langle r, s\rangle \text{ in } \mathbf{I}^2.$$

Thus,

$$J_{\mathbf{f}} = \begin{bmatrix} \bar{x}_r & \bar{x}_s \\ \bar{y}_r & \bar{y}_s \\ (f_x \circ \mathbf{h})\bar{x}_r + (f_y \circ \mathbf{h})\bar{y}_r & (f_x \circ \mathbf{h})\bar{x}_s + (f_y \circ \mathbf{h})\bar{y}_s \end{bmatrix},$$

and it may be verified that

$$\begin{aligned} |J_{\mathbf{f}}| &= |\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r| \sqrt{1 + (f_x \circ \mathbf{h})^2 + (f_y \circ \mathbf{h})^2} \\ &= |J_{\mathbf{h}}| \sqrt{1 + (f_x \circ \mathbf{h})^2 + (f_y \circ \mathbf{h})^2}. \end{aligned}$$

Therefore, the integral formula for g on $\mathscr{S}$ is

$$\int_{\mathscr{S}} g\, dA = \int_{\mathbf{I}^2} (g \circ \mathbf{f})\sqrt{1 + (f_x \circ \mathbf{h})^2 + (f_y \circ \mathbf{h})^2}\, |J_{\mathbf{h}}|\, dA$$

$$= \int_{\mathbf{I}^2} g(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))\, \sqrt{1 + f_x(\bar{x}, \bar{y})^2 + f_y(\bar{x}, \bar{y})^2}\, |J_{\mathbf{h}}|\, dA.$$

Since $\mathbf{h} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and $\mathbf{I}^2$ represents $\mathscr{R}$, this may be written

Integral Formula for a Surface Graph

$$\int_{\mathscr{S}} g\, dA = \int_{\mathscr{R}} g(x, y, f)\sqrt{1 + (f_x)^2 + (f_y)^2}\, dA.$$

Example 3.2 Let $g(x, y, z) = z^2$ on the surface graph $\mathscr{S}$ of $z = f(x, y) = \sqrt{x^2 + y^2}$, which is on the region $\mathscr{R}$ bounded by $x^2 + y^2 = 1$. From

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y = \frac{y}{\sqrt{x^2 + y^2}},$$

we obtain

$$\int_{\mathscr{S}} g\, dA = \int_{\mathscr{R}} (\sqrt{x^2 + y^2})^2 \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\, dA$$

$$= \int_{\mathscr{R}} \sqrt{2}(x^2 + y^2)\, dA.$$

Representing $\mathscr{R}$ by

$$\mathbf{h}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi,$$

gives $|J_{\mathbf{h}}| = r$. Thus

$$\int_{\mathscr{S}} g\, dA = \int_0^1 dr \int_0^{2\pi} \sqrt{2}\,(r^2\cos^2\theta + r^2\sin^2\theta) r\, d\theta = \frac{\pi}{\sqrt{2}}.$$

We next consider the integral of g on the surface of revolution obtained by revolving about the x axis a curve $\mathscr{C}$ which lies in the upper half of the xy plane and is represented by

$$\mathbf{h}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}, \quad a \le r \le b.$$

Then $\mathscr{C}$ is represented by

$$\mathbf{f}(r, \phi) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\cos\phi\mathbf{j} + \bar{y}(r)\sin\phi\mathbf{k}, \qquad a \le r \le b, \quad 0 \le \phi \le 2\pi.$$

Then

$$J_{\mathbf{f}} = \begin{bmatrix} \dfrac{d\bar{x}}{dr} & 0 \\ \dfrac{d\bar{y}}{dr}\cos\phi & -\bar{y}\sin\phi \\ \dfrac{d\bar{y}}{dr}\sin\phi & \bar{y}\cos\phi \end{bmatrix}$$

and it may be verified that

$$|J_{\mathbf{f}}| = \bar{y}\sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2} = \bar{y}|J_{\mathbf{h}}|.$$

Thus,

$$\int_{\mathscr{S}} g\, dA = \int_a^b dr \int_0^{2\pi} g(\bar{x}, \bar{y}\cos\phi, \bar{y}\sin\phi)\bar{y}|J_{\mathbf{h}}|\, d\phi.$$

Since $\mathbf{h} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and $[a, b]$ represents $\mathscr{C}$, this may be written

$$\int_{\mathscr{S}} g\, dA = \int_{\mathscr{C}} \left(\int_0^{2\pi} yg(x, y\cos\phi, y\sin\phi)\, d\phi\right) dL.$$

It is convenient to write this integral in the following form.

Integral Formula for a Surface of Revolution

$$\int_{\mathscr{S}} g\, dA = \int_{\mathscr{C}} dL \int_0^{2\pi} yg(x, y\cos\phi, y\sin\phi)\, d\phi.$$

Example 3.3 Let

$$g(x, y, z) = \frac{y^2 + z^2}{x^5}$$

on the surface of revolution obtained by revolving the curve $\mathscr{C}$ described by $y = h(x) = x^2$, $1 \le x \le 2$, about the x axis. Then, by the above integral formula,

$$\int_{\mathscr{S}} g\, dA = \int_{\mathscr{C}} dL \int_0^{2\pi} \frac{y(y^2\cos^2\phi + y^2\sin^2\phi)}{x^5}\, d\phi = \int_{\mathscr{C}} \frac{2\pi y^3}{x^5}\, dL.$$

Representing $\mathscr{C}$ by $\mathbf{h}(x) = x\mathbf{i} + x^2\mathbf{j}$, $1 \le x \le 2$, gives $|J_{\mathbf{h}}| = \sqrt{1 + 4x^2}$; hence

$$\int_{\mathscr{S}} g\, dA = \int_1^2 2\pi x\sqrt{1 + 4x^2}\, dx = \frac{\pi}{6}(17^{3/2} - 5^{3/2}).$$

The *cylindrical surface* having the base curve $\mathscr{C}$ described by $\mathbf{h}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \leq r \leq b$, consists of all vectors $\langle x, y, z\rangle$ where $\langle x, y\rangle$ is on $\mathscr{C}$. Let $\mathscr{S}$ denote the portion of this surface lying between the surface graphs of $z = f_1(x, y)$ and $z = f_2(x, y)$, where $f_2(x, y) \geq f_1(x, y)$ above $\mathscr{C}$ (see Figure 14.8). The segment joining

$$\mathbf{u}_1 = \langle \bar{x}(r_o), \bar{y}(r_o), f_1 \circ \mathbf{h}(r_o)\rangle \quad \text{and} \quad \mathbf{u}_2 = \langle \bar{x}(r_o), \bar{y}(r_o), f_2 \circ \mathbf{h}(r_o)\rangle$$

is $\mathbf{u}_1 + [s(\mathbf{u}_2 - \mathbf{u}_1)]$. From this it follows that a representation of $\mathscr{S}$ is

$$\mathbf{f}(r, s) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + Z(r, s)\mathbf{k}, \qquad a \leq r \leq b, \quad 0 \leq s \leq 1,$$

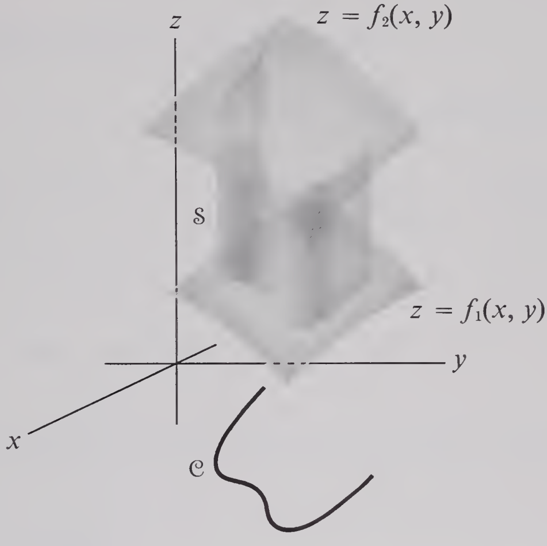


Figure 14.8

where $Z(r, s) = f_1 \circ \mathbf{h}(r) + s(f_2 \circ \mathbf{h}(r) - f_1 \circ \mathbf{h}(r))$. Thus

$$J_{\mathbf{f}} = \begin{bmatrix} \dfrac{d\bar{x}}{dr} & 0 \\ \dfrac{d\bar{y}}{dr} & 0 \\ - & f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h} \end{bmatrix},$$

from which it may be verified that

$$|J_{\mathbf{f}}| = (f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h})\sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2} = (f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h})|J_{\mathbf{h}}|.$$

This gives the formula

$$\int_{\mathscr{S}} g\, dA = \int_a^b dr \int_0^1 (g \circ \mathbf{f})(f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h})|J_{\mathbf{h}}|\, ds.$$

For r fixed, a change of variable $z = Z(r, s)$ and substitution of coordinate functions converts this formula to

$$\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dr \int_{f_1(\bar{x},\bar{y})}^{f_2(\bar{x},\bar{y})} g(\bar{x}, \bar{y}, \bar{z})|J_{\mathbf{h}}|\,dz.$$

Using notation similar to that for the integral formula for a surface of revolution, this may be further simplified to the equation

Integral Formula for a Cylindrical Surface

$$\int_{\mathscr{S}} g\,dA = \int_a^b dL \int_{f_1(x,y)}^{f_2(x,y)} g(x, y, z)\,dz.$$

Example 3.4 Given that $g(x, y, z) = z$ and $\mathscr{S}$ is the portion of the cylindrical surface having the base $x^2 + y^2 = 4$ and lying between $z = 2$ and $z = 6$, then application of the above integral formula with $\mathbf{h}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $0 \le \theta \le 2\pi$ and $f_1 = 2, f_2 = 6$, gives

$$\int_{\mathscr{S}} g\,dA = \int_0^{2\pi} d\theta \int_2^6 z\,dz = 32\pi.$$

The *surface area* of a surface $\mathscr{S}$ represented by $\mathbf{f}(r, s)$, $\langle r, s\rangle$ in $\mathbf{I}^2$, is given next.

Definition of Area (Surface)

$$\begin{aligned} A(\mathscr{S}) &= \int_{\mathscr{S}} 1\,dA \\ &= \int_{\mathbf{I}^2} |J_{\mathbf{f}}|\,dA. \end{aligned}$$

Each of the special formulas for $\int_{\mathscr{S}} g\,dA$ yields a particular surface-area formula. If $\mathscr{S}$ is the surface graph of $z = f(x, y)$ on a region $\mathscr{R}$, then, by the integral formula for a surface graph, we have the following result.

Area of a Surface Graph

$$A(\mathscr{S}) = \int_{\mathscr{R}} \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA.$$

Example 3.5 The surface area of the graph of $z=3x+4y$ on the rectangle $[1, 3] \times [0, 2]$ is given by

$$A(\mathscr{S}) = \int_1^3 dx \int_0^2 \sqrt{1+(3)^2+(4)^2}\, dy = 4\sqrt{26}.$$

If the curve $\mathscr{C}$ in the upper half of the xy plane is revolved about the x axis, then by the integral formula for a surface of revolution the given surface of revolution has area shown below.

$$A(\mathscr{S}) = \int_{\mathscr{C}} dL \int_0^{2\pi} y\, d\phi.$$

A simple computation gives the result shown below.

Area of a Surface of Revolution

$$A(\mathscr{S}) = 2\pi \int_{\mathscr{C}} y\, dL.$$

Example 3.6 A sphere of radius c is obtained by revolving

$$\mathbf{h}(\theta) = c\cos\theta\mathbf{i} + c\sin\theta\mathbf{j}, \qquad 0 \le \theta \le \pi,$$

about the x axis. Its area is, by the preceding integral formula,

$$A(\mathscr{S}) = 2\pi \int_0^{\pi} c\sin\theta\sqrt{(-c\sin\theta)^2+(c\cos\theta)^2}\, d\theta = 4\pi c^2.$$

The cylindrical surface having base $\mathscr{C}$ and lying between $z=f_1(x, y)$ and $z=f_2(x, y)$ has, from the integral formula for a cylindrical surface, the area given next.

Area of a Cylindrical Surface

$$A(\mathscr{S}) = \int_{\mathscr{C}} (f_2 - f_1)\, dL.$$

Example 3.7 The surface area of the cylinder having the base $x^2+y^2=c^2$ and lying between $z=0$ and $z=\delta$ may be found from the above integral formula using $\mathbf{h}(\theta) = c\cos\theta\mathbf{i} + c\sin\theta\mathbf{j}$, $0 \le \theta \le 2\pi$. The area is

$$A(\mathscr{S}) = \int_0^{2\pi} (\delta - 0)c\, d\theta = 2\pi\, c\delta.$$

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Find $\int_{\mathscr{S}} g\, dA$ if $g(x, y, z) = z^2/\sqrt{x^2 + y^2}$ and $\mathscr{S}$ is the surface of revolution about the z axis of the xz-plane curve $z = e^x$, $1 \leq x \leq 2$. (*Hint*: Rename the xyz system as the yzx system and apply an integral formula from this section.)
2. Find the area of the surface cut out of the sphere $x^2 + y^2 + z^2 = 16$ by the cylinder $y^2 + z^2 = 4$. (*Hint*: Consider both the top and bottom pieces. Each is a surface graph.)
3. Let $\mathscr{S}$ be the skew cylindrical surface between $z = 2$ and $z = 4$, generated by a line with direction vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$ having for its base the circle $x^2 + y^2 = 1$ in the xy plane. Find $\int_{\mathscr{S}} g\, dA$ if $g(x, y, z) = x^2 - y^2 - 2z(x - y)$.

Proofs

1. Let $\mathscr{S}$ be the cylinder lying between $z = c_1$ and $z = c_2$ and having as base an xy plane curve $\mathscr{C}$. Prove directly from the definition of an integral on a surface that if $g(x, y, z) = g'(x, y)$ for each $\langle x, y\rangle$ in $\mathscr{C}$ and each z in $[c_1, c_2]$, then $\int_{\mathscr{S}} g\, dA = (c_2 - c_1) \int_{\mathscr{C}} g'\, dL$.
2. Let $\mathscr{C}$ be a curve in the right half of the xy plane represented by $\mathbf{h}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \leq r \leq b$. Obtain a formula for $\int_{\mathscr{S}} g\, dA$ if $\mathscr{S}$ is the surface of revolution of $\mathscr{C}$ about the y axis.
3. Let $\mathscr{S}$ be the triangular surface $\mathbf{u}_o + [r\mathbf{u}_1 + rs\mathbf{u}_2]$.
   (a) Obtain a representation $\mathbf{f}(r, s)$, $[0, 1] \times [0, 1]$, from this form and show $|J_{\mathbf{f}}| = r|\langle \mathbf{u}_1, \mathbf{u}_2\rangle|$.
   (b) From (a) prove that $A(\mathscr{S}) = \frac{1}{2}|\langle \mathbf{u}_1, \mathbf{u}_2\rangle|$.

4. *Integrals on Solids*

We now consider the integral of a continuous function $g(x, y, z)$ on a solid $\mathscr{T}$ which is the $\mathbf{f}$-image of a 3-interval $\mathbf{I}^3$ where, with conditions as in previous sections,

(a) $\mathbf{f}$ is of class C^1,
(b) $\mathbf{f}$ is injective on the interior of $\mathbf{I}^3$, and
(c) $|J_{\mathbf{f}}(\mathbf{u}_o)| \neq 0$ for each $\mathbf{u}_o$ in the interior of $\mathbf{I}^3$.

As before, the integral of g on $\mathcal{T}$ is given by the definition

Definition of an Integral on a Solid

$$\int_{\mathcal{T}} g\, dV = \int_{\mathbf{I}^3} (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dV.$$

Since $J_{\mathbf{f}}$ is a 3×3 matrix, the equality $|J_{\mathbf{f}}| = |\det J_{\mathbf{f}}|$ holds.

Example 4.1 Given that $g(x, y, z) = (x - y)z$ and $\mathcal{T}$ is represented by

$$\mathbf{f}(r, s, t) = (r + s)\mathbf{i} + r\mathbf{j} + t\mathbf{k}, \qquad 0 \le r, s, t \le 1,$$

then $g \circ \mathbf{f} = (r + s - r)t = st$ and

$$|J_{\mathbf{f}}| = \left\| \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\| = 1.$$

Hence,

$$\int_{\mathcal{T}} (g \circ \mathbf{f})\, dV = \int_0^1 dr \int_0^1 ds \int_0^1 (st)1\, dt = \tfrac{1}{4}.$$

We next obtain integral formulas for particular solids. Let $\mathcal{T}$ be the solid of revolution obtained by revolving about the x axis the region $\mathcal{R}$ in the upper half of the xy plane represented by $\mathbf{h}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, $a_1 \le r \le b_1$ and $a_2 \le s \le b_2$. If $\langle x_0, y_0 \rangle$ in $\mathcal{R}$ is revolved about the x axis through an angle ϕ, then $\langle x_0, y_0 \cos \phi, y_0 \sin \phi \rangle$ is obtained (see Figure 14.9). It therefore follows

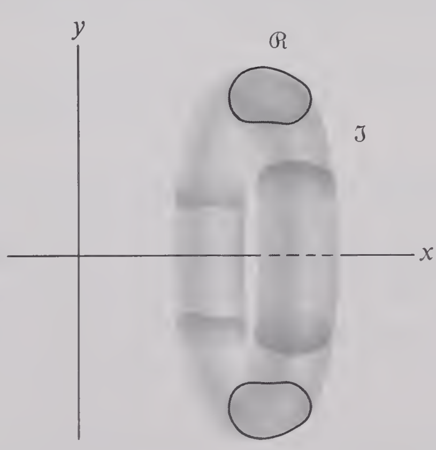


Figure 14.9

that a representation for $\mathcal{T}$ is

$$\mathbf{f}(r, s, \phi) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\cos\phi\mathbf{j} + \bar{y}(r, s)\sin\phi\mathbf{k},$$

where $a_1 \le r \le b_1$, $a_2 \le s \le b_2$, and $0 \le \phi \le 2\pi$. From

$$J_{\mathbf{f}} = \begin{bmatrix} \bar{x}_r & \bar{x}_s & 0 \\ \bar{y}_r \cos\phi & \bar{y}_s \cos\phi & -\bar{y}\sin\phi \\ \bar{y}_r \sin\phi & \bar{y}_s \sin\phi & \bar{y}\cos\phi \end{bmatrix},$$

it may be verified that $|J_{\mathbf{f}}| = \bar{y}|\bar{x}_r\bar{y}_s - \bar{x}_s\bar{y}_r| = \bar{y}|J_{\mathbf{h}}|$. Therefore,

$$\int_{\mathcal{T}} g\, dV = \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} ds \int_0^{2\pi} g(\bar{x}, \bar{y}\cos\phi, \bar{y}\sin\phi)\bar{y}|J_{\mathbf{h}}|\, d\phi.$$

Since $\mathbf{h} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and $[a_1, b_1] \times [a_2, b_2]$ represents $\mathcal{R}$, using notation similar to that in the previous section, this gives the formula

Integral Formula for a Solid of Revolution

$$\int_{\mathcal{T}} g\, dV = \int_{\mathcal{R}} dA \int_0^{2\pi} yg(x, y\cos\phi, y\sin\phi)\, d\phi.$$

Example 4.2 Let $g(x, y, z) = x^2$ on the solid $\mathcal{T}$ obtained by revolving about the x axis the region $\mathcal{R}$ in the upper half of the xy plane and bounded by $x^2/9 + y^2/4 = 1$, $y = 0$. Then by the above formula,

$$\int_{\mathcal{T}} g\, dV = \int_{\mathcal{R}} dA \int_0^{2\pi} yx^2\, d\phi = \int_{\mathcal{R}} 2\pi x^2 y\, dA.$$

Using the representation $\mathbf{h}(r, \theta) = 3r\cos\theta\mathbf{i} + 2r\sin\theta\mathbf{j}$, $0 \le r \le 1$, $0 \le \theta \le \pi$, which yields $|J_{\mathbf{h}}| = 6r$, we have

$$\int_{\mathcal{T}} g\, dV = \int_0^1 dr \int_0^{\pi} 2\pi(3r\cos\theta)^2(2r\sin\theta)6r\, d\theta = \frac{144\pi}{5}.$$

The *cylinder* having as its base the region $\mathcal{R}$ in the xy plane consists of all $\langle x, y, z\rangle$ where $\langle x, y\rangle$ is in $\mathcal{R}$ and z is a real number. Let $\mathcal{T}$ denote the portion of this cylinder lying between the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $f_2 > f_1$ above $\mathcal{R}$ (see Figure 14.10). If $\mathcal{R}$ is represented by $\mathbf{h}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, $a_1 \le r \le b_1$ and $a_2 \le s \le b_2$, then consideration of the vertical line segment from

$$\langle \bar{x}(r, s), \bar{y}(r, s), f_1 \circ \mathbf{h}(r, s)\rangle \quad \text{to} \quad \langle \bar{x}(r, s), \bar{y}(r, s), f_2 \circ \mathbf{h}(r, s)\rangle$$

yields for $\mathcal{T}$ the representation,

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + Z(r, s, t)\mathbf{k},$$

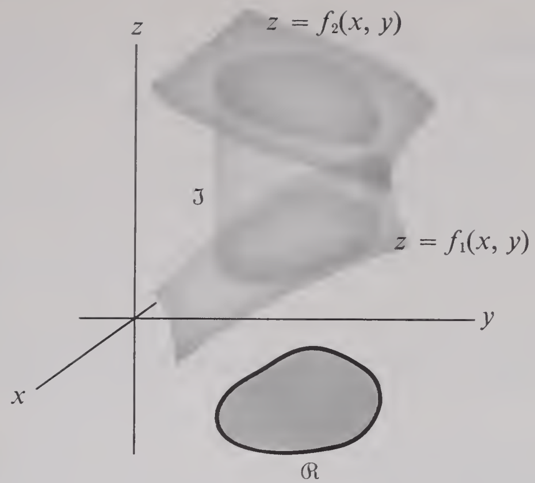


Figure 14.10

where $a_1 \leq r \leq b_1$, $a_2 \leq s \leq b_2$, $0 \leq t \leq 1$ and

$$Z(r, s, t) = f_1 \circ \mathbf{h}(r, s) + t(f_2 \circ \mathbf{h}(r, s) - f_1 \circ \mathbf{h}(r, s)).$$

Hence,

$$J_{\mathbf{f}} = \begin{bmatrix} \bar{x}_r & \bar{x}_s & 0 \\ \bar{y}_r & \bar{y}_s & 0 \\ — & — & f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h} \end{bmatrix},$$

and it may be seen by inspection that

$$|J_{\mathbf{f}}| = |\bar{x}_r \bar{y}_s - \bar{x}_s \bar{y}_r|(f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h}) = |J_{\mathbf{h}}|(f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h}).$$

Thus,

$$\int_{\mathscr{T}} g\, dV = \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} ds \int_0^1 (g \circ \mathbf{f})(f_2 \circ \mathbf{h} - f_1 \circ \mathbf{h})|J_{\mathbf{h}}|\, dt.$$

For r and s fixed, a change of variable $z = Z(r, s, t)$ and substitution of coordinate functions converts this equation to

$$\int_{\mathscr{T}} g\, dV = \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} ds \int_{f_1(\bar{x}, \bar{y})}^{f_2(\bar{x}, \bar{y})} g(\bar{x}, \bar{y}, z)|J_{\mathbf{h}}|\, dz.$$

Since $\mathbf{h} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, and $[a_1, b_1] \times [a_2, b_2]$ represents $\mathscr{R}$, we have the equation

Integral Formula for a Solid Cylinder

$$\int_{\mathscr{T}} g\, dV = \int_{\mathscr{R}} dA \int_{f_1(x,y)}^{f_2(x,y)} g(x, y, z)\, dz.$$

Example 4.3 Let $g(x, y, z) = x$ on the cylinder $\mathcal{T}$ having as a base the region $\mathcal{R}$ bounded by $y = x$, $y = x^2$ and lying between the surfaces $z = x + 2y$ and $z = 3x + 6y$. Then

$$\int_{\mathcal{T}} g\, dV = \int_{\mathcal{R}} dA \int_{x+2y}^{3x+6y} x\, dz = \int_{\mathcal{R}} (2x^2 + 4xy)\, dA$$

$$= \int_0^1 dx \int_{x^2}^{x} (2x^2 + 4xy) dy = \frac{4}{15}.$$

In addition to the Cartesian system there are two other important coordinate systems in $\mathbf{R}^3$,

(I) the spherical system and
(II) the cylindrical system.

If the region in the xz plane represented by $\rho \sin\phi\mathbf{i} + \rho\cos\phi\mathbf{k}$, $0 \le \rho \le a$, $0 \le \phi \le \pi$, is revolved about the z axis, then a (solid) sphere of radius a results with representation

$$\mathbf{f}(\rho, \phi, \theta) = \rho \sin\phi \cos\theta\mathbf{i} + \rho\sin\phi\sin\theta\mathbf{j} + \rho\cos\phi\mathbf{k},$$

where $0 \le \rho \le a$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. Any point in space is enclosed in a sufficiently large sphere; we call (ρ, ϕ, θ) its spherical coordinates (see Figure 14.11). They are related to the rectangular coordinates by the relationships

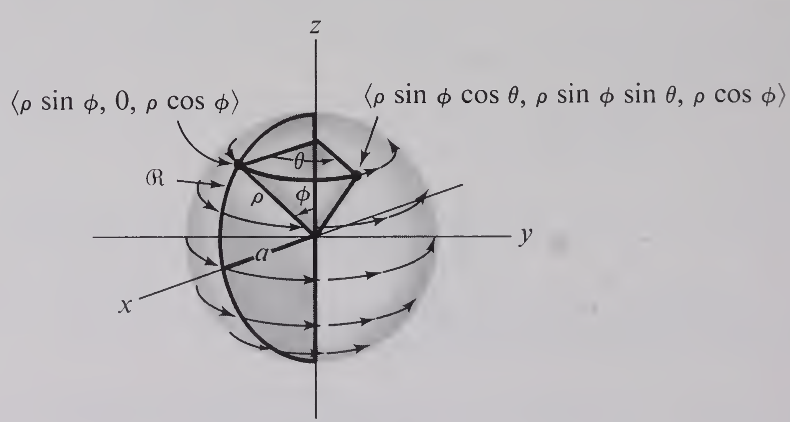


Figure 14.11

$$x = \rho \sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arctan\frac{y}{x}, \qquad \phi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

It can be shown that $|J_{\mathbf{f}}| = \rho^2 \sin\phi$; therefore, given $g(x, y, z)$ on the sphere $\mathcal{T}$ of radius a centered at the origin, then there is the formula

Integral Formula for a Solid Sphere

$$\int_{\mathscr{T}} g\, dV = \int_0^a d\rho \int_0^\pi d\phi \int_0^{2\pi} (g\circ\mathbf{f})\rho^2 \sin\phi\, d\theta,$$

where $g\circ\mathbf{f}(\rho, \phi, \theta) = g(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$.

We next study cylindrical coordinates. The cylinder $\mathscr{T}$ having as a base the disk $x^2 + y^2 \leq a^2$ and between the planes $z = c$ and $z = d$ has for a representation

$$\mathbf{f}(r, \theta, z) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k},$$

where $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $c \leq z \leq d$. We call (r, θ, z) the *cylindrical coordinates* of a point in space (see Figure 14.12). It may be verified that $|J_{\mathbf{f}}| = r$, and

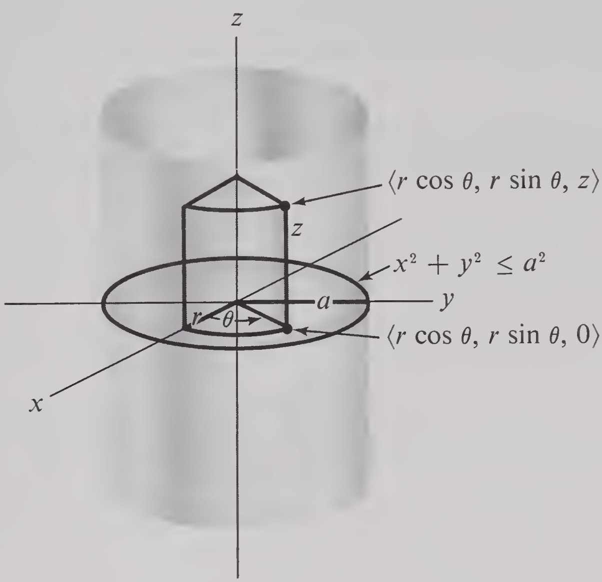


Figure 14.12

hence, given $g(x, y, z)$ on $\mathscr{T}$, then there results the formula

Integral Formula for a Solid Cylinder

$$\int_{\mathscr{T}} g\, dV = \int_0^a dr \int_0^{2\pi} d\theta \int_c^d (g\circ\mathbf{f})r\, dz,$$

where $g\circ\mathbf{f}(r, \theta, z) = g(r\cos\theta, r\sin\theta, z)$.

The *volume* of a solid $\mathscr{T}$ represented by $\mathbf{f}(r, s, t)$, $\langle r, s, t \rangle$ in $\mathbf{I}^3$, is defined next.

Definition of Volume

$$V(\mathscr{T}) = \int_{\mathscr{T}} 1 \, dV$$

$$= \int_{\mathbf{I}^3} |J_{\mathbf{f}}| \, dV.$$

Example 4.4 The volume of a sphere $\mathscr{T}$ of radius a is, by the integral formula for a solid sphere,

$$V(\mathscr{T}) = \int_0^a d\rho \int_0^\pi d\phi \int_0^{2\pi} \rho^2 \sin\phi \, d\theta = \frac{4\pi a^3}{3}.$$

If $\mathscr{T}$ is the solid of revolution obtained by revolving about the x axis the region $\mathscr{R}$, then the volume of $\mathscr{T}$ is, by the integral formula for a solid of revolution,

$$V(\mathscr{T}) = \int_{\mathscr{R}} dA \int_0^{2\pi} y \, d\phi.$$

By a simple computation there results the formula

Volume of a Solid of Revolution

$$V(\mathscr{T}) = 2\pi \int_{\mathscr{R}} y \, dA.$$

Example 4.5 Let $\mathscr{T}$ be the solid of revolution of the first quadrant region bounded by $x = y^3$, $x = 0$, and $y = 1$ about the x axis. The volume of $\mathscr{T}$ is, by the preceding formula,

$$V(\mathscr{T}) = 2\pi \int_0^1 dy \int_0^{y^3} y \, dx = \frac{2\pi}{5}.$$

The volume of the cylinder $\mathscr{T}$ having as a base the region $\mathscr{R}$ and lying between the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$ is, from the integral formula for a solid cylinder,

Volume of a Solid Cylinder

$$V(\mathscr{T}) = \int_{\mathscr{R}} (f_2 - f_1)\, dA.$$

Problems

1. Do Problem Set D at the end of the chapter.

Exercises

1. Let $\mathscr{R}$ be the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$. Find the volume of the solid of revolution of $\mathscr{R}$ about the y axis. (*Hint*: Rename the xy axis system as the y, $-x$ axis system and apply a volume formula from this section.)
2. Let $\mathscr{R}$ be the region bounded by the polar equation curves $\rho = \theta^{1/3}$, $\theta = 0$, $\theta = \pi/2$. Find the volume of the solid of revolution of $\mathscr{R}$ about the polar axis.
3. Find the volume of the skew cylinder lying between $z = 0$ and $z = 1$ and generated by a line having direction vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and having as a base the xy-plane disk $x^2 + y^2 \le 1$. (*Hint*: An arbitrary element of the skew cylinder has the form $\mathbf{OP} + s(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $0 \le s \le 1/2$ where $P = (r \cos\theta, r \sin\theta, 0)$ is in the disk.)
4. Find the volume of the skew cone with vertex $(1, 0, 1)$ and base the xy plane disk $x^2 + y^2 \le 1$. (*Hint*: An arbitrary element of the cone has the form $\mathbf{OP} + s\mathbf{PQ}$, where $P = (1, 0, 1)$ and $Q = (r \cos\theta, r \sin\theta, 0)$ is in the disk.)

Proofs

1. Let $\mathscr{T}$ be the solid cylinder which is between $z = c_1$, and $z = c_2$ and which has for its base a region $\mathscr{R}$ in the xy plane. Prove directly from the definition of an integral on a solid that if $g(x, y, z) = g'(x, y)$ for each $\langle x, y\rangle$ in $\mathscr{R}$ and each z in $[c_1, c_2]$, then
$$\int_{\mathscr{T}} g\, dV = (c_2 - c_1) \int_{\mathscr{R}} g'\, dA.$$
2. Let $\mathscr{T}$ be the solid tetrahedron $\mathbf{u}_0 + [r\mathbf{u}_1 + rs\mathbf{u}_2 + rst\mathbf{u}_3]$:
   (a) Obtain a representation $\mathbf{f}(r, s, t)$, $[0, 1] \times [0, 1] \times [0, 1]$, from this form and show that $|J_{\mathbf{f}}| = r^2 s |\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\rangle|$.
   (b) From (a) prove that $V(\mathscr{T}) = \frac{1}{6}|\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\rangle|$.

3. Let $\mathscr{T}$ be the solid skew cylinder lying between $z = c_1$ and $z = c_2$, and generated by a line with direction vector $\mathbf{u}$ having base region $\mathscr{R}$ in the xy plane. Prove $V(\mathscr{T}) = (c_2 - c_1)A(\mathscr{R})$.
4. Let $\mathscr{T}$ be a solid skew cone with vertex (a, b, c) and base region $\mathscr{R}$ in the xy plane. Prove
$$V(\mathscr{T}) = \frac{c}{3} A(\mathscr{R}).$$
5. Let $\mathscr{R}$ be a disk in the upper half of the xy plane bounded by $(x - a)^2 + (y - b)^2 = c^2$. Prove that the volume of the solid of revolution of $\mathscr{R}$ about the x axis is $2\pi^2 bc^2$.
6. Let $\mathscr{R}$ be a region in the right half of the xy plane, and $\mathscr{T}$ the solid of revolution of $\mathscr{R}$ about the y axis. Prove $V(\mathscr{T}) = 2\pi \int_{\mathscr{R}} x\, dA$.

5. *Integrals on n-Surfaces in* $\mathbf{R}^m$

The integrals of the previous sections will now be generalized to include nongeometric domains. In each of the previous geometric studies the emphasis was placed on sets that are suitable images of n-intervals. These yield a develompent which is limited, but nevertheless suitable for our purposes. We now proceed to a formal construction of our theory.

Definition of n-Surface in $\mathbf{R}^m$

An n-surface $\mathscr{S}^n$ in $\mathbf{R}^m$, $n \leq m$, is a set which is the $\mathbf{f}$*-image of an n-interval $\mathbf{I}^n$, where*

(a) $\mathbf{f}$ *is of class* C^1 *from an open set in* $\mathbf{R}^n$ *to* $\mathbf{R}^m$,
(b) $\mathbf{f}$ *is injective on the interior of* $\mathbf{I}^n$, *and*
(c) $|J_{\mathbf{f}}(\mathbf{u})| \neq 0$ *for each* $\mathbf{u}$ *in the interior of* $\mathbf{I}^n$.

The n–surface $\mathscr{S}^n$ is said to be *represented* by $\{\mathbf{f}, \mathbf{I}^n\}$. The significance of the properties (a), (b), and (c) has already been noted in certain geometric instances. It is implied that $\mathbf{I}^n$ is a subset of the domain of $\mathbf{f}$. An exact domain for $\mathbf{f}$ is not usually specified, since only the vectors in $\mathbf{I}^n$ are used to describe $\mathscr{S}^n$. However, it must be assumed that the domain of $\mathbf{f}$ is an open set in order for the C^1 property to be meaningful. Alternately, the definition of n–surface could require that $\mathbf{f}$ be a function that has $\mathbf{I}^n$ as its domain, and is extendible to a class-C^1 function on some open set containing $\mathbf{I}^n$. It would not suffice for $\mathbf{f}$ to be continuous on $\mathbf{I}^n$ and of class C^1 on the interior of $\mathbf{I}^n$; there are examples, even for $n = 1$, which

show that the integral of a continuous function g on the $\mathbf{f}$-image of $\mathbf{I}^n$ might then fail to exist.† We now define the integral of a continuous function g on $\mathcal{S}^n$ by generalizing the integrals of the previous sections in an obvious way.

Definition of Integrals on n-Surfaces

The integral of a continuous function g on an n-surface $\mathcal{S}^n$ represented by $\{\mathbf{f}, \mathbf{I}^n\}$ is

$$\int_{\mathcal{S}^n} g\,dV = \int_{\mathbf{I}^n} (g\circ\mathbf{f})|J_{\mathbf{f}}|\,dV.$$

Since this definition uses a particular representation of $\mathcal{S}^n$, it is natural to ask if two different representations of $\mathcal{S}^n$ necessarily give the same value for $\int_{\mathcal{S}^n} g\,dV$ according to this definition. We shall postpone a discussion of this until later in the section and, for the present, proceed as though all representations give the same value for $\int_{\mathcal{S}^n} g\,dV$. Using corresponding properties of the integral on $\mathbf{I}^n$, some properties of the integral on $\mathcal{S}^n$ will now be established.

Proposition 5.1

(a) $\int_{\mathcal{S}^n} (g_1 + g_2)\,dV = \int_{\mathcal{S}^n} g_1\,dV + \int_{\mathcal{S}^n} g_2\,dV$,
(b) $\int_{\mathcal{S}^n} cg\,dV = c\int_{\mathcal{S}^n} g\,dV$.

For a proof of (b) let $\{\mathbf{f}, \mathbf{I}^n\}$ represent $\mathcal{S}^n$. Then

$$\begin{aligned}\int_{\mathcal{S}^n} cg\,dV &= \int_{\mathbf{I}^n} (cg\circ\mathbf{f})|J_{\mathbf{f}}|\,dV\\ &= \int_{\mathbf{I}^n} c(g\circ\mathbf{f})|J_{\mathbf{f}}|\,dV\\ &= c\int_{\mathbf{I}^n} (g\circ\mathbf{f})|J_{\mathbf{f}}|\,dV\\ &= c\int_{\mathcal{S}^n} g\,dV.\end{aligned}$$

A similar proof can be made for (a) and also each part of the next result.

Proposition 5.2

(a) If $g_2 \geq g_1$, then $\int_{\mathcal{S}^n} g_2\,dV \geq \int_{\mathcal{S}^n} g_1\,dV$;
(b) if $g_2 \geq g_1$ and $g_2 \neq g_1$, then $\int_{\mathcal{S}^n} g_2\,dV > \int_{\mathcal{S}^n} g_1\,dV$; and
(c) $\left|\int_{\mathcal{S}^n} g\,dV\right| \leq \int_{\mathcal{S}^n} |g|\,dV$.

† As an illustration let $g = 1$, and let $\mathscr{C}$ be the graph of f where $f(x) = 1/x \sin 1/x$ for $0 < x \leq 1$ and $f(0) = 0$. In this case it can be shown that $\mathscr{C}$ has infinite length.

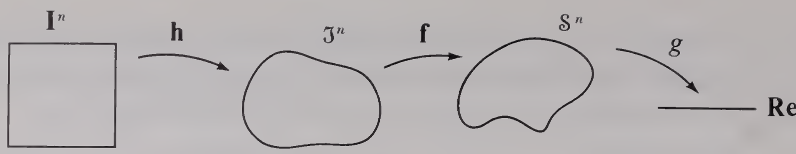


Figure 14.13

We now obtain an important result involving the composition of representing functions (see Figure 14.13).

Change of Domain Theorem for Integrals

Let $\{\mathbf{h}, \mathbf{I}^n\}$ *represent an n–surface* $\mathscr{T}^n$ *in* $\mathbf{R}^n$, *and let* $\{\mathbf{f}\circ\mathbf{h}, \mathbf{I}^n\}$ *represent an n–surface* $\mathscr{S}^n$ *in* $\mathbf{R}^m$. *If g is continuous on* $\mathscr{S}^n$, *then*

$$\int_{\mathscr{S}^n} g\,dV = \int_{\mathscr{T}^n} (g\circ\mathbf{f})|J_\mathbf{f}|\,dV.$$

A proof using $J_{\mathbf{f}\circ\mathbf{h}} = J_\mathbf{f}\circ\mathbf{h}$ (Proposition 4.5(c) in Chapter XI) and $|BA| = |B|\,|A|$ (Proposition 3.2 in Chapter VII) is

$$\begin{aligned}\int_{\mathscr{S}^n} g\,dV &= \int_{\mathbf{I}^n} (g\circ(\mathbf{f}\circ\mathbf{h}))|J_{\mathbf{f}\circ\mathbf{h}}|\,dV\\ &= \int_{\mathbf{I}^n} [(g\circ\mathbf{f})\circ\mathbf{h}]|(J_\mathbf{f}\circ\mathbf{h})J_\mathbf{h}|\,dV\\ &= \int_{\mathbf{I}^n} [(g\circ\mathbf{f})\circ\mathbf{h}](|J_\mathbf{f}|\circ\mathbf{h})|J_\mathbf{h}|\,dV\\ &= \int_{\mathscr{T}^n} (g\circ\mathbf{f})|J_\mathbf{f}|\,dV.\end{aligned}$$

Some of our previously established integral formulas are essentially specific instances of the change of domain theorem for integrals. In some cases a generalization is needed for the case in which the embedding space of $\mathscr{T}^n$ has dimension other than n. This extension is valid provided the equality $|J_{\mathbf{f}\circ\mathbf{h}}| = (|J_\mathbf{f}|\circ\mathbf{h})|J_\mathbf{h}|$ holds, since the foregoing proof then carries over without change (see Proofs, exercises 1, 2).

Example 5.1 Given the surface graph $\mathscr{S}$ of the function $z = f(x, y)$ on a region $\mathscr{R}$ in the xy plane represented by $\{\mathbf{h}, \mathbf{I}^2\}$, then, using

$$\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k},$$

we obtain by the change of domain theorem for integrals

$$\begin{aligned}\int_{\mathscr{S}} g\,dA &= \int_{\mathscr{R}} (g\circ\mathbf{f})|J_\mathbf{f}|\,dV\\ &= \int_{\mathscr{R}} g(x, y, f)\,\sqrt{1 + (f_x)^2 + (f_y)^2}\,dA.\end{aligned}$$

This is the integral formula which was derived by another method in Section 3.

We now consider the question of whether two different representations of $\mathscr{S}^n$ might yield different values for $\int_{\mathscr{S}^n} g\,dV$. A similar problem arose in the study of tangent n-planes; we shall proceed in an analogous way.

Definition of Equivalent Representations

Two representations $\{\mathbf{f}, \mathbf{I}^n\}$ *and* $\{\mathbf{f}', (\mathbf{I}^n)'\}$ *of* $\mathscr{S}^n$ *are* equivalent *provided there is an injective function* $\mathbf{h}$ *from an open set in* $\mathbf{R}^n$ *to* $\mathbf{R}^n$ *such that*

(a) $\mathbf{h}$ *and* $\mathbf{h}^{-1}$ *are both of class* C^1,
(b) $\mathbf{h}(\mathbf{I}^n) = (\mathbf{I}^n)'$, *and*
(c) $\mathbf{f} = \mathbf{f}' \circ \mathbf{h}$.

The next result says that equivalent representations assign the same integral.

Proposition 5.3 If $\{\mathbf{f}, \mathbf{I}^n\}$ and $\{\mathbf{f}', (\mathbf{I}^n)'\}$ are equivalent representations of $\mathscr{S}^n$, then for each continuous function g on $\mathscr{S}^n$,

$$\int_{\mathbf{I}^n} (g \circ \mathbf{f})\,dV = \int_{(\mathbf{I}^n)'} (g \circ \mathbf{f}')\,dV.$$

The nucleus of an argument for Proposition 5.3 will be described for $n = 2$. Let $\mathbf{f} = \mathbf{f}' \circ \mathbf{h}$, $p = \{\mathbf{I}_1{}^2, \ldots, \mathbf{I}_k{}^2\}$ be a partition of $\mathbf{I}^2$, and $p' = \{\mathbf{P}_1{}', \ldots, \mathbf{P}_k{}'\}$, where $\mathbf{P}_j{}' = \mathbf{h}(\mathbf{I}_j{}^2)$(see Figure 14.14). Also let $\mathbf{u}_j$ be arbitrary in $\mathbf{I}_j{}^2$ and $\mathbf{h}(\mathbf{u}_j) = \mathbf{u}_j{}'$.

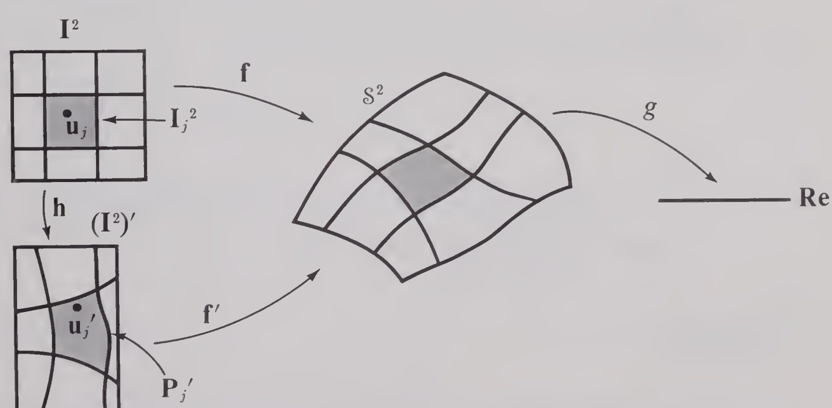


Figure 14.14

Then, using the same properties as in the proof of the change of domain theorem for integrals, together with the magnification theorem,

$$\begin{aligned}\int_{\mathbf{I}^2} (g\circ\mathbf{f})|J_{\mathbf{f}}|\,dV &\approx \sum_{j=1}^{k} [(g\circ\mathbf{f})|J_{\mathbf{f}}|]\Big|_{\mathbf{u}_j} A(\mathbf{I}_j{}^2)\\ &= \sum_{j=1}^{k} [(g\circ\mathbf{f}')|J_{\mathbf{f}'}|]\Big|_{\mathbf{u}_j'} |J_{\mathbf{h}}|\Big|_{\mathbf{u}_j} A(\mathbf{I}_j{}^2)\\ &\approx \sum_{j=1}^{k} [(g\circ\mathbf{f}')|J_{\mathbf{f}'}|]\Big|_{\mathbf{u}_j'} A(\mathbf{P}_j)\\ &\approx \int_{(\mathbf{I}^2)'} (g\circ\mathbf{f}')|J_{\mathbf{f}'}|\,dA.\end{aligned}$$

Questions

1. Two representations for an n-surface give the same integral provided they are ________.
2. The absolute determinant of the Jacobian matrix of a representing function is nonzero in the ________ of the representing interval.

Exercises

1. Find $\int_{\mathscr{S}^3} g\,dV$ given that $g(x_1, x_2, x_3, x_4) = x_1 - x_3$ and $\mathscr{S}^3$ is represented by

$$\mathbf{f}(r_1, r_2, r_3) = \langle r_1 - r_2, r_3, r_1 + r_2, 3r_3 - r_2\rangle, \qquad 0 \le r_1, r_2, r_3 \le 1.$$

2. Let $g(x, y) = xy$ on the line segment $\mathscr{C}$ with end points $(-1, -1)$ and $(1, 1)$. Compute $\int_{\mathscr{C}} g\,dL$ using representing functions (a) $\mathbf{f}(r) = r\mathbf{i} + r\mathbf{j}$ and (b) $\mathbf{f}'(r) = r^3\mathbf{i} + r^3\mathbf{j}$. Are these representations equivalent?
3. Let $g(x, y) = xy$ on the first-quadrant portion $\mathscr{R}$ of the disk $x^2 + y^2 \le 1$. Compute $\int_{\mathscr{R}} g\,dA$ using the representations

$$\text{(a)}\quad \mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\mathbf{j},\ [0, 1] \times \left[0, \frac{\pi}{2}\right]$$

and

$$\text{(b)}\quad \mathbf{f}'(r, s) = r\mathbf{i} + s\sqrt{1 - r^2}\mathbf{j},\ [0, 1] \times [0, 1].$$

Are these representations equivalent?

Proofs

1. Let $\mathscr{S}$ be the surface of revolution about the x axis of a curve $\mathscr{C}$ in the upper half of the xy plane represented by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$.

Prove the formula $A(\mathscr{S}) = 2\pi \int_{\mathscr{C}} y \, dL$ as follows:
(a) Show $\mathscr{S}$ is represented by the composition $\mathbf{f} \circ \mathbf{h}$ of

$$\mathbf{h}(r, \phi) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \phi\mathbf{k}$$

and

$$\mathbf{f}(x, y, z) = x\mathbf{i} + y \cos z\mathbf{j} + y \sin z\mathbf{k}, \qquad a \le r \le b \quad \text{and} \quad 0 \le \phi \le 2\pi.$$

(b) Verify $|J_{\mathbf{f} \circ \mathbf{h}}| = |J_{\mathbf{f}}| \, |J_{\mathbf{h}}|$.
(c) Apply Exercise 1 of the proofs in Section 3 and a modification of the change of domain theorem.

2. Let $\mathscr{T}$ be the solid of revolution about the x axis of a region $\mathscr{R}$ in the upper half of the xy plane. Prove that $V(\mathscr{T}) = 2\pi \int_{\mathscr{R}} y \, dA$ by a method similar to that in 1. Use Exercise 1 of the proofs in Section 4.

6. Mass and Moments

A basic physical interpretation of our integrand function g is that of *mass density*. We associate with each physical object a number called its *mass*, which relates to its weight or gravitational force. The *average mass density* of the object is given by the equation

$$\text{average mass density} = \frac{\text{mass}}{\text{volume}}.$$

Let $\mathscr{T}$ be a solid object and $\mathbf{u}$ a point in $\mathscr{T}$. If $\sigma[\mathbf{u} : \delta]$ denotes a small solid sphere in $\mathscr{T}$ having radius δ and centered at $\mathbf{u}$ and if m_δ is the average mass density of $\sigma[\mathbf{u} : \delta]$, then the *mass density* or *density* of $\mathscr{T}$ at $\mathbf{u}$ is

$$\text{density} = \lim_{\delta \to 0} m_\delta .$$

Thus if the function g describes the mass density of $\mathscr{T}$, then for a small piece $\mathscr{T}_j$ of $\mathscr{T}$ the estimate

$$\text{mass of } \mathscr{T}_j \approx g(\mathbf{u}_j) V(\mathscr{T}_j)$$

holds for any $\mathbf{u}_j$ in $\mathscr{T}_j$. Hence, if $\mathscr{T}$ is subdivided into k pieces $\{\mathscr{T}_1, \ldots, \mathscr{T}_k\}$, then the mass $m(\mathscr{T})$ of $\mathscr{T}$ is approximated by the Riemann sum

$$m(\mathscr{T}) \approx g(\mathbf{u}_1) V(\mathscr{T}_1) + \cdots + g(\mathbf{u}_k) V(\mathscr{T}_k).$$

Since the right side tends to the integral of g on $\mathscr{T}$, we have the basic formula

$$m(\mathscr{T}) = \int_{\mathscr{T}} g \, dV,$$

where g is the density function of $\mathscr{T}$.

Example 6.1 Given a cube $[0, 1] \times [0, 1] \times [1, 2]$ with density equal to the square of the distance from the origin, its mass is

$$\int_0^1 dx \int_0^1 dy \int_1^2 (x^2 + y^2 + z^2)\, dz = 3.$$

The mass of a thin rod may be evaluated by treating it as a curve. Likewise the mass of a flat sheet of material may be obtained by treating it as a plane region or surface. In general the formula

$$m(\mathscr{S}^n) = \int_{\mathscr{S}^n} g\, dV$$

is valid, where $\mathscr{S}^n$ is a physical object having the shape of an n–surface and g is its density.

Example 6.2 Let $\mathscr{R}$ be a flat circular disk having radius 1 and density function $r + 1$, where r denotes distance from the center. We introduce an xy coordinate system with origin at the center of the disk. Since $g(x, y) = \sqrt{x^2 + y^2} + 1$, we have

$$m(\mathscr{R}) = \int_{\mathscr{R}} (\sqrt{x^2 + y^2} + 1)\, dA.$$

Using the representation $\mathbf{f}(r, \theta) = r \cos \theta\mathbf{i} + r \sin \theta\mathbf{j}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, it may be verified that

$$m(\mathscr{R}) = \int_0^1 dr \int_0^{2\pi} (r + 1)r\, d\theta = \frac{5\pi}{3}.$$

The physical concept of *moment* measures a tendency to produce rotation. The moment M about a line or plane of a point object having mass m is, from physics,

$$M = md,$$

where d is the distance from the point object to the line or plane. Let $\mathscr{R}$ be a flat plate in the xy plane having the shape of a region and density function $g(x, y)$. If $\{\mathscr{R}_1, \ldots, \mathscr{R}_k\}$ is a subdivision of $\mathscr{R}$ and $\mathbf{u}_j = \langle a_j, b_j\rangle$ is in $\mathscr{R}_j$, $j = 1, 2, \ldots, k$, then the moment M_j of the mass m_j of $\mathscr{R}_j$ about the x axis is estimated by

$$M_j \approx m_j \cdot b_j \approx g(\mathbf{u}_j) \cdot A(\mathscr{R}_j) \cdot b_j .$$

The total moment $M_x(\mathscr{R})$ of the plate about the x axis equals the sum of the moments of the separate pieces. Thus,

$$\begin{aligned} M_x(\mathscr{R}) &= M_1 + \cdots + M_k \\ &\approx b_1 g(\mathbf{u}_1)A(\mathscr{R}_1) + \cdots + b_k g(\mathbf{u}_k)A(\mathscr{R}_k). \end{aligned}$$

In the limit, since b_j is the y coordinate of $\mathbf{u}_j$, this gives the formula

$$M_x(\mathscr{R}) = \int_{\mathscr{R}} yg\, dA.$$

Similarly the moment of $\mathscr{R}$ about the y axis is

$$M_y(\mathscr{R}) = \int_{\mathscr{R}} xg\, dA,$$

Example 6.3 Let $\mathscr{R}$ be the upper half of the flat circular disk in Example 6.2. Using the representation

$$\mathbf{f}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le \pi,$$

we obtain

$$M_x(\mathscr{R}) = \int_0^1 dr \int_0^\pi (r\sin\theta)(r+1)\, r\, d\theta = \frac{7}{6}.$$

It may be verified directly, or by using symmetry, that $M_y(\mathscr{R}) = 0$.

A similar moment formula holds for a curve $\mathscr{C}$ lying in the xy plane and having density function $g(x, y)$. Thus,

$$M_x(\mathscr{C}) = \int_{\mathscr{C}} yg\, dL\,;$$

$$M_y(\mathscr{C}) = \int_{\mathscr{C}} xg\, dL.$$

For objects in Cartesian space the analogous formulas hold for moments with respect to coordinate planes. The moment of an object having the shape of an n–surface $\mathscr{S}^n$ and density $g(x, y, z)$ relative to the xy, xz, and yz planes, respectively, is

$$M_{xy}(\mathscr{S}^n) = \int_{\mathscr{S}^n} zg\, dA,$$

$$M_{xz}(\mathscr{S}^n) = \int_{\mathscr{S}^n} yg\, dA,$$

$$M_{yz}(\mathscr{S}^n) = \int_{\mathscr{S}^n} xg\, dA.$$

Example 6.4 Let $\mathscr{S}$ be the cylindrical surface having the base $x^2 + y^2 = 1$ and lying between $z = 0$ and $z = 1$. If $\mathscr{S}$ has density function $g(x, y, z) = z + 1$, then, by the integral formula for a cylindrical surface,

$$M_{xy}(\mathscr{S}) = \int_{\mathscr{C}} dL \int_0^1 z(z+1)\, dz = \int_{\mathscr{C}} \frac{5}{6}\, dL = \frac{5\pi}{3}.$$

By symmetry, $M_{xz}(\mathscr{S}) = M_{yz}(\mathscr{S}) = 0$.

Thus far we have considered moments with respect to hyperplanes. Such moments have a special character, since hyperplanes divide the space into two parts and permit a *signed* distance from a point to the hyperplane. Other moments, such as those with respect to a point or a line in space, may be defined. These definitions employ a distance which is necessarily positive (not signed). In general, if $\mathbf{P}$ is a k-plane ($k \geq 0$) other than a hyperplane, then the moment about $\mathbf{P}$ of an object $\mathscr{S}^n$ having density $g(x_1, \ldots, x_n)$ is given by

$$M_{\mathbf{P}}(\mathscr{S}^n) = \int_{\mathscr{S}^n} \delta g \, dA,$$

where $\delta(x_1, \ldots, x_n)$ is the distance from $\langle x_1, \ldots, x_n \rangle$ to $\mathbf{P}$.

Example 6.5 Let $\mathscr{C}$ be the line segment from (0, 0) to (1, 3) having density given by $g(x, y) = xy$. Using $\delta(x, y) = \sqrt{x^2 + y^2}$ and the representation $\mathbf{f}(r) = r\mathbf{i} + 3r\mathbf{j}$, $0 \leq r \leq 1$, for $\mathscr{C}$, the moment of $\mathscr{C}$ about the origin is

$$\int_0^1 (\sqrt{r^2 + 9r^2}) r(3r) \sqrt{10} \, dr = \frac{15}{2}.$$

Moments of higher degree also have important applications. We shall restrict ourselves here to the case of second-order moments. The signed distance is then not relevant, since $(-1)^2 = 1$, and we define the *second moment* about $\mathbf{P}$ of an n-surface $\mathscr{S}^n$ having density g by

$$I_{\mathbf{P}}(\mathscr{S}^n) = \int_{\mathscr{S}^n} \delta^2 \, g \, dA,$$

where δ is again the distance function to $\mathbf{P}$. Second moments, often called *moments of inertia*, have valuable interpretations which may be found in elementary physics texts. The second moment I_o with respect to the origin is called the *polar moment* of *inertia*.

Example 6.6 Let $\mathscr{R}$ be the disk $x^2 + y^2 \leq 1$ having density $r + 1$ as in Example 6.2. The polar moment of inertia of $\mathscr{R}$ is

$$\begin{aligned} I_o(\mathscr{R}) &= \int_{\mathscr{R}} (x^2 + y^2)(\sqrt{x^2 + y^2} + 1) \, dA \\ &= \int_0^1 dr \int_0^{2\pi} r^2(r + 1) r \, d\theta = \frac{9\pi}{10}. \end{aligned}$$

The *centroid* (or *center of mass*) of a physical object is its balance point. For an object $\mathscr{S}^n$ in the Cartesian plane it is defined to be the point

$$\frac{1}{m(\mathscr{S}^n)} \langle M_y(\mathscr{S}^n), M_x(\mathscr{S}^n) \rangle.$$

For an object $\mathscr{S}^n$ in Cartesian space it is

$$\frac{1}{m(\mathscr{S}^n)}\langle M_{yz}(\mathscr{S}^n), M_{xz}(\mathscr{S}^n), M_{xy}(\mathscr{S}^n)\rangle.$$

Example 6.7 Let $\mathscr{R}$ be the upper half of the disk $x^2+y^2\le 1$ having density $r+1$, as in Example 6.3. Then

$$m(\mathscr{R})=\frac{1}{2}\left(\frac{5\pi}{3}\right)=\frac{5\pi}{6}$$

using Example 6.2 and symmetry. Hence, by Example 6.3, the centroid is

$$\frac{6}{5\pi}\left\langle 0,\frac{7}{6}\right\rangle=\left\langle 0,\frac{7}{5\pi}\right\rangle.$$

The *radius of gyration* of an object $\mathscr{S}^n$ about a k-plane $\mathbf{P}$ is the number such that if the entire mass of the object were placed at that number distance from $\mathbf{P}$, then the moment of inertia of the point mass would be the same as the moment of inertia of the object. It is given by the formula

$$\sqrt{\frac{I_{\mathbf{P}}(\mathscr{S}^n)}{m(\mathscr{S}^n)}}.$$

Example 6.8 Let $\mathscr{R}$ be the disk $x^2+y^2\le 1$ having density $r+1$ as in Example 6.2. The moment of inertia about the x axis is

$$\begin{aligned}I_x(\mathscr{R})&=\int_{\mathscr{R}} y^2(\sqrt{x^2+y^2}+1)\,dA\\&=\int_0^1 dr\int_0^{2\pi} r^2\sin^2\theta\,(r+1)r\,d\theta=\frac{9\pi}{20}.\end{aligned}$$

From $m(\mathscr{S}^n)=5\pi/3$, the radius of gyration is

$$\sqrt{\left(\frac{9\pi}{20}\right)\left(\frac{3}{5\pi}\right)}=\frac{3\sqrt{3}}{10}.$$

Questions

1. Mass density is a quotient of mass and ______.
2. The concept of ______ measures tendency to produce rotation.
3. Second moments are called moments of ______.
4. A mathematical word (or expression) for balance point is ______.

Problems

1. Do Problem Set E at the end of the chapter.

Exercises

1. Find the moment about the x axis of the rectangular region $[0, 1] \times [0, 2]$ if the density equals the square of the distance from the origin.
2. Find the polar moment of inertia of the disk $x^2 + y^2 \leq 1$ having constant density 1.
3. Find the moment about the x axis of the line segment with end points $(0, 2)$ and $(1, 5)$ and having constant density 1.
4. Find the moment about the xy plane of the helix coil
$$\mathbf{f}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \theta\mathbf{k}, \quad 0 \leq \theta \leq 2\pi,$$
with constant density 1.
5. Find the polar moment of inertia of the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$, with constant density 1.

Proofs

1. Let $\mathscr{T}$ be the solid of revolution about the x axis of a region $\mathscr{R}$ in the upper half of the xy plane.
   (a) Prove $M_{yz}(\mathscr{T}) = 2\pi \int_{\mathscr{R}} xy\, dA$.
   (b) Derive formulas for $M_{xz}(\mathscr{T})$, $M_{xy}(\mathscr{T})$, $I_{xy}(\mathscr{T})$, $I_{xz}(\mathscr{T})$, and $I_{yz}(\mathscr{T})$.
2. Assuming constant density 1, prove the centroid of
   (a) the semicircle $x^2 + y^2 = a^2$, $y \geq 0$, is $(0, 2a/\pi)$,
   (b) the first-quadrant portion of $x^2 + y^2 \leq a^2$ is $(4a/3\pi, 4a/3\pi)$,
   (c) a triangular region is the intersection of its medians,
   (d) the first octant portion of $x^2 + y^2 + z^2 \leq a^2$ is $(3a/8, 3a/8, 3a/8)$.
3. Assuming constant density 1, prove the radius of gyration of
   (a) a solid circular cylinder of base radius a about its axis is $a/\sqrt{2}$, and
   (b) a solid cone with base radius a about an axis through the vertex and base center is $\sqrt{3/10}\, a$.
4. Prove, using a suitable coordinate system, the following two theorems of Pappus (Pappus was an early Greek mathematician).
   (a) First theorem. If a region $\mathscr{R}$ lies entirely on one side of a line and is revolved about that line, then the volume of the generated solid of revolution is the product of the area of $\mathscr{R}$ and the length of the path of the centroid of $\mathscr{R}$.
   (b) Second theorem. If a plane curve $\mathscr{C}$ lies entirely on one side of a line and is revolved about that line, then the area of the generated surface of revolution is the product of the length of $\mathscr{C}$ and the length of the path of the centroid of $\mathscr{C}$.
   (*Hint*: These are easy corollaries of integral formulas for solids and surfaces of revolution.)

7. Integrals on Other Domains

Thus far we have considered integrals only on sets which are **f**-images of n-intervals, where **f** is of class C^1 and satisfies certain other conditions. Many simple geometric configurations cannot be represented easily, if at all, in such a manner. Now we extend our techniques of integration to include certain other sets. We shall give special attention to sets which are

(I) a set union of n-surfaces where any two of the n-surfaces have for their intersection either the empty set or a set consisting only of boundary points of the two sets. In this case the integral on the union is the sum of integrals on the separate n-surfaces.

(II) a limit of an increasing collection of n-surfaces. The integral on the limit set is then a limit (if it exists) of the integrals on the n-surfaces in the collection.

The 1-surfaces considered thus far include only smooth curves. The perimeter of a triangle is not smooth, since it has a corner at each vertex. They are examples of *piecewise smooth curves.* A curve is *piecewise smooth* if it is composed of finitely many smooth curves joined end to end. If $\mathscr{C}$ is a piecewise smooth plane curve composed of smooth curves $\mathscr{C}_1, \ldots, \mathscr{C}_k$, then the integral of $g(x, y)$ on $\mathscr{C}$ is given by

$$\int_{\mathscr{C}} g\,dL = \int_{\mathscr{C}_1} g\,dL + \cdots + \int_{\mathscr{C}_k} g\,dL.$$

Example 7.1 Let $g(x, y) = xy$ on the perimeter of the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. If $\mathscr{C}_1$ joins $(0, 0)$ and $(1, 0)$, $\mathscr{C}_2$ joins $(1, 0)$ and $(0, 1)$, and $\mathscr{C}_3$ joins $(0, 0)$ and $(0, 1)$, then $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$ have respective representations $\mathbf{f}(r) =$

$$r\mathbf{i}, \quad (1-r)\mathbf{i} + r\mathbf{j}, \quad \text{and} \quad r\mathbf{j},$$

where $0 \le r \le 1$ in each case. Thus

$$\int_{\mathscr{C}} g\,dL = \int_0^1 r(0)\sqrt{1}\,dr + \int_0^1 (1-r)r\sqrt{2}\,dr + \int_0^1 0(r)\sqrt{1}\,dr = \frac{\sqrt{2}}{6}.$$

We next consider plane regions which are the union of two or more 2-surfaces. The set $\mathscr{R}$ in Figure 14.15 can be described in the two ways pictured as a union of rectangles whose intersection contains only boundary points of each. The integral of a continuous function $g(x, y)$ on $\mathscr{R}$ is given by

$$\int_{\mathscr{R}} g\,dA = \int_{\mathscr{R}_1} g\,dA + \int_{\mathscr{R}_2} g\,dA,$$

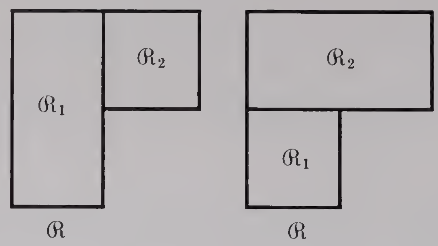


Figure 14.15

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are 2-surfaces which combine to give $\mathscr{R}$. It is not evident that this integral is independent of the choices of the subdividing 2-intervals; more advanced treatments of the integral study this question. We shall assume that the given formula for $\int_{\mathscr{R}} g\, dA$ is well defined, and extends to the more general equation

$$\int_{\mathscr{R}} g\, dA = \int_{\mathscr{R}_1} g\, dA + \cdots + \int_{\mathscr{R}_k} g\, dA.$$

Example 7.2 Let $g(x, y) = xy^2$ on the region $\mathscr{R}$ bounded by the line segments joining successively $(0, 0)$, $(1, 0)$, $(1, 2)$, $(2, 2)$, $(2, 3)$, $(0, 3)$, and $(0, 0)$. Then $\mathscr{R}$ is the union of $\mathscr{R}_1 = [0, 1] \times [0, 3]$ and $\mathscr{R}_2 = [1, 2] \times [2, 3]$. Hence,

$$\begin{aligned}\int_{\mathscr{R}} g\, dA &= \int_{\mathscr{R}_1} g\, dA + \int_{\mathscr{R}_2} g\, dA \\ &= \int_0^1 dx \int_0^3 xy^2\, dy + \int_1^2 dx \int_2^3 xy^2\, dy = \frac{9}{2} + \frac{19}{2} = 14.\end{aligned}$$

The same answer may be obtained by writing $\mathscr{R}$ as a union of $[0, 1] \times [0, 2]$ and $[0, 3] \times [2, 3]$.

Similar considerations apply to surfaces and solids.

Example 7.3 Let $\mathscr{S}$ be the entire surface, including the two ends, of the cylinder having the base $x^2 + y^2 = 1$ and lying between $z = 0$ and $z = 2$. The three pieces of $\mathscr{S}$ are represented by

(a) $\mathbf{f}(\theta, z) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2;$
(b) $\mathbf{f}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + 0\mathbf{k}, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi;$
(c) $\mathbf{f}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + 2\mathbf{k}, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$

If $\mathscr{S}$ has density given by the constant function $g=1$, then its moment about the xy plane is

$$M_{xy}(\mathscr{S})=\int_{\mathscr{S}} z\,dA=\int_0^{2\pi} d\theta \int_0^2 z\,dz+\int_0^1 dr\int_0^{2\pi} 0r\,d\theta$$
$$+\int_0^1 dr\int_0^{2\pi} 2r\,d\theta=4\pi+0+2\pi=6\pi.$$

We next give attention to integrals on sets which are limits of n-surfaces. Of particular interest is the case in which the integrand function or the domain set is unbounded. The integral is then called an *improper integral.* An improper integral is defined as a limit and called *convergent* if the limit exists; otherwise it is *divergent.*

Let $\mathscr{C}$ be a plane curve, perhaps unbounded, and $\mathscr{C}_x$ a variable 1-surface which tends to $\mathscr{C}$ as x approaches a limiting value. Then the integral of a continuous, perhaps unbounded, function $g(x, y)$ on $\mathscr{C}$ is given by

$$\int_{\mathscr{C}} g\,dL=\lim_{x}\int_{\mathscr{C}_x} g\,dL.$$

Example 7.4 Let $\mathscr{C}$ be the line $y=x$, $x\geq 1$, and $g(x, y)=1/x^2y$. We define

$$\int_{\mathscr{C}} g\,dL=\lim_{x\to\infty}\int_{\mathscr{C}_x} g\,dL$$

where $\mathscr{C}_x$ is the portion of $\mathscr{C}$ from (1, 1) to (x, x). Then $\mathscr{C}_x$ is represented by $\mathbf{f}(r)=r\mathbf{i}+r\mathbf{j}$, $1\leq r\leq x$, and hence,

$$\int_{\mathscr{C}} g\,dA=\lim_{x\to\infty}\int_1^x \frac{1}{(r^2)r}\sqrt{2}\,dr=\frac{1}{\sqrt{2}}.$$

In the preceding example $\mathscr{C}$ was unbounded; in the next example $\mathscr{C}$ is bounded but g is unbounded.

Example 7.5 Let $\mathscr{C}$ be the line segment $y=x, 0<x\leq 1$ and $g(x, y)=(xy)^{-1/4}$. We define

$$\int_{\mathscr{C}} g\,dL=\lim_{x\to 0}\int_{\mathscr{C}_x} g\,dL,$$

where $\mathscr{C}_x$ is the portion of $\mathscr{C}$ from (x, x) to (1, 1).Then $\mathscr{C}_x$ is represented by $\mathbf{f}(r)=r\mathbf{i}+r\mathbf{j}$, $x\leq r\leq 1$, and

$$\int_{\mathscr{C}} g\,dL=\lim_{x\to 0}\int_x^1 g\,dL=\lim_{x\to 0}\int_x^1 (r^2)^{-1/4}\sqrt{2}\,dr=2\sqrt{2}.$$

Similar techniques apply to integrals on regions, surfaces, and solids which can be described as limits.

Example 7.6 Let $g(x, y) = 1/\sqrt{x^2 + y^2}$ on the set $\mathscr{R}$ which is the disk $x^2 + y^2 \leq 1$ with the origin deleted. We define

$$\int_{\mathscr{R}} g\, dA = \lim_{\varepsilon \to 0} \int_{\mathscr{R}_\varepsilon} g\, dA,$$

where $\mathscr{R}_\varepsilon$ is the annulus lying between $x^2 + y^2 = \varepsilon^2$ and $x^2 + y^2 = 1$. Since $\mathscr{R}_\varepsilon$ is represented by $\mathbf{f}(r, \theta) = r \cos \theta\mathbf{i} + r \sin \theta\mathbf{j}$, $\varepsilon \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, we obtain

$$\int_{\mathscr{R}} g\, dA = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} dr \int_{0}^{2\pi} \frac{1}{r} r\, d\theta = 2\pi.$$

Example 7.7 Let $g(x, y, z) = 1/z^2$ on the cylindrical surface $\mathscr{S}$ described by $x^2 + y^2 = 1$, $z \geq 1$. Then

$$\int_{\mathscr{S}} g\, dA = \lim_{Z \to \infty} \int_{\mathscr{S}_Z} g\, dA,$$

where $\mathscr{S}_Z$ is the portion of $\mathscr{S}$ bounded above by the plane $z = Z$, $Z > 1$. Since $\mathscr{S}_Z$ is represented by $\mathbf{f}(\theta, z) = \cos \theta\mathbf{i} + \sin \theta\mathbf{j} + z\mathbf{k}$, $0 \leq \theta \leq 2\pi$, $1 \leq z \leq Z$, we have

$$\int_{\mathscr{S}} g\, dA = \lim_{Z \to \infty} \int_{0}^{2\pi} d\theta \int_{1}^{Z} \frac{1}{z^2} 1\, dz = 2\pi.$$

Questions

1. A ______ smooth curve is made up of finitely many smooth curves joined end to end.
2. An integral is improper if the integrand function or its domain is not ______.
3. An improper integral which can be evaluated as a limit is said to ______.

Problems

1. Do Problem Set F at the end of the chapter.

Exercises

1. Find the moment about the yz plane of the conical surface of revolution about the x axis of $y = x$, $0 \leq x \leq 1$, together with its top disk (assume constant density 1).

Proofs

1. Find values of c and d so that $\int_{\mathscr{R}} x^c y^d \, dA$ converges if $\mathscr{R} =$
   (a) $[1, \infty) \times (-\infty, 1]$, (b) $(0, 1] \times [-1, 0)$.
2. Let $\mathscr{R}_c$ be the xy plane region having constant density 1 and bounded by $x = 1$, $y = 0$, and $y = x^c$ $(x \geq 1)$. Determine for which values of c the integral converges for the (a) first, (b) second moment of the solid of revolution of $\mathscr{R}_c$ about the x axis.

Problems

A. *Computation of Integrals on Curves*

In the evaluation of integrals on curves in the Cartesian plane, we shall be required to describe a given curve $\mathscr{C}$ as the image of a real interval. This description involves

(I) a vector equation $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, and
(II) a real interval $[a, b]$ acting as the domain of $\mathbf{f}$.

The vector equation and domain interval combination is called a *representation* of $\mathscr{C}$. If $\mathscr{C}$ is a curve with end points, then a and b may be determined by the property that $\mathbf{f}(a)$ and $\mathbf{f}(b)$ are the end points of $\mathscr{C}$. The evaluation of the integral of a function $g(x, y)$ on $\mathscr{C}$ proceeds as follows:

A.1 (I) Obtain for $\mathscr{C}$ a representation consisting of a function $\mathbf{f}(r)$ and an interval $\mathbf{I} = [a, b]$.
(II) Find the absolute determinant $|J_{\mathbf{f}}|$ of the Jacobian matrix of $\mathbf{f}$, and
(III) Evaluate $\int_{\mathscr{C}} g \, dL = \int_{\mathbf{I}} (g \circ \mathbf{f})|J_{\mathbf{f}}| \, dr$.

If $\mathscr{C}$ is represented by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \leq r \leq b$, then the equality in A.1(III) becomes

A.2 $$\int_{\mathscr{C}} g \, dL = \int_a^b g(x, y)\sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2} \, dr.$$

1. Given $g(x, y) = x^3/y$ on the curve $\mathscr{C}$ which is the arc of the parabola $y = x^2$ with end points (1, 1) and (2, 4):

   (a) Find a representation for $\mathscr{C}$. Obtain both a vector equation $\mathbf{f}(r)$ and representing interval $[a, b]$.
   (b) Find $g \circ \mathbf{f}$ and $|J_{\mathbf{f}}|$ from (a).
   (c) Find $\int_{\mathscr{C}} g\, dL$.

2. Find $\int_{\mathscr{C}} g\, dL$ given the following conditions.

   (a) $g(x, y) = xy^2$ and $\mathscr{C}$ is the right half-plane portion of $x^2 + y^2 = 4$;
   (b) $g(x, y) = e^{x+y}$ and $\mathscr{C}$ is the line segment with end points (1, 2) and (7, 5). Obtain the representation from the vector algebra form $\mathbf{u}_o + [r\mathbf{u}_1]$ of a line segment;
   (c) $g(x, y) = xy$ and $\mathscr{C}$ is the first-quadrant portion of the ellipse $x^2/4 + y^2/9 = 1$.

If $\mathscr{C}$ is the graph of $y = f(x)$, $a \leq x \leq b$, then applying the representation $\mathbf{f}(x) = x\mathbf{i} + f(x)\mathbf{j}$, $a \leq x \leq b$, to A.2 gives

A.3 $$\int_{\mathscr{C}} g\, dL = \int_a^b g(x, s(x)) \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx.$$

3. Find $\int_{\mathscr{C}} g\, dL$ when

   (a) $g(x, y) = x/\sqrt{2 - y^2}$ and $\mathscr{C}$ is the graph of $y = \cos x$, $0 \leq x \leq \pi/2$, and
   (b) $g(x, y) = \sqrt{1 + y^2}$ and $\mathscr{C}$ is the graph of $y = e^x$, $0 \leq x \leq 1$.

If g is given as a function $g(\rho, \theta)$ of polar coordinates ρ and θ on the curve $\rho = h(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, then the formula in A.1(III) becomes

A.4 $$\int_{\mathscr{C}} g\, dL = \int_{\theta_1}^{\theta_2} g(h(\theta), \theta) \sqrt{h^2 + \left(\frac{dh}{d\theta}\right)^2}\, d\theta.$$

4. Find $\int_{\mathscr{C}} g\, dV$ given the following conditions.

   (a) $g(\rho, \theta) = \sqrt{\rho + 4/\theta}$ and $\mathscr{C}$ is the curve of $\rho = \theta^2$, $\pi \leq \theta \leq 2\pi$,
   (b) $g(\rho, 0) = 3\rho$ and $\mathscr{C}$ is the curve of $\rho = 0$, $0 \leq \theta \leq \pi$.

The procedure for finding the integral of $g(x, y, z)$ on a curve $\mathscr{C}$ in Cartesian space described by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$, $a \leq r \leq b$, follows the same pattern. The formula in A.2 then becomes

A.5 $$\int_{\mathscr{C}} g\, dL = \int_a^b g(\bar{x}, \bar{y}, \bar{z}) \sqrt{\left(\frac{d\bar{x}}{dr}\right)^2 + \left(\frac{d\bar{y}}{dr}\right)^2 + \left(\frac{d\bar{z}}{dr}\right)^2}\, dr.$$

5. Find $\int_{\mathscr{C}} g\, dV$ given that

   (a) $g(x, y, z) = xyz$ and $\mathscr{C}$ is represented by $r\mathbf{i} + 3r^2\mathbf{j} + 6r^3\mathbf{k}$, $0 \leq r \leq 1$,
   (b) $g(x, y, z) = \sqrt{x + z^2}$ and $\mathscr{C}$ is represented by $\mathbf{i} + r\mathbf{j} + e^r\mathbf{k}$, $0 \leq r \leq 1$.

The *arc length* of a curve $\mathscr{C}$ in the Cartesian plane or space is

A.6 $\quad L(\mathscr{C}) = \int_{\mathscr{C}} 1\, dL.$

6. Find the arc length of the curve represented by

 (a) $\mathbf{f}(r) = r^2\mathbf{i} + r^3\mathbf{k}$, $0 \le r \le 1$,
 (b) $\mathbf{f}(r) = r\mathbf{i} + 3r^2\mathbf{j} + 6r^3\mathbf{k}$, $0 \le r \le 1$.

7. Give as an integral the arc length of

 (a) the ellipse $x^2/16 + y^2/9 = 1$,
 (b) the graph of $y = xe^x$, $0 \le x \le 4$.

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8. Find $\int_{\mathscr{C}} g\, dL$ for each of the following sets of conditions.

 (a) $g(x, y) = y$ and $\mathscr{C}$ is the arc of $y = x^3$ having end points $(0, 0)$ and $(-1, -1)$.
 (b) $g(x, y) = xy$ and $\mathscr{C}$ is the line segment with end points $(0, 2)$ and $(3, 4)$.
 (c) $g(x, y) = xy$ and $\mathscr{C}$ is the third-quadrant portion of $x^2 + y^2 = 1$.
 (d) $g(\rho, \theta) = \rho \sin\theta$ and $\mathscr{C}$ is the arc of $\rho = \cos\theta$, where $\pi \le \theta \le 5\pi/4$.
 (e) $g(x, y, z) = 2x + 9z$ and $\mathscr{C}$ is represented by $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j} + r^3\mathbf{k}$, $0 \le r \le 1$.

9. Write as an integral the arc length of

 (a) the ellipse $x^2/1 + y^2/9 = 1$,
 (b) the curve $x = e^{2y}$, $0 \le y \le 3$.

B. *Computation of Integrals of Regions*

The procedure for evaluating the integral of $g(x, y)$ on a region $\mathscr{R}$ in the Cartesian plane proceeds in a similar manner to finding the integral on a curve.

B.1 (I) Obtain for $\mathscr{R}$ a representation vector function $\mathbf{f}(r, s)$, with associated rectangular region domain $\mathbf{I} = [a_1, b_1] \times [a_2, b_2]$.
 (II) Find $g \circ \mathbf{f}$.
 (III) Find the absolute determinant $|J_{\mathbf{f}}|$ of the Jacobian matrix of $\mathbf{f}$.
 (IV) Evaluate $\int_{\mathscr{R}} g\, dA = \int_{\mathbf{I}} (g \circ \mathbf{f})|J_{\mathbf{f}}|\, dA$.

If $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j}$, then B.1(IV) becomes

B.2 $$\int_{\mathscr{R}} g\, dA = \int_{a_1}^{b_1} dr \int_{a_2}^{b_2} g(\bar{x}, \bar{y})\,|\bar{x}_r\bar{y}_s - \bar{x}_s\bar{y}_r|\, ds.$$

1. Find $\int_{\mathscr{R}} g\, dA$ if $g(x, y) = x^2/y^2$ and $\mathscr{R}$ is represented by

$$\mathbf{f}(r, s) = rs\mathbf{i} + r\mathbf{j}, \qquad \mathbf{I} = [0, 1] \times [1, 2].$$

2. Find $\int_{\mathscr{R}} g\, dA$ for each condition indicated.
   (a) $g(x, y) = 2x + 1$ and $\mathscr{R}$ is the region bounded by $x^2 + y^2 = 4$.
   (b) $g(x, y) = 9x^2 + 4y^2$ and $\mathscr{R}$ is the region bounded by $x^2/4 + y^2/9 = 1$.

3. Find $\int_{\mathscr{R}} g\, dV$ for each case.
   (a) $g(x, y) = x + y$ and $\mathscr{R}$ is the parallelogram region with vertices $(1, 4)$, $(2,5)$, $(4, 3)$, and $(3, 2)$. Obtain the representation from the vector algebra form $\mathbf{u}_o + [r\mathbf{u}_1 + s\mathbf{u}_2]$.
   (b) $g(x, y) = 3x - 5y$ and $\mathscr{R}$ is the triangular region with vertices $(1, 5)$, $(2, 0)$, and $(7, 3)$. Obtain the representation from the vector algebra form $\mathbf{u}_o + [r\mathbf{u}_1 + rs\mathbf{u}_2]$.

In some cases a change of variable will produce a simpler formula for computing. If $\mathscr{R}$ is the region bounded by the graphs of $y = h_1(x)$, $y = h_2(x)$, $x = a$, and $x = b$ with $h_1(x) < h_2(x)$ for $a < x < b$, then B.1(IV) becomes

B.3 $$\int_{\mathscr{R}} g\, dA = \int_{a}^{b} dx \int_{h_1(x)}^{h_2(x)} g(x, y)\, dx.$$

4. Find $\int_{\mathscr{R}} g\, dA$ in both cases.
   (a) $g(x, y) = xy$ and $\mathscr{R}$ is the region bounded by $y = x$, $y = x^2 + 2$, $x = 0$, and $x = 1$.
   (b) $g(x, y) = x^2y$ and $\mathscr{R}$ is the region bounded by $y = x^2$ and $y = 1$.

Similarly if $\mathscr{R}$ is bounded by $x = h_1(y)$, $x = h_2(y)$, $y = a$, and $y = b$, where $h_1(y) < h_2(y)$ for $a < y < b$, then B.1(IV) becomes

B.4 $$\int_{\mathscr{R}} g\, dA = \int_{a}^{b} dy \int_{h_1(y)}^{h_2(y)} g(x, y)\, dx.$$

5. Find $\int_{\mathscr{R}} g\, dA$ if $g(x, y) = xy^2$ and $\mathscr{R}$ is the region bounded by $y = x^3$, $x = -3$, $y = 1$, $y = -1$.

Using polar coordinates, for a given $g(\rho, \theta)$ on the region $\mathscr{R}$ bounded by $\rho = h(\theta)$, $\theta = \theta_1$, and $\theta = \theta_2$ the formula in B.1(IV) becomes

B.5 $$\int_{\mathscr{R}} g\, dA = \int_{\theta_1}^{\theta_2} d\theta \int_{0}^{h(\theta)} g(\rho, \theta)\, \rho\, d\rho.$$

6. Find $\int_{\mathscr{R}} g\, dA$ given that

   (a) $g(\rho, \theta) = \rho\, \theta$ on the region bounded by $\rho = \theta$, $\theta = 0$, and $\theta = \pi/2$;
   (b) $g(\rho, \theta) = \cos\theta$ on the first-quadrant portion of the circle $x^2 + y^2 = 9$.

The *area* of a region $\mathscr{R}$ is given by

B.6 $$A(\mathscr{R}) = \int_{\mathscr{R}} 1\, dA.$$

7. Find $A(\mathscr{R})$ if $\mathscr{R}$ is bounded by

   (a) $y = x$, $y = x^4$,
   (b) $x = y^2$, $x = 4$,
   (c) $y = e^x$, $y = x$, $x = 0$, $x = 2$,
   (d) $\rho = \sqrt{\cos\theta}$, $\theta = 0$, $\theta = \pi/2$.

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8. Find $\int_{\mathscr{R}} g\, dA$ in each case below.

   (a) $g(x, y) = 3x - y$ and $\mathscr{R}$ is the triangle with vertices (1, 0), (2, 3), (1, 5).
   (b) $g(x, y) = xy^2$ and $\mathscr{R}$ is the right half-plane region bounded by $x^2 + y^2 = 1$ and $x = 0$.
   (c) $g(x, y) = x/y$ and $\mathscr{R}$ is the region bounded by $x = y$, $x = y^3$, $y = 1$, and $y = 2$.
   (d) $g(x, y) = x$ and $\mathscr{R}$ is the region bounded by $y = x$ and $y = x^2$.
   (e) $g(\rho, \theta) = \sin\theta$ and $\mathscr{R}$ is the region bounded by $\rho = \cos^2\theta$, $\theta = 0$, and $\theta = \pi/3$.

9. Find the area of

   (a) the region bounded by $y = e^x$, $y = e^{-x}$, $x = 1$,
   (b) the first-quadrant region bounded by $y = \sqrt{3}\, x$, $y = 0$, and $x^2/4 + y^2/9 = 1$,
   (c) the region bounded by $\rho = \sqrt{\sin\theta}$, $\theta = 0$, $\theta = \pi/2$.

C. *Computation of Integrals on Surfaces*

The evaluation of the integral of $g(x, y, z)$ on a surface $\mathscr{S}$ proceeds as with the integrals on curves and regions.

C.1 (I) Obtain for $\mathscr{S}$ a representation consisting of a vector function $\mathbf{f}(r, s)$ and an associated rectangular region domain $\mathbf{I}$.
(II) Find $g \circ \mathbf{f}$.
(III) Find the absolute determinant $|J_{\mathbf{f}}|$ of the Jacobian matrix of $\mathbf{f}$.
(IV) Evaluate $\int_{\mathscr{S}} g\, dA = \int_{\mathbf{I}} (g \circ \mathbf{f})\, |J_{\mathbf{f}}|\, dA$.

1. Find $\int_{\mathscr{S}} g\,dA$ if $g(x, y, z) = y(z - x)$ and $\mathscr{S}$ is represented by $\mathbf{f}(r, s) = (r - s)\mathbf{i} + r\mathbf{j} + (r + 5s)\mathbf{k}$, $\mathbf{I} = [0, 1] \times [0, 2]$.

2. Given $P = (2, 1, 5)$, $Q = (3, 2, 0)$, and $R = (7, 1, -2)$, find $\int_{\mathscr{S}} g\,dA$ where

   (a) $g(x, y, z) = x - y$ and $\mathscr{S}$ is the parallelogram surface with sides $\overline{PQ}$ and $\overline{QR}$,
   (b) $g(x, y, z) = x + 2y + z$ and $\mathscr{S}$ is the triangle surface with vertices P, Q, and R.

If $\mathscr{S}$ is the graph of $z = f(x, y)$ on a region $\mathscr{R}$ in the xy plane, then C.1(IV) becomes

C.2 $$\int_{\mathscr{S}} g\,dA = \int_{\mathscr{R}} g(x, y, f(x, y))\sqrt{1 + (f_x)^2 + (f_y)^2}\,dA.$$

3. Let $g(x, y, z) = x + y^2 + y - z$ be given on the surface $\mathscr{S}$ which is the graph of $z = f(x, y) = x + y^2$ on $[0, 1] \times [0, 2]$.

   (a) Find $g(x, y, f(x, y))$. This is obtained by substitution of $f(x, y)$ for z in the rule for g.
   (b) Find $\sqrt{1 + (f_x)^2 + (f_y)^2}$.
   (c) Find $\int_{\mathscr{R}} g(x, y, f(x, y))\sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$ by the methods of Problem Set B, or Problem Set A in chapter XIII.

4. Find $\int_{\mathscr{S}} g\,dA$ given that $g(x, y, z) = y + z$ and $\mathscr{S}$ is the graph of $z = 3x - y$ on the region bounded by $y = x^2$ and $y = x$.

Next, let $\mathscr{S}$ be the surface of revolution about the x axis of a curve $\mathscr{C}$ in the upper half of the xy plane. Then C.1(IV) yields the equality,

C.3 $$\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dL \int_0^{2\pi} yg(x, y\cos\phi, y\sin\phi)\,d\phi.$$

5. Let $g(x, y, z) = x$ on $\mathscr{S}$, where $\mathscr{S}$ is the surface of revolution about the x axis of the first quadrant portion of $x^2 + y^2 = 1$.

   (a) Find $g(x, y\cos\phi, y\sin\phi)$. This is obtained by substitution of $y\cos\phi$ for y and $y\sin\phi$ for z in the rule for g.
   (b) Find $\int_0^{2\pi} yg(x, y\cos\phi, y\sin\phi)\,d\phi$. Note that x and y are treated as constants, and the result is a function of the form $F(x, y)$.
   (c) Obtain from (b), by the methods of Problem Set A, the integral $\int_{\mathscr{C}} F(x, y)\,dL$.

6. Find $\int_{\mathscr{S}} g\,dA$ given that $g(x, y, z) = z^2/x$ and $\mathscr{S}$ is the surface of revolution about the x axis of $x = y^2$, $1 \le y \le 2$.

We now let $\mathscr{S}$ be the cylindrical surface having as its base the xy plane curve $\mathscr{C}$ and lying between the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $f_1(x, y) \le f_2(x, y)$ above $\mathscr{C}$. Then C.1(IV) becomes

C.4 $$\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dL \int_{f_1(x,y)}^{f_2(x,y)} g(x, y, z)\,dz.$$

7. Find $\int_{\mathscr{S}} g\,dA$ for each of the following conditions.

   (a) $g(x, y, z) = y/x$ and $\mathscr{S}$ is the cylindrical surface having the base $y = x^2$, $1 \le x \le 2$, and lying between $z = 3$ and $z = 4$.
   (b) $g(x, y, z) = y$ and $\mathscr{S}$ is the cylindrical surface having the base $x^2 + y^2 = 4$ and lying between $z = x - y$ and $z = x + y + 6$.

The *surface area* of a surface $\mathscr{S}$ is given by

C.5 $$A(\mathscr{S}) = \int_{\mathscr{S}} 1\,dA.$$

8. Find the surface area of

   (a) the graph of $z = 2x - y$ on the rectangle $[0, 2] \times [1, 2]$,
   (b) the graph of $z = x - y$ on the region bounded by $x^2 + y^2 = 1$,
   (c) the surface of revolution of $(x - 2)^2 + (y - 3)^2 = 1$ about the x axis,
   (d) the cylindrical surface having the base $x^2 + y^2 = 1$ and lying between $z = x - 3$ and $z = 2y + 4$.

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9. Find $\int_{\mathscr{S}} g\,dA$ for each case.
   (a) $g(x, y, z) = yz$ and $\mathscr{S}$ is the graph of $z = x - 2y$ having as its domain the parallelogram region $\mathscr{R}$ in the xy plane with vertices $(1, 1)$, $(1, 5)$, $(2, 2)$, and $(2, 6)$.
   (b) $g(x, y, z) = 2x - z$ and $\mathscr{S}$ is the triangular surface with vertices $(2, 0, 1)$, $(3, 1, 2)$, and $(5, 4, 6)$.
   (c) $g(x, y, z) = x + y + z$ and $\mathscr{S}$ is the surface of revolution of $y = x$, $0 \le x \le 1$ about the x axis.
   (d) $g(x, y, z) = \sqrt{64 - 3y^2}$ and $\mathscr{S}$ is the cylindrical surface which has as its base $x^2/4 + y^2/16 = 1$ and which lies between $z = 1$ and $z = 2$.
10. Find the surface area of $\mathscr{S}$ as described below.
   (a) $\mathscr{S}$ is the graph of $z = 4x + 2y$ on $[0, 1] \times [0, 2]$.
   (b) $\mathscr{S}$ is the surface of revolution of $y = x^3$, $0 \le x \le 1$, about the x axis.
   (c) $\mathscr{S}$ is the surface of revolution of $y = 3x$, $1 \le x \le 2$, about the x axis.
   (d) $\mathscr{S}$ is the cylindrical surface which has for a base $y = x^2$, $0 \le x \le 1$, and which lies between $z = 0$ and $z = x$.

D. *Computation of Integrals on Solids*

The evaluation of the integral of $g(x, y, z)$ on a solid $\mathscr{T}$ proceeds as with integrals on curves, regions, and surfaces.

D.1 (I) Obtain for $\mathscr{T}$ a representation consisting of a vector function $\mathbf{f}(r, s, t)$ and an associated solid rectangular parallelepiped domain $\mathbf{I}$.
(II) Find $g \circ \mathbf{f}$.
(III) Find the absolute determinant $|J_\mathbf{f}|$ of the Jacobian matrix of $\mathbf{f}$.
(IV) Evaluate $\int_\mathscr{T} g\, dV = \int_\mathbf{I} (g \circ \mathbf{f})\, |J_\mathbf{f}|\, dV$.

1. Find $\int_\mathscr{T} g\, dV$ given that
   (a) $g(x, y, z) = x + y + z$ and $\mathscr{T}$ is represented by
   $\mathbf{f}(r, s, t) = (2r - s + t)\mathbf{i} + (r + s - t)\mathbf{j} + 3s\mathbf{k}$, $[0, 1] \times [0, 1] \times [0, 2]$,
   (b) $g(x, y, z) = x + y$ and $\mathscr{T}$ is the parallelepiped with sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RS}$ where $P = (1, 0, 2)$, $Q = (2, -1, 3)$, $R = (3, 1, 7)$, and $S = (2, -1, 5)$.

If $\mathscr{T}$ is the solid of revolution about the x axis of the region $\mathscr{R}$ in the upper half of the xy plane, then D.1(IV) becomes

D.2 $$\int_\mathscr{T} g\, dV = \int_\mathscr{R} dA \int_0^{2\pi} y g(x, y\cos\phi, y\sin\phi)\, d\phi.$$

2. Find $\int_\mathscr{T} g\, dV$ for each of the following cases.
   (a) $g(x, y, z) = x^2$ and $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $y = x^2$, $y = 0$, and $x = 1$,
   (b) $g(x, y, z) = x(y^2 + z^2)$ and $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $x = 0$, $x = y^2$, $y = 1$, and $y = 2$.

If $\mathscr{T}$ is the solid cylinder having as its base the region $\mathscr{R}$ in the xy plane and lying between $z = f_1(x, y)$ and $z = f_2(x, y)$, where $f_1(x, y) \le f_2(x, y)$ on $\mathscr{R}$, then D.1(IV) becomes

D.3 $$\int_\mathscr{T} g\, dV = \int_\mathscr{R} dA \int_{f_1}^{f_2} g\, dz.$$

3. Find $\int_\mathscr{T} g\, dV$ given that
   (a) $g(x, y, z) = x^2 + y^2$ and $\mathscr{T}$ is the cylinder having as its base the disk $x^2 + y^2 \le 1$ and lying between the surfaces $z = x - y$ and $z = 3 + x$,
   (b) $g(x, y, z) = x + y$ and $\mathscr{T}$ is the cylinder having as its base the triangle with vertices $(1, 0)$, $(0, 1)$, $(1, 1)$ and lying between the surfaces $z = xy$ and $z = y(x + 1)$.

The *volume* of a solid $\mathscr{T}$ is

D.4 $$V(\mathscr{T}) = \int_\mathscr{T} 1\, dV.$$

Formulas D.2 and D.3 give, for $g = 1$,

D.5 $\quad V(\mathscr{T}) = 2\pi \int_{\mathscr{R}} y\, dA \qquad \mathscr{T}$ the solid of revolution of $\mathscr{R}$ about the x axis.

D.6 $\quad V(\mathscr{T}) = \int_{\mathscr{R}} (f_2 - f_1)\, dA \qquad \mathscr{T}$ the solid cylinder having base $\mathscr{R}$ and between f_1, f_2.

4. Find $V(\mathscr{T})$ for each case.
   (a) $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $x^2 + (y - 2)^2 = 1$.
   (b) $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $y = x^3$, $y = 0$, $x = 0$, and $x = 1$.
   (c) $\mathscr{T}$ is the cylinder having the base $[0, 1] \times [0, 1]$ and lying between the graphs of $z = 3x + 2y + 1$ and $z = 5x + 3y + 4$.

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5. Find $\int_{\mathscr{T}} g\, dV$ where
   (a) $g(x, y, z) = (3x + 2y)z$ and $\mathscr{T}$ is represented by
   $$(r - 2s)\mathbf{i} + 3s\mathbf{j} + (2 - t)\mathbf{k}, \quad [0, 2] \times [0, 1] \times [0, 1],$$
   (b) $g(x, y, z) = y^2 + z^2$ and $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$,
   (c) $g(x, y, z) = x$ and $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $x = \sqrt{y^2 + y}$, $x = 0$, $y = 0$, and $y = 1$,
   (d) $g(x, y, z) = y$ and $\mathscr{T}$ is the solid cylinder having as a base the region bounded by $y = x$ and $y = x^2$ and lying between the surfaces $z = x$ and $z = x + y$.

6. Find $V(\mathscr{T})$ given that
   (a) $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $x = y^6$, $x = 0$, $y = 0$, and $y = 1$,
   (b) $\mathscr{T}$ is the cylinder having as a base the region bounded by $x^2 + y^2 = 1$ and lying between $z = 0$ and $z = 2 - x$.

E. *Moments of Regions and Solids*

If $\mathscr{R}$ is a region in the Cartesian plane, then the moments of $\mathscr{R}$ about the x and y axes are respectively given by

E.1 $\quad M_x(\mathscr{R}) = \int_{\mathscr{R}} y\, dA,$

$\quad M_y(\mathscr{R}) = \int_{\mathscr{R}} x\, dA,$

These moments describe a physical tendency to produce rotation about the respective axes by a flat plate having density 1 and the shape of the region $\mathscr{R}$.

1. Find the moment of $\mathscr{R}$ about the x and y axes if

   (a) $\mathscr{R}$ is the rectangular region $[0, 1] \times [0, 2]$,
   (b) $\mathscr{R}$ is the triangular region with vertices $(1, 0)$, $(2, 3)$, $(-1, 5)$,
   (c) $\mathscr{R}$ is the upper half-plane portion of the disk $x^2 + y^2 \leq 1$.

If $A(\mathscr{R})$ denotes the area of $\mathscr{R}$, then the *centroid* of $\mathscr{R}$ is the point

E.2 $$\left\langle \frac{M_y(\mathscr{R})}{A(\mathscr{R})}, \frac{M_x(\mathscr{R})}{A(\mathscr{R})} \right\rangle.$$

The centroid gives the balance point of a homogeneous flat plate having the shape of $\mathscr{R}$.

2. Find the centroid of $\mathscr{R}$ given that

   (a) $\mathscr{R}$ is the triangular region with vertices $(0, 0)$, $(2, 0)$, $(0, 3)$,
   (b) $\mathscr{R}$ is the portion of the disk $x^2 + y^2 \leq 1$ between the polar angles $\theta = 0$ and $\theta = \pi/6$,
   (c) $\mathscr{R}$ is the region bounded by $y = x$ and $y = x^2$.

If $\mathscr{T}$ is a solid, then the moments about the xy, xz, and yz planes are, respectively,

E.3 $$M_{xy}(\mathscr{T}) = \int_{\mathscr{T}} z\, dV,$$
$$M_{xz}(\mathscr{T}) = \int_{\mathscr{T}} y\, dV,$$
$$M_{yz}(\mathscr{T}) = \int_{\mathscr{T}} x\, dV.$$

3. Find $M_{xy}(\mathscr{T})$, $M_{xz}(\mathscr{T})$, $M_{yz}(\mathscr{T})$ when

   (a) $\mathscr{T}$ is the solid parallelepiped $[0, 1] \times [1, 2] \times [2, 3]$,
   (b) $\mathscr{T}$ is the solid of revolution of the region bounded by $y = x$, $y = 0$, $x = 1$, and $x = 0$ about the x axis, and
   (c) $\mathscr{T}$ is the solid hemisphere $x^2 + y^2 + z^2 \leq 1$, $x \geq 0$. (*Note*: This is a solid of revolution.)

If $V(\mathscr{T})$ is the volume of a solid $\mathscr{T}$, then the *centroid* of $\mathscr{T}$ is the point,

E.4 $$\left\langle \frac{M_{yz}(\mathscr{T})}{V(\mathscr{T})}, \frac{M_{xz}(\mathscr{T})}{V(\mathscr{T})}, \frac{M_{xy}(\mathscr{T})}{V(\mathscr{T})} \right\rangle.$$

4. Find the centroid of $\mathscr{T}$ for each of the following cases.

   (a) $\mathscr{T}$ is the solid cone of revolution of the region bounded by $y = 2x$, $y = 0$, and $x = 1$ about the x axis.
   (b) $\mathscr{T}$ is the solid cylinder lying between $z = 0$ and $z = x + 3$ and having the base $x^2 + y^2 \leq 1$.

The *moments of inertia* of a region $\mathscr{R}$ and a solid $\mathscr{T}$ are given as follows:

E.5
$$I_x(\mathscr{R}) = \int_{\mathscr{R}} y^2\, dA,$$
$$I_y(\mathscr{R}) = \int_{\mathscr{R}} x^2\, dA,$$
$$I_{xy}(\mathscr{T}) = \int_{\mathscr{T}} z^2\, dV,$$
$$I_{xz}(\mathscr{T}) = \int_{\mathscr{T}} y^2\, dV,$$
$$I_{yz}(\mathscr{T}) = \int_{\mathscr{T}} x^2\, dV.$$

5. Find $I_x(\mathscr{R})$ and $I_y(\mathscr{R})$ given the following conditions.
   (a) $\mathscr{R}$ is the rectangular region $[0, 1] \times [0, 2]$.
   (b) $\mathscr{R}$ is the triangular region with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.
   (c) $\mathscr{R}$ is the portion of the disk $x^2 + y^2 \le 1$ in the right half-plane.

6. Find $I_{xy}(\mathscr{T})$, $I_{xz}(\mathscr{T})$, and $I_{yz}(\mathscr{T})$ given that
   (a) $\mathscr{T}$ is the solid cylinder which lies between $z = 0$ and $z = 1$ and has for its base $x^2 + y^2 \le 1$,
   (b) $\mathscr{T}$ is the solid of revolution of the region bounded by $y = x^3$, $y = 0$, and $x = 1$ about the x axis.

The radius of gyration of $\mathscr{R}$ about the x and y axes is given respectively by

E.6
$$\sqrt{\frac{I_x(\mathscr{R})}{A(\mathscr{R})}}, \qquad \sqrt{\frac{I_y(\mathscr{R})}{A(\mathscr{R})}}.$$

7. Find the radius of gyration about the x and y axes for each region below.
   (a) $\mathscr{R}$ is the rectangular region $[0, 1] \times [I, 2]$.
   (b) $\mathscr{R}$ is the portion of $x^2 + y^2 \le 1$ in the upper half-plane.

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8. Find $M_x(\mathscr{R})$, $M_y(\mathscr{R})$, $I_x(\mathscr{R})$, $I_y(\mathscr{R})$, the centroid, and the radius of gyration about the x axis when
   (a) $\mathscr{R}$ is the rectangular region $[1, 2] \times [3, 4]$,
   (b) $\mathscr{R}$ is the region bounded by $y = x$ and $y = x^3$.

9. Find $M_{xy}(\mathscr{T})$, $M_{xz}(\mathscr{T})$, $M_{yz}(\mathscr{T})$, $I_{xy}(\mathscr{T})$, $I_{xz}(\mathscr{T})$, $I_{yz}(\mathscr{T})$, and the centroid for the cases in which
   (a) $\mathscr{T}$ is the parallelepiped $\mathbf{j} + [r(\mathbf{i} - \mathbf{j}) + s\mathbf{k} + t\mathbf{j}]$, and
   (b) $\mathscr{T}$ is the solid of revolution about the x axis of the region bounded by $y = x$, $y = 0$, $x = 1$, and $x = 2$.

F. *Other Integrals*

We first consider integrals on domains which are made up of finitely many pieces of the types of domain previously studied. If no two of the finitely many pieces have interior points in common, then the integral on the total domain is the sum of the integrals on the separate pieces.

1. Let $\mathscr{C}$ consist of two line segments in the Cartesian plane, the first $\mathscr{C}_1$ with end points (0, 0) and (1, 1) and the second $\mathscr{C}_2$ with end points (1, 1) and (2, 0). If $g(x, y) = xy^2$, find
$$\int_{\mathscr{C}} g\,dL = \int_{\mathscr{C}_1} g\,dL + \int_{\mathscr{C}_2} g\,dL.$$

2. Find $\int_{\mathscr{C}} g\,dL$ if $g(x, y) = x^2y$ and $\mathscr{C}$ is the boundary (perimeter) of the rectangle $[0, 1] \times [0, 1]$.

3. Find $\int_{\mathscr{C}} g\,dL$ if $g(x, y) = x$ and $\mathscr{C}$ is the perimeter of the triangle with vertices (1, 1), (2, 1), (2, 2).

4. Find $\int_{\mathscr{R}} g\,dA$ if $g(x, y) = xy$ and $\mathscr{R}$ is the region bounded by the line segments successively joining the points (0, 0), (1, 0), (1, 1), (2, 1), (2, 2), (0, 2), (0, 0).

5. Find the moment about the x axis of the region bounded by $y = |x| - 1$ and $y = \sqrt{1 - x^2}$, $-1 \le x \le 1$.

We next consider integrals on domains which are limits of increasing curves or regions. In this case the integral on the limit set may be evaluated as a limit of integrals provided the limit exists, in which case the integral is said to *converge*; otherwise it *diverges*.

6. Given $\mathscr{R} = [0, 1] \times [1, \infty)$ and $g(x, y) = x/y^2$,

   (a) let $\mathscr{R}_Y = [0, 1] \times [1, Y]$ and find $\int_{\mathscr{R}_Y} g\,dA$,
   (b) evaluate $\int_{\mathscr{R}} g\,dA = \lim_{Y \to \infty} \int_{\mathscr{R}_Y} g\,dA$.

7. Define $\int_{\mathscr{R}} g\,dA$ as a limit, and evaluate the integral if it converges.

   (a) $g(x, y) = ye^{-x}$, $\mathscr{R} = [0, \infty) \times [1, 3]$,
   (b) $g(x, y) = xye^{-x^2y}$, $\mathscr{R} = (-\infty, 0) \times [1, 2]$.

8. Find $\int_{\mathscr{R}} g\,dA$ if $g(x, y) = xy^2$ and $\mathscr{R}$ is the region bounded by $x = 1$, $y = 0$, and $y = 1/x$.

9. Find $\int_{\mathscr{C}} g\,dL$ if

   (a) $g(x, y) = 1/(xy\sqrt{1 + 4y})$ and $\mathscr{C}$ is the curve $y = x^2$, $x \ge 1$,
   (b) $g(x, y) = 1/(x^2y^3)$ and $\mathscr{C}$ is the curve $y = x$, $x \ge 2$.

We next consider integrals on curves which are limits of a decreasing set of curves, along with regions which are limits of a decreasing set of regions. The previous method again applies.

10. Given $g(x, y) = y/\sqrt{x}$ on $\mathscr{R} = (0, 1] \times [1, 2]$,

    (a) let $\mathscr{R}_X = [X, 1] \times [1, 2]$ and find $\int_{\mathscr{R}_X} g\, dA$,
    (b) evaluate $\int_{\mathscr{R}} g\, dA = \lim_{X \to 0} \int_{\mathscr{R}_X} g\, dA$.

11. Define $\int_{\mathscr{R}} g\, dA$ as a limit and evaluate the integral if it converges.

    (a) $g(x, y) = (xy)^{-1/3}$, $\mathscr{R} = [1, 8] \times (0, 1]$.
    (b) $g(x, y) = (xy)^{-1/2}$, $\mathscr{R} = (0, 1] \times (0, 1]$.
    (c) $g(x, y) = (x^2 + y^2)^{-1/3}$ and $\mathscr{R}$ is the disk $x^2 + y^2 \leq 1$ with the origin deleted.

12. Find $\int_{\mathscr{C}} g\, dL$ if $g(x, y) = (x^2 + y^2)^{-1/3}$ and $\mathscr{C}$ is the line segment from $(0, 0)$ to $(1, 1)$ which includes $(1, 1)$ but not $(0, 0)$.

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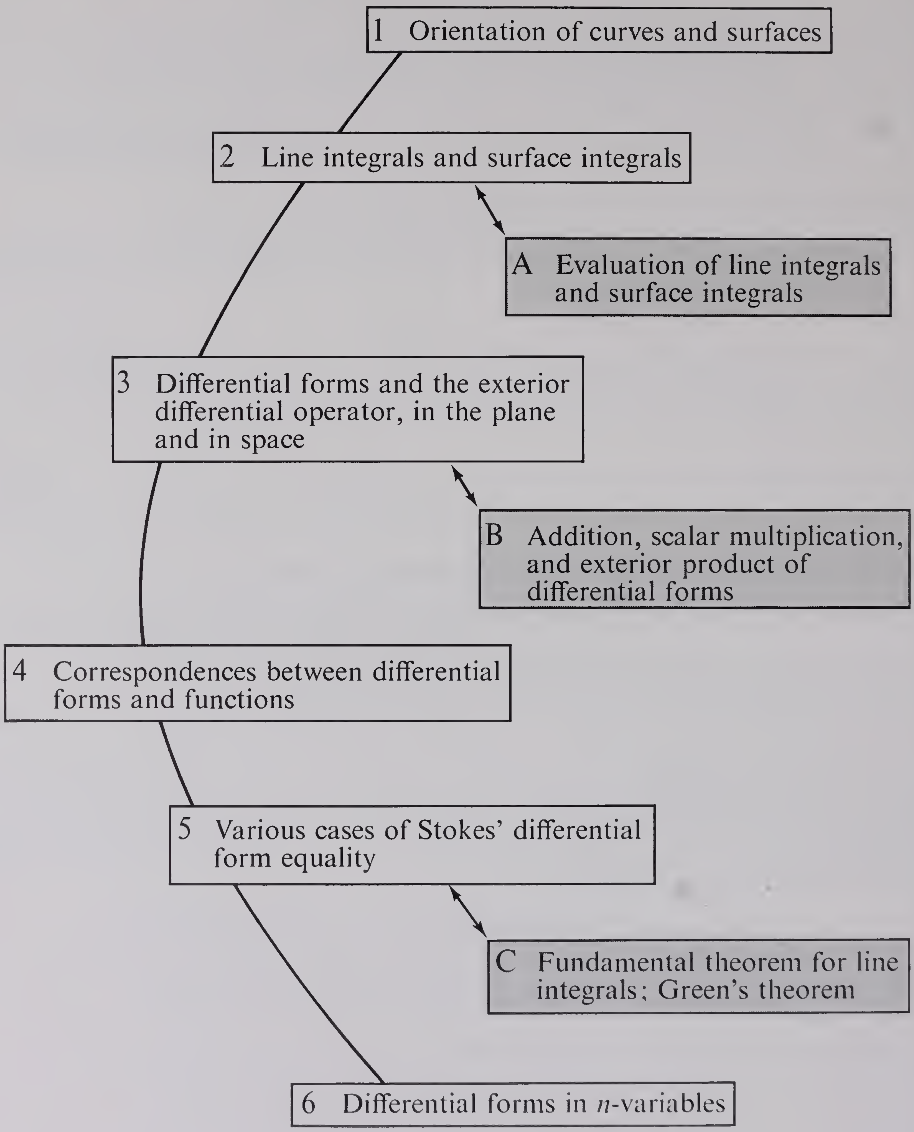
13. Evaluate $\int_{\mathscr{R}} g\, dA$ given the following conditions.

    (a) $g(x, y) = y$, $\mathscr{R}$ is the region bounded by line segments successively joining $(0, 0)$, $(2, 0)$, $(2, 1)$, $(1, 2)$, $(0, 1)$, and $(0, 0)$.
    (b) $g(x, y) = 1/(xy^2)$, $\mathscr{R} = [1, 2] \times [1, \infty)$.
    (c) $g(x, y) = 1\sqrt{x}$, $\mathscr{R}$ is the bounded region between $x = y^2$ and $x = 1$.
    (d) $g(x, y) = (x^2 + 4y^2)^{-1/3}$, $\mathscr{R}$ is the region $x^2/4 + y^2/1 \leq 1$ with the origin deleted.

14. Find $\int_{\mathscr{C}} g\, dL$ given

    (a) $g(x, y) = x + 2y$, $\mathscr{C}$ is the curve $y = |x|$, $-1 \leq x \leq 1$;
    (b) $g(x, y) = (x + y)^{-1/2}$, $\mathscr{C}$ is the curve $y = 2x$, $0 < x \leq 3$;
    (c) $g(x, y) = y/\sqrt{x^4 + 1}$, $\mathscr{C}$ is the curve $y = 1/x$, $x \geq 1$.


1 Orientation of curves and surfaces
2 Line integrals and surface integrals
A Evaluation of line integrals and surface integrals
3 Differential forms and the exterior differential operator, in the plane and in space
B Addition, scalar multiplication, and exterior product of differential forms
4 Correspondences between differential forms and functions
5 Various cases of Stokes' differential form equality
C Fundamental theorem for line integrals; Green's theorem
6 Differential forms in n-variables

XV

Vector Analysis and Differential Forms

The previous two chapters considered integrals of scalar functions. In this chapter we study integrals of vector functions. The 2– and 3–dimensional study of vector-function integrals found here is rooted in physics problems of work and fluid flow, and it forms a basic part of the subject area called *vector analysis*. Vector analysis assumed its present form in the 1880's in the work of the American mathematician J. W. Gibbs (1839–1903). Gibbs' work was not highly original, but was the culmination of a development which was begun in Ireland by Hamilton (1805–1865) and had received valuable contributions from many other mathematicians. An independent contemporary exposition of vector analysis was made in England by Oliver Heaviside (1850–1925), who received inspiration from the work in electromagnetic theory of James Clerk Maxwell (Scotch : 1831–1879).

The notation of vector analysis is limited to 3–dimensional space. It is a practical notation, but the concepts of vector analysis do not lend themselves easily to an elegant structure. With vector-analysis symbolism there is no natural extension to dimensions greater than three. Such an extension does exist, however, in the work of Grassman, which contains the basic structure of *differential*

forms. The theory of differential forms includes the integrals of vector functions and unifies many of the concepts of vector analysis. It also provides significant insight into various vector analysis relationships. For example, Green's theorem, Stokes' theorem, and the divergence theorem of vector analysis are all seen to be specific instances of a single theorem in the theory of differential forms.

We shall first study the integrals of vector functions. For this purpose it is necessary to introduce the idea of an *oriented set*. Then the basic algebra of differential forms is introduced and related to certain vector analysis concepts. Finally, various properties, including Stokes' equality for differential forms, are discussed and illustrated.

1. *Oriented Sets*

For many purposes it is desired to give direction to a smooth curve $\mathscr{C}$. For instance, if $\mathscr{C}$ represents the path of a moving particle, then there may be a need to distinguish mathematically the two directions of movement along $\mathscr{C}$. In physics or engineering applications this assignment of direction, called an *orientation*, is made by associating a *tangent vector* to each point of $\mathscr{C}$. Geometrically, a tangent vector at $\mathbf{u}_o$ on $\mathscr{C}$ is a directed segment originating at $\mathbf{u}_o$ and lying in the tangent line to $\mathscr{C}$ at $\mathbf{u}_o$ (see Figure 15.1). There are obviously two such choices of direction. We shall now observe that a representation for $\mathscr{C}$ gives an orientation to $\mathscr{C}$. If $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$, represents $\mathscr{C}$, then

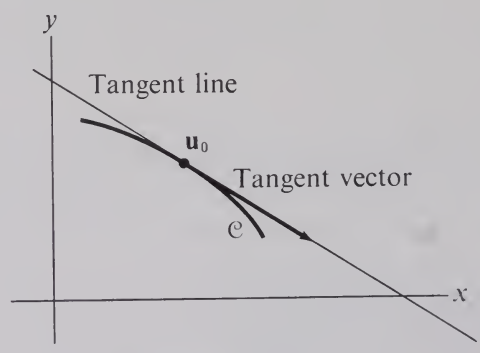


Figure 15.1

$$\frac{d\mathbf{f}}{dr}(r_o) = \frac{d\bar{x}}{dr}(r_o)\mathbf{i} + \frac{d\bar{y}}{dr}(r_o)\mathbf{j}$$

is a vector in the direction of the line tangent to $\mathscr{C}$ at $\mathbf{u}_o = \mathbf{f}(r_o)$.

Furthermore $d\mathbf{f}/dr(r_o)$ is in the direction of movement along $\mathscr{C}$ for increasing r. Since multiplication by a positive scalar preserves direction, we have the following definitions.

(I) *A* tangent vector *assigned by* $\mathbf{f}$ *to* $\mathscr{C}$ *at* $\mathbf{f}(r_o)$ *is any vector* $c\,d\mathbf{f}/dr(r_o)$ *where* $c > 0$.

(II) *Two representations of* $\mathscr{C}$ *have the* same orientation *if they assign the same tangent vectors at each point of* $\mathscr{C}$. *Otherwise they have* opposite orientation.

Example 1.1 The curve $\mathscr{C}$ described by $y = x^2$, $0 \le x \le 1$, is represented by

$$\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j}, \quad 0 \le r \le 1$$

and

$$\mathbf{f}'(r) = (1-r)\mathbf{i} + (1-r)^2\mathbf{j}, \quad 0 \le r \le 1.$$

From $d\mathbf{f}/dr = \mathbf{i} + 2r\mathbf{j}$ we see that a tangent vector of $\mathbf{f}$ at $\langle \frac{1}{2}, \frac{1}{4}\rangle$ is $\mathbf{i}+\mathbf{j}$. From $d\mathbf{f}'/dr = -\mathbf{i} - 2(1-r)\mathbf{j}$ we know that a tangent vector of $\mathbf{f}'$ at $\langle \frac{1}{2}, \frac{1}{4}\rangle$ is $-\mathbf{i}-\mathbf{j}$. Thus at $\langle \frac{1}{2}, \frac{1}{4}\rangle$ the tangent vectors assigned by $\mathbf{f}$ and $\mathbf{f}'$ have opposite direction.

The notions of tangent vector and same orientation apply equally well to closed curves and to curves in space. Some special conclusions can be made for simple closed curves in the plane. Direction of movement in this case can be distinguished by *clockwise* and *counterclockwise*. The concept of clockwise is intuitively simple, but difficult to formulate precisely for general curves. If $\mathbf{f}(r)$, $a \le r \le b$, represents a simple closed curve $\mathscr{C}$, then as r increases from a to b, its image $\mathbf{f}(r)$ has a net turn of one revolution in either a clockwise or a counterclockwise direction. A related criterion uses a theorem, called the *Jordan curve theorem*, which says that $\mathscr{C}$ divides the plane into two parts which we call the *inside* and *outside* of $\mathscr{C}$. Let $\mathbf{n}(\mathbf{u}_o)$ be the unit vector which initiates at $\mathbf{u}_o$ on $\mathscr{C}$ and extends towards the outside of $\mathscr{C}$ along the direction of the normal line to $\mathscr{C}$ at $\mathbf{u}_o$. It is geometrically evident that if $\mathbf{f}$ assigns counterclockwise direction to $\mathscr{C}$, then a tangent vector assigned by $\mathbf{f}$ has the direction of $\pi/2$ radians counterclockwise from $\mathbf{n}(\mathbf{u}_o)$. If $\mathbf{f}$ assigns clockwise direction, then the tangent vector is $\pi/2$ radians clockwise from $\mathbf{n}(\mathbf{u}_o)$ (see Figure 15.2).

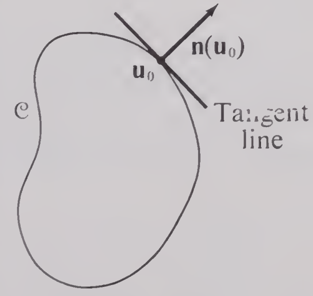


Figure 15.2

We next consider orientations of smooth surfaces. Such orientations might be used to distinguish fluid flow through the surface in the two different directions. Whereas tangent vectors are used to characterize orientations of curves, for surfaces we use *normal vectors.* If $\mathbf{f}(r, s)$, $\langle r, s\rangle$ in $\mathbf{I}^2$, represents a surface $\mathscr{S}$, then the normal line to $\mathscr{S}$ at $\mathbf{u}_o = \mathbf{f}(r_o, s_o)$ has a direction vector $\mathbf{f}_r \times \mathbf{f}_s(r_o)$. Proceeding similarly to the curve analysis, we have the following definitions.

A normal vector *assigned by* $\mathbf{f}$ *to* $\mathscr{S}$ *at* $\mathbf{f}(r_o, s_0)$ *is any vector* $\mathbf{f}_r \times \mathbf{f}_s(r_o)$ *where* $c > 0$.

Two representations of $\mathscr{S}$ *have the* same orientation *if they assign the same normal vectors at each point of* $\mathscr{S}$; *otherwise they have* opposite orientation.

Example 1.2 The surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$, is represented by

$$\mathbf{f}(\phi, \theta) = \cos\phi\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} + \sin\phi\sin\theta\,\mathbf{k}, \qquad 0 \leq \phi \leq \pi,\quad 0 \leq \theta \leq 2\pi,$$

and

$$\mathbf{f}'(\phi, \theta) = \cos\phi\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\theta\,\mathbf{k}, \qquad 0 \leq \phi \leq \pi,\quad 0 \leq \theta \leq 2\pi.$$

It may be verified that $\mathbf{f}(\pi/2, \pi/4) = \mathbf{f}'(\pi/2, \pi/4)$. From

$$\mathbf{f}_\phi \times \mathbf{f}_\theta\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = -\mathbf{i} \times \left(-\frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\right) = \frac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k})$$

and

$$\mathbf{f}'_\phi \times \mathbf{f}'_\theta\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = -\mathbf{i} \times \left(\frac{\sqrt{2}}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}\right) = -\frac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k}),$$

we see that $\mathbf{f}$ and $\mathbf{f}'$ assign opposite orientations.

A *closed surface* is a surface with no boundary. An example of a closed surface is a sphere. Smooth closed surfaces divide Cartesian space into two parts, the inside and outside. The *unit normal* assigned by a representing function $\mathbf{f}$ to a surface $\mathscr{S}$ (not necessarily closed) at $\mathbf{f}(r_o, s_o) = \mathbf{u}_o$ is

$$\mathbf{n}(\mathbf{u}_o) = \frac{\mathbf{f}_r \times \mathbf{f}_s(r_o, s_o)}{|\mathbf{f}_r \times \mathbf{f}_s(r_o, s_o)|}.$$

If $\mathscr{S}$ is closed, then $\mathbf{n}(\mathbf{u}_o)$ is directed to either the inside or the outside of $\mathscr{S}$. Thus $\mathbf{f}$ can be said to direct the unit normal inward or outward.

For the development of our theory it is also necessary to consider orientations of regions and solids. If $\{\mathbf{f}, \mathbf{I}^2\}$ represents a region $\mathscr{R}$, then $\mathbf{f}$ is said to assign *positive orientation* to $\mathscr{R}$ provided that $\det J_{\mathbf{f}} > 0$ on the interior of $\mathbf{I}^2$. If $\det J_{\mathbf{f}} < 0$, then $\mathbf{f}$ assigns *negative orientation.* Two representations of $\mathscr{R}$ assign the *same orientation* if they both assign positive or both assign negative orientations. The same definitions apply for representations of solids.

Example 1.3 Given the representation $\mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, it is seen from $\det J_{\mathbf{f}} = r > 0$ on the interior that $\mathbf{f}$ assigns positive orientation.

Questions

1. The mathematical term for assigned direction is ______.
2. The two directions of movement about a closed curve are ______ and ______.
3. Curves are directed by tangent vectors, whereas surfaces are directed by ______ vectors.
4. If $\mathbf{f}(r, s)$ is a representing function for $\mathscr{S}$, then a normal vector at $\mathbf{f}(r_o, s_o)$ is ______.
5. A representing function $\mathbf{f}$ of a region $\mathscr{S}$ assigns positive orientation provided ______ > 0.

Exercises

1. Determine whether the following pairs of representations assign the same or opposite orientations.
   (a) $\mathbf{f}(r) = r\mathbf{i} + 3r\mathbf{j}$, $0 \le r \le 1$;
   $\mathbf{f}'(r) = -r\mathbf{i} - 3r\mathbf{j}$, $-1 \le r \le 0$.
   (b) $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$;
   $\mathbf{f}'(\theta) = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$.
   (c) $\mathbf{f}(r, s) = (r - s)\mathbf{i} + (r + 2s)\mathbf{j} + (s - r)\mathbf{k}$, $[0, 1] \times [0, 1]$;
   $\mathbf{f}'(r, s) = (s - r)\mathbf{i} + (3 - r - 2s)\mathbf{j} + (r - s)\mathbf{k}$, $[0, 1] \times [0, 1]$.
2. Determine whether the following representations assign positive or negative orientation.
   (a) $\mathbf{f}(r, \theta) = r\theta^2\cos\theta\mathbf{i} + r\theta^2\sin\theta\mathbf{j}$, $[0, 1] \times [0, \pi/2]$;
   (b) $\mathbf{f}(r, s, t) = (r - 2s)\mathbf{i} + (s + t)\mathbf{j} + (t - 4r)\mathbf{k}$, $[0, 1] \times [0, 1] \times [0, 1]$.
3. Given that $\mathbf{f}(r)$, $a \le r \le b$ and $\mathbf{f}'(r) = \mathbf{f}(-r)$, $-b \le r \le -a$, assign opposite orientations of a curve $\mathscr{C}$ (see Proofs, exercise 1), for each of the following find a representation which gives opposite orientation.
   (a) $\mathbf{f}(r) = r^2\mathbf{i} + r\mathbf{j}$, $1 \le r \le 2$,
   (b) $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} - 3\sin\theta\,\mathbf{j}$, $0 \le \theta \le \pi$.

Proofs

1. Prove that $\mathbf{f}(r)$, $a \le r \le b$, and $\mathbf{f}'(r) = \mathbf{f}(-r)$, $-b \le r \le -a$, give opposite orientations to the same curve. (*Hint*: $\mathbf{f}' = h \circ \mathbf{f}$ where $h(r) = -r$, $a \le r \le b$.)

2. Prove that
$$\mathbf{f}(r, s),\ [a_1, b_1] \times [a_2, b_2],$$
and
$$\mathbf{f}'(r, s) = \mathbf{f}(-r, s),\ [-b_1, -a_1] \times [a_2, b_2],$$
give opposite orientations to the same surface.

3. Let $\mathbf{h}$ be an injective, linear function from $\mathbf{R}^2$ to $\mathbf{R}^2$ and
$$\mathscr{C} = \{\langle x, y\rangle : x^2 + y^2 = 1\}.$$
Prove that if $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$ represents $\mathscr{C}$, then $\mathbf{h} \circ \mathbf{f}$ assigns counterclockwise orientation to $\mathbf{h}(\mathscr{C})$ if and only if $\det J_{\mathbf{h}} > 0$. (*Hint*: Let $A(\theta)$ be the polar angle of $\mathbf{h} \circ \mathbf{f}(\theta)$; then movement about $\mathbf{h}(\mathscr{C})$ is counterclockwise provided $dA/d\theta > 0$.)

2. *Line Integrals and Surface Integrals*

In this section we define the integral of vector functions on oriented curves and surfaces. These integrals have important applications in physics and engineering. The line integral definition is motivated by a problem in work, and the surface integral definition by a problem in fluid flow.

A smooth *oriented curve* $\mathscr{C}^\circ$ is a curve with a representation $\{\mathbf{f}, \mathbf{I}\}$, which gives it an assigned orientation. The necessity for the orientation condition will be seen later, when it is observed that reversing orientation causes the integrals of this section to be changed by a (-1) factor (see Proofs, exercises 1,2). We consider a particle moving along a oriented plane curve $\mathscr{C}^\circ$ and acted upon by a force field
$$\mathbf{g}(x, y) = \bar{g}_1(x, y)\mathbf{i} + \bar{g}_2(x, y)\mathbf{j}.$$
Let $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$, represent $\mathscr{C}^\circ$ and $p = [r_0, r_1, \ldots, r_k]$ be a partition of $[a, b]$. As the particle moves from $\mathbf{f}(r_{j-1})$ to $\mathbf{f}(r_j)$, the work W_j done on the particle by the force field $\mathbf{g}$ is approximately the dot product of the force vector $\mathbf{g}(\mathbf{f}(r_j))$ and the displacement vector $\mathbf{f}(r_j) - \mathbf{f}(r_{j-1})$. Thus, (see Figure 15.3)
$$W_j \approx \bar{g}_1 \circ \mathbf{f}(r_j)(\bar{x}(r_j) - \bar{x}(r_{j-1})) + \bar{g}_2 \circ \mathbf{f}(r_j)(\bar{y}(r_j) - \bar{y}(r_{j-1})).$$

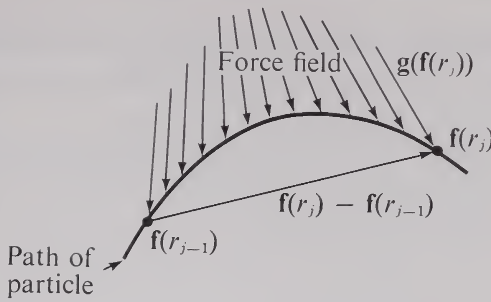


Figure 15.3

By the mean value theorem there exist r_j' and r_j'' in $[r_{j-1}, r_j]$ such that

$$\bar{x}(r_j) - \bar{x}(r_{j-1}) = \frac{d\bar{x}}{dr}(r_j')(r_j - r_{j-1}),$$

$$\bar{y}(r_j) - \bar{y}(r_{j-1}) = \frac{d\bar{y}}{dr}(r_j'')(r_j - r_{j-1}).$$

Substitution gives

$$W_j \approx \left[\bar{g}_1 \circ \mathbf{f}(r_j)\frac{d\bar{x}}{dr}(r_j') + \bar{g}_2 \circ \mathbf{f}(r_j)\frac{d\bar{y}}{dr}(r_j'')\right](r_j - r_{j-1}).$$

Summing over p gives a sum lying between the upper and lower Riemann sums of

$$(\bar{g}_1 \circ \mathbf{f})\frac{d\bar{x}}{dr} + (\bar{g}_2 \circ \mathbf{f})\frac{d\bar{y}}{dr}.$$

This suggests the following definition formula for the *line integral* of a continuous function $\mathbf{g} = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j}$ on $\mathscr{C}^\circ$.

Definition of a Line Integral (Plane)

$$\begin{aligned}\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f} &= \int_a^b (\mathbf{g} \circ \mathbf{f}) \cdot \frac{d\mathbf{f}}{dr}\, dr, \\ &= \int_a^b \left[(\bar{g}_1 \circ \mathbf{f})\frac{d\bar{x}}{dr} + (\bar{g}_2 \circ \mathbf{f})\frac{d\bar{y}}{dr}\right] dr.\end{aligned}$$

If $\mathbf{g}$ is a force field, this integral gives the work done by $\mathbf{g}$.

Example 2.1 Let $\mathscr{C}^\circ$ be the line segment oriented from $\langle 0, 0\rangle$ to $\langle 1, 1\rangle$, and $\mathbf{g}(x, y) = xy^2\mathbf{i} + (x - 2y)\mathbf{j}$ on $\mathscr{C}^\circ$. Then $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = r\mathbf{i} + r\mathbf{j}$, $0 \le r \le 1$, and therefore,

$$\mathbf{g} \circ \mathbf{f} = r^3\mathbf{i} - r\mathbf{j} \quad \text{and} \quad \frac{d\mathbf{f}}{dr} = \mathbf{i} + \mathbf{j}.$$

Thus,

$$\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f} = \int_0^1 (r^3\mathbf{i} - r\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})dr = \int_0^1 (r^3 - r)dr = -\frac{1}{4}.$$

The defining formula for a line integral in the plane is also valid for the integral of a vector function $\mathbf{g} = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}$ on an oriented curve $\mathscr{C}^\circ$ in space represented by $\{\mathbf{f}, [a, b]\}$.

An *oriented surface* $\mathscr{S}^\circ$ is a surface with a representation $\{\mathbf{f}(r, s), \mathbf{I}^2\}$, which gives it an assigned orientation. We suppose a fluid flows through an oriented surface $\mathscr{S}^\circ$ and has movement described by the velocity vector

$$\mathbf{g}(x, y, z) = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}.$$

Let $p = \{\mathbf{I}_1{}^2, \ldots, \mathbf{I}_k{}^2\}$ be a partition of $\mathbf{I}^2$, $\mathscr{S}_j = \mathbf{f}(\mathbf{I}_j{}^2)$, and $\mathbf{u}_j$ be in $\mathbf{I}_j{}^2$, $j = 1, \ldots, k$. The rate of flow (flow volume per unit time) F_j of the fluid through $\mathscr{S}_j$ is the component of fluid flow perpendicular to $\mathscr{S}_j$. Since this component can be expressed as the dot product of the velocity vector $\mathbf{g}(\mathbf{f}(\mathbf{u}_j))$ and the unit normal $\mathbf{n}(\mathbf{f}(\mathbf{u}_j))$ to $\mathscr{S}^\circ$ (see Figure 15.4), we have

$$\begin{aligned} F_j &\approx \left[\mathbf{g}\circ\mathbf{f}(\mathbf{u}_j)\cdot\frac{\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)}{|\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)|}\right] A(\mathscr{S}_j), \\ &\approx \left[\mathbf{g}\circ\mathbf{f}(\mathbf{u}_j)\cdot\frac{\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)}{|\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)|}\right] |J_{\mathbf{f}}(\mathbf{u}_j)| A(\mathbf{I}_j{}^2) \\ &= [\mathbf{g}\circ\mathbf{f}(\mathbf{u}_j)\cdot\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)] A(\mathbf{I}_j{}^2), \end{aligned}$$

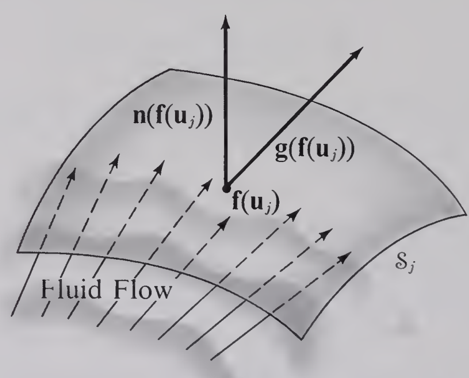


Figure 15.4

since $|J_{\mathbf{f}}(\mathbf{u}_j)| = |\mathbf{f}_r \times \mathbf{f}_s(\mathbf{u}_j)|$ (see Proposition 4.2 in Chapter VII). This suggests defining the *surface integral* of $\mathbf{g}$ on $\mathscr{S}^\circ$ as follows.

Definition of a Surface Integral

$$\int_{\mathscr{S}^\circ} \mathbf{g}\cdot\mathbf{n}\, dA = \int_{\mathbf{I}^2} (\mathbf{g}\circ\mathbf{f})\cdot(\mathbf{f}_r \times \mathbf{f}_s)\, dA.$$

Applying the definition of $\mathbf{f}_r \times \mathbf{f}_s$, the right side of the above equality may be written when $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}$, as

$$\int_{\mathbf{I}^2} \left[(\bar{g}_1 \circ \mathbf{f}) \det \begin{bmatrix} \bar{y}_r & \bar{y}_s \\ \bar{z}_r & \bar{z}_s \end{bmatrix} + (\bar{g}_2 \circ \mathbf{f}) \det \begin{bmatrix} \bar{z}_r & \bar{z}_s \\ \bar{x}_r & \bar{x}_s \end{bmatrix} \right.$$
$$\left. + (\bar{g}_3 \circ \mathbf{f}) \det \begin{bmatrix} \bar{x}_r & \bar{x}_s \\ \bar{y}_r & \bar{y}_s \end{bmatrix} \right] dA.$$

Example 2.2 Let $\mathbf{g} = xy\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$ on the surface $\mathscr{S}°$ represented by

$$\mathbf{f}(r, s) = (r + s)\mathbf{i} + rs\mathbf{j} + s^2\mathbf{k}, \qquad 0 \leq r, s \leq 1.$$

Then

$$\mathbf{g} \circ \mathbf{f} = rs(r + s)\mathbf{i} + rs^3\mathbf{j} + r^2 s^2\mathbf{k}$$

and

$$\mathbf{f}_r \times \mathbf{f}_s = (\mathbf{i} + s\mathbf{j}) \times (\mathbf{i} + r\mathbf{j} + 2s\mathbf{k}) = 2s^2\mathbf{i} - 2s\mathbf{j} + (r - s)\mathbf{k}.$$

Therefore,

$$\int_{\mathscr{S}°} \mathbf{g} \cdot \mathbf{n}\, dA = \int_0^1 dr \int_0^1 (\mathbf{g} \circ \mathbf{f}) \cdot (\mathbf{f}_r \times \mathbf{f}_s) ds$$
$$= \int_0^1 dr \int_0^1 [2rs^3 (r + s) - 2rs^4 + r^2 s^2 (r - s)] ds = \frac{1}{6}.$$

In addition to the line and surface integrals we shall also have need for the integral of a scalar function on an oriented region or solid. Given $g(x, y)$ on a region $\mathscr{R}^0$ oriented by $\{\mathbf{f}, \mathbf{I}^2\}$, then by definition

$$\int_{\mathscr{R}^0} g\, dA = \int_{\mathbf{I}^2} (g \circ \mathbf{f}) \det J_{\mathbf{f}}\, dA.$$

This integral agrees with our previously defined integral on a region provided $\mathbf{f}$ assigns positive orientation to $\mathscr{R}°$; otherwise it differs by a (-1) factor. Similarly, given $g(x, y, z)$ on a solid $\mathscr{T}°$ oriented by $\{\mathbf{f}, \mathbf{I}^3\}$, then

$$\int_{\mathscr{T}°} g\, dV = \int_{\mathbf{I}^3} (g \circ \mathbf{f}) \det J_{\mathbf{f}}\, dV.$$

Questions

1. The line integral of a force field gives the ______ done by the force.
2. If $\{\mathbf{f}, \mathbf{I}^2\}$ represents an unoriented region $\mathscr{R}$ and an oriented region $\mathscr{R}°$, then $\int_{\mathscr{R}°} g\, dA = \int_{\mathscr{R}} g\, dA$ provided ______.

Problems

1. Do Problem Set A at the end of the chapter.

Exercises

1. Let $g(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ on the portion $\mathscr{C}^\circ$ of the parabola $y = x^2$ with end points $(-1, 1)$ and $(0, 0)$.
   (a) Compute $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ if $\mathbf{f}$ orients $\mathscr{C}^\circ$ from $(-1, 1)$ to $(0, 0)$.
   (b) Compute $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ if $\mathbf{f}$ orients $\mathscr{C}^\circ$ from $(0, 0)$ to $(-1, 1)$.
2. Find $\int_{\mathscr{S}^\circ} \mathbf{g} \cdot \mathbf{n}\, dA$ if $\mathbf{g}(x, y, z) = x\mathbf{i} - (y - 1)\mathbf{j} + 2z\mathbf{k}$ and $\mathscr{S}^\circ$ is the triangular surface with vertices $(2, 1, 0)$, $(0, 1, 0)$, $(0, 0, 3)$ and is directed away from the origin. (You must choose $\mathbf{f}$ so that $\mathbf{f}_r \times \mathbf{f}_s$ is directed away from the origin.)
3. Find $\int_{\mathscr{S}^\circ} \mathbf{g} \cdot \mathbf{n}\, dA$ if $\mathbf{g}(x, y, z) = x\mathbf{i} + (y - z)\mathbf{j} + x\mathbf{k}$ and $\mathscr{S}^\circ$ is the cylindrical surface $x^2 + y^2 = 1$, $0 \leq z \leq 2$ with outwardly directed normal.
4. Find $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ if $\mathbf{g}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ and $\mathscr{C}^\circ$ is the perimeter of the rectangle $[0, 1] \times [0, 1]$ oriented counterclockwise. (Proceed as with integrals of scalar functions in Section 7 of Chapter XIV.)
5. Find $\int_{\mathscr{S}^\circ} \mathbf{g} \cdot \mathbf{n}\, dA$ if $\mathbf{g}(x, y, z) = x^2\mathbf{i} - z\mathbf{k}$ and $\mathscr{S}^\circ$ is the surface of the cube $[0, 1] \times [0, 1] \times [0, 1]$ with outwardly directed normal.

Proofs

1. Let $\mathscr{C}^\circ$ be represented by $\mathbf{f}(r)$, $a \leq r \leq b$, and $(\mathscr{C}^\circ)'$ by $\mathbf{f}'(r) = \mathbf{f}(-r)$, $-b \leq r \leq -a$. Prove $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f} = -\int_{(\mathscr{C}^\circ)'} \mathbf{g} \cdot d\mathbf{f}'$. Thus observe that opposite orientations give integrals which differ by a (-1) factor.
2. Let $\mathscr{S}^\circ$ be represented by $\mathbf{f}(r, s)$, $[a_1, b_1] \times [a_2, b_2]$ and $(\mathscr{S}^\circ)'$ by $\mathbf{f}'(r, s) = \mathbf{f}(-r, s)$, $[-b_1, -a_1] \times [a_2, b_2]$. Prove
$$\int_{\mathscr{S}^\circ} \mathbf{g} \cdot \mathbf{n}\, dA = -\int_{(\mathscr{S}^\circ)'} \mathbf{g} \cdot \mathbf{n}\, dA.$$
   Thus observe that opposite orientations of a surface give integrals which differ by a (-1) factor.

3. *Differential Forms* ($\mathbf{R}^2$, $\mathbf{R}^3$)

Before proceeding with the properties of the line and surface integrals, we shall study the algebra of differential forms. A differential form can be defined in a variety of ways, each involving a complicated mathematical

structure. We shall not attempt to say what a differential form is, however, but merely regard it as a collection of symbols that represents some kind of function. Operations which yield vector spaces with product operations will be introduced; these give mathematical systems called *algebras.*

Differential forms in $\mathbf{R}^2$ are symbolized

(a) g, 0-form,
(b) $g_1\,dx + g_2\,dy$, 1-form,
(c) $g\,dxdy$, 2-form,

where g, g_1, and g_2 are real functions of class C^∞ having as their domain an open set in $\mathbf{R}^2$. Differential forms in $\mathbf{R}^3$ are symbolized

(a) g, 0-form,
(b) $g_1\,dx + g_2\,dy + g_3\,dz$, 1-form,
(c) $g_1\,dydz + g_2\,dzdx + g_3\,dxdy$, 2-form,
(d) $g\,dxdydz$, 3-form,

where g, g_1, g_2, and g_3 are of class C^∞ on an open set in $\mathbf{R}^3$. Zero forms are symbolized and manipulated as functions, although they are conceptually different. The scalar multiple of a k-form and the addition of two k-forms are defined in a natural way. Particular instances are

(I) (a) $c(g_1dx + g_2dy) = cg_1dx + cg_2dy$,
(b) $(g_1dx + g_2\,dy) + (g_1'dx + g_2'dy) = (g_1 + g_1')dx + (g_2 + g_2')dy$,
(c) $c(g\,dxdy) = cg\,dxdy$,
(d) $g\,dxdy + g'dxdy = (g + g')dxdy$.

Example 3.1

$$(x^2dx + xy\,dy) + 3(y\,dx + 2xy\,dy) = (x^2 + 3y)dx + 7xy\,dy.$$

These operations give as a vector space the set of all k-forms with k fixed and having domain a given open set in $\mathbf{R}^2$ (or $\mathbf{R}^3$) (see Proofs, exercise 1). The zero vector is $g = 0$ for 0-forms, $0\,dx + 0\,dy$ for 1-forms in $\mathbf{R}^2$, $0\,dxdy$ for 2-forms in $\mathbf{R}^2$, and so forth.

There is a third operation on differential forms called the *product* or *exterior product*, and denoted $(\cdot)$. Before describing this operation, it is convenient to introduce some notation and a definition. Let $dx = 1\,dx$, and $dxdy = 1\,dxdy$, and so forth. An *elementary k-form* is a k-form with at most one nonzero coefficient function. Every k-form is a sum of elementary k-forms.

Example 3.2 The 1-form $x^2dx + 3xy\,dy + xyz\,dz$ is the sum of the three elementary 1-forms x^2dx, $3xy\,dy$, and $xyz\,dz$.

The exterior product is defined by the following characterizing properties.

(II) (a) If ω_1 and ω_2 are elementary forms, then $\omega_1 \cdot \omega_2 = 0$ provided dx, dy, or dz occurs in both ω_1 and ω_2,
(b) $dx \cdot dy = - dy \cdot dx = dxdy$,
$dz \cdot dx = - dx \cdot dz = dzdx$,
$dy \cdot dz = - dz \cdot dy = dydz$,
(c) $g \cdot g' = gg'$,
(d) $g \cdot dx = dx \cdot g = g\, dx$ (similarly for $g \cdot dy$, $g \cdot dz$),
(e) $g \cdot dydz = dydz \cdot g = g\, dydz$ (similarly for $g \cdot dzdx$, $g \cdot dxdy$),
(f) $g \cdot dxdydz = dxdydz \cdot g = g\, dxdydz$,
(g) the exterior product is associative, and
(h) it is also distributive about addition.

From (II) it is seen that $(\cdot)$ satisfies the usual laws of a product except that repetition of a dx, dy, or dz produces 0, whereas reversal of order introduces a (-1) factor. The product of dx, dy, and dz in any order is $dxdydz$ or $-dxdydz$ according to whether the final order can be obtained from the natural order by an even or odd number of interchanges. Thus

$$dx \cdot dy \cdot dz = dy \cdot dz \cdot dx = dz \cdot dx \cdot dy = dxdydz,$$

while

$$dx \cdot dz \cdot dy = dz \cdot dy \cdot dx = dy \cdot dx \cdot dz = - dxdydz.$$

Example 3.3
(a) $g \cdot (g_1{}'dx + g_2{}'dy + g_3{}'dz) = gg_1{}'dx + gg_2{}'dy + gg_3{}'dz$,
(b) $(xy^2dx + x\, dy) \cdot (y^3dx - y^2dy)$
$= (xy^2)y^3dx \cdot dx - (xy^2)y^2dx \cdot dy + xy^3dy \cdot dx - xy^2dy \cdot dy$
$= 0 - xy^4dxdy - xy^3dxdy + 0 = -xy^3(y+1)dxdy$,
(c) $(x\, dx - dz) \cdot (y\, dydz + x^3dzdx + xz\, dxdy)$
$= xy(dx \cdot dydz) - xz(dz \cdot dxdy) = x(y - z)dxdydz$.

A basic property of the exterior product is

Proposition 3.1 If ω is a k-form and ω' an l-form, then $\omega \cdot \omega' = (-1)^{kl}\omega' \cdot \omega$.

A proof may be obtained by considering various cases. For instance, if $\omega = g\, dx$ and $\omega' = g'dydz$, then

$$\omega \cdot \omega' = g\, dx \cdot g'dydz = gg' \cdot dx \cdot dy \cdot dz = g'dydz \cdot g\, dx = \omega' \cdot \omega$$

as desired, since $kl = 1(2) = 2$ in this case. If $\omega = g\,dx$ and $\omega' = g'\,dy$, then

$$\begin{aligned}\omega \cdot \omega' &= g\,dx \cdot g'dy = gg' \cdot dx \cdot dy = -gg' \cdot dy \cdot dx\\ &= -g'dy \cdot g\,dx = -\omega' \cdot \omega\end{aligned}$$

as desired, since $kl = 1(1) = 1$ in this case.

We now introduce on the system of differential k-forms an operator D which behaves as the derivative operator. In fact, with our structure more thoroughly defined, it becomes an extension of the derivative. It is called the (*exterior*) *differential* and in $\mathbf{R}^2$ is defined by the properties

(III) (a) $Dg = g_x dx + g_y dy$,
(b) $D(g_1 dx + g_2\,dy) = Dg_1 \cdot dx + Dg_2 \cdot dy$,
(c) $D(g\,dxdy) = 0$,

and in $\mathbf{R}^3$ by the properties

(IV) (a) $Dg = g_x dx + g_y dy + g_z dz$,
(b) $D(g_1 dx + g_2 dy + g_3\,dz) = Dg_1 \cdot dx + Dg_2 \cdot dy + Dg_3 \cdot dz$,
(c) $D(g_1 dydz + g_2\,dzdx + g_3\,dxdy) = Dg_1 \cdot dydz + Dg_2 \cdot dzdx + Dg_3 \cdot dxdy$,
(d) $D(g\,dxdydz) = 0$.

Example 3.4

(a) $$\begin{aligned}D(xy\,dx) &= D(xy) \cdot dx = ((xy)_x dx + (xy)_y\,dy) \cdot dx\\ &= (y\,dx + x\,dy) \cdot dx = -x\,dxdy.\end{aligned}$$

(b) $$\begin{aligned}D(xz\,dydz + y^3 dzdx) &= (z\,dx + x\,dz) \cdot dydz\\ &\quad + 3y^2 dy \cdot dzdx = (z + 3y^2)dxdydz.\end{aligned}$$

We now observe some properties of D.

Proposition 3.2 (a) $D(c\omega) = cD\omega$,
(b) $D(\omega + \omega') = D\omega + D\omega'$.

Proposition 3.3 If ω is a k-form and ω' an l-form, then $D(\omega \cdot \omega') = (-1)^k \omega \cdot D\omega' + D\omega \cdot \omega'$.

Proposition 3.4 $D(D\omega) = 0$ for all ω.

For Proposition 3.2 (a) see Proofs, exercise 2. Proposition 3.3 is an extension of the formula for the derivative of the product of two functions. The $(-1)^k$ factor arises because the exterior product produces a negative factor according to Proposition 3.1. A proof may be obtained by considering various cases of k (see Proofs, exercise 3). The proof of $D(D\omega) = 0$ essentially rests on the formula $f_{xy} = f_{yx}$ (see Proofs, exercise 4).

Example 3.5 Given $\omega = xz\,dx$ and $\omega' = y^2z$, then

$$D(\omega \cdot \omega') = D(xy^2z^2\,dx) = -2xyz^2dxdy + 2xy^2z\,dzdx,$$
$$D\omega = x\,dzdx,\ D\omega' = 2yz\,dy + y^2dz.$$

Hence,

$$\begin{aligned}(-1)^k\omega \cdot D\omega' + D\omega \cdot \omega' &= -xz\,dx \cdot (2yz\,dy + y^2dz) + x\,dzdx \cdot y^2z \\ &= -2xyz^2dxdy + 2xy^2z\,dzdx = D(\omega \cdot \omega')\end{aligned}$$

verifying Proposition 3.3.

Example 3.6 Given $\omega = yz\,dx$, then

$$D\omega = y\,dzdx - z\,dxdy \text{ and } D(D\omega) = dy \cdot dzdx - dz \cdot dxdy = 0.$$

This verifies Proposition 3.4.

Questions

1. The three operations on differential forms are ______, ______, and ______.

2. The sum of a k-form and an l-form is ______.
   (a) a $(k + l)$-form,
   (b) a kl-form,
   (c) defined only when $k = l$.

3. The product of a k-form and an l-form is ______.
   (a) a $(k + l)$-form or 0,
   (b) a kl-form or 0,
   (c) defined only when $k = l$.

4. The basic operator on differential forms is called the ______.

5. If ω is a k-form, then $D\omega$ is either 0 or a ______ -form.
   (a) k,
   (b) $k - 1$,
   (c) $k + 1$.

6. If ω is a 1-form and ω' a 2-form, then $\omega \cdot \omega' =$ ______.
   (a) $\omega' \cdot \omega$,
   (b) $(-1)\omega' \cdot \omega$,
   (c) 0.

Problems

1. Do Problem Set B at the end of the chapter.

Proofs

1. Prove that the set of all 1-forms in $\mathbf{R}^2$ forms a vector space with the addition and scalar multiplication operations.
2. Prove $D(c\omega) = cD\omega$ for differential forms in $\mathbf{R}^2$.
3. Prove $D(\omega \cdot \omega') = (-1)^k \omega \cdot D\omega' + D\omega \cdot \omega'$, where ω is a k-form, given (a) $\omega = g_1 dx$, $\omega' = g_2\, dy$, (b) $\omega = g_1 dxdy$, $\omega' = g_2\, dz$.
4. Prove $D(D\omega) = 0$ for the $\mathbf{R}^3$ case given (a) $\omega = g$, (b) $\omega = g\ dy$.

4. *Scalar and Vector Functions and Differential Forms*

In this section we assert various correspondences between functions and differential forms. These correspondences are not one-to-one, in that a scalar function or vector function may correspond to more than one differential form. It then follows that a single property of differential forms may correspond to more than one property for functions. To simplify the discussion, all given functions will be assumed of class C^∞. The basic correspondences for the $\mathbf{R}^3$ domain case are (Note that g_i and $\bar{g}_i$ denote the same function).

(I)	Differential Form		Scalar, Vector Function
(a)	g (0-form)	$\leftrightarrow$	g,
(b)	$g_1 dx + g_2\, dy + g_3\, dz$	$\leftrightarrow$	$\bar{g}_1\mathbf{i} + \bar{g}_2\,\mathbf{j} + \bar{g}_3\,\mathbf{k}$,
(c)	$g_1 dydz + g_2\, dzdx + g_3\, dxdy$	$\leftrightarrow$	$\bar{g}_1\,\mathbf{i} + \bar{g}_2\,\mathbf{j} + \bar{g}_3\,\mathbf{k}$,
(d)	$g\ dxdydz$	$\leftrightarrow$	g.

Thus for the $\mathbf{R}^3$ case each scalar function and each vector function corresponds to two differential forms. The analogous correspondence for the $\mathbf{R}^2$ case is

(II)				
	(a)	g (0-form)	$\leftrightarrow$	g,
	(b)	$g_1 dx + g_2 dy$	$\leftrightarrow$	$\bar{g}_1\mathbf{i} + \bar{g}_2\,\mathbf{j}$,
	(c)	$g\ dxdy$	$\leftrightarrow$	g.

In combining correspondences with the various operations, it must be recognized which correspondence is being applied. By comparing the various operations, we observe the following induced correspondences for the $\mathbf{R}^3$ case.

(III)	Differential Forms		Scalar, Vector Functions
(a)	Product of two 0-forms	$\leftrightarrow$	Product of scalar fields,
(b)	Product of 0-form and k-form, $k = 1,2,3$	$\leftrightarrow$	Product of scalar field and vector field,
(c)	Product of two 1-forms	$\leftrightarrow$	Cross product of vector fields,
(d)	Product of 1-form and 2-form	$\leftrightarrow$	Dot product of vector fields.

For a proof of (d), let

$$\begin{aligned}
\omega &= g_1 dx + g_2\, dy + g_3\, dz,\\
\omega' &= g_1' dydz + g_2' dzdx + g_3' dxdy,\\
\mathbf{g} &= \bar{g}_1 \mathbf{i} + \bar{g}_2 \mathbf{j} + \bar{g}_3 \mathbf{k}, \text{ and}\\
\mathbf{g}' &= \bar{g}_1' \mathbf{i} + \bar{g}_2' \mathbf{j} + \bar{g}_3' \mathbf{k};
\end{aligned}$$

then $\omega \leftrightarrow \mathbf{g}$ and $\omega' \leftrightarrow \mathbf{g}'$. From

$$\begin{aligned}
\omega \cdot \omega' &= (g_1 g_1' + g_2 g_2' + g_3 g_3')dxdydz,\\
\mathbf{g} \cdot \mathbf{g}' &= \bar{g}_1 \bar{g}_1' + \bar{g}_2 \bar{g}_2' + \bar{g}_3 \bar{g}_3',
\end{aligned}$$

it is seen that $\omega \cdot \omega' \leftrightarrow \mathbf{g} \cdot \mathbf{g}'$ as desired.

Example 4.1 We verify (III)(c) for the case

$$\omega = x^2 dx + xy\, dz \quad \text{and} \quad \omega' = xy\, dx + y^2 dy + z\, dz.$$

Then $\omega \leftrightarrow x^2\mathbf{i} + xy\mathbf{k} = \mathbf{g}$ and $\omega' \leftrightarrow xy\mathbf{i} + y^2\mathbf{j} + z\mathbf{k} = \mathbf{g}'$.

From

$$\omega \cdot \omega' = -xy^3 dydz + (x^2y^2 - x^2z)dzdx + x^2y^2 dxdy$$

and

$$\begin{aligned}
\mathbf{g} \times \mathbf{g}' &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 & 0 & xy \\ xy & y^2 & z \end{bmatrix}\\
&= -xy^3\mathbf{i} + (x^2y^2 - x^2z)\mathbf{j} + x^2y^2\mathbf{k},
\end{aligned}$$

it is seen that $\omega \cdot \omega' \leftrightarrow \mathbf{g} \times \mathbf{g}'$.

In order to establish correspondences for the differential operator D on differential forms, we define two new operators on the vector function $\mathbf{g} = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}$.

(IV) The ***curl*** of **g** is

$$\begin{aligned}
\mathbf{curl}\ \mathbf{g} = {} & [(\bar{g}_3)_y - (\bar{g}_2)_z]\mathbf{i} + [(\bar{g}_1)_z - (\bar{g}_3)_x]\mathbf{j}\\
& + [(\bar{g}_2)_x - (\bar{g}_1)_y]\mathbf{k}.
\end{aligned}$$

(V) The *divergence* of **g** is

$$\operatorname{div} \mathbf{g} = (\bar{g}_1)_x + (\bar{g}_2)_y + (\bar{g}_3)_z.$$

Thus, the **curl** is a vector function and the divergence is a scalar function. These are often written using the *del operator* symbol

$$\nabla = (\ \ )_x\mathbf{i} + (\ \ )_y\mathbf{j} + (\ \ )_z\mathbf{k}.$$

It may be verified by expansion along the top row that

$$\mathbf{curl}\ \mathbf{g} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (\ \ )_x & (\ \ )_y & (\ \ )_z \\ \bar{g}_1 & \bar{g}_2 & \bar{g}_3 \end{bmatrix}.$$

The matrix entries are manipulated here as if they were real numbers. Using similar methods for the divergence, we arrive at the following useful alternate formulations,

$$\mathbf{curl\ g} = \nabla \times \mathbf{g},$$
$$\text{div}\ \mathbf{g} = \nabla \cdot \mathbf{g}.$$

Example 4.2 Let $\mathbf{g} = xz\mathbf{i} + y^2z\mathbf{j} + x^3z\mathbf{k}$; then

$$\mathbf{curl\ g} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (\ )_x & (\ )_y & (\ )_z \\ xz & y^2z & x^3z \end{bmatrix} = -y^2\mathbf{i} + (x - 3x^2z)\mathbf{j},$$

$$\text{div}\ \mathbf{g} = ((\ )_x\mathbf{i} + (\ )_y\mathbf{j} + (\ )_z\mathbf{k}) \cdot \mathbf{g} = z + 2yz + x^3.$$

We recall that the gradient of g is given by $\mathbf{grad}\ g = \nabla g = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$. The correspondences for D in the $\mathbf{R}^3$ case can now be formulated.

(VI) (a) If ω(0-form) $\leftrightarrow$ g, then $D\omega$ $\leftrightarrow$ $\mathbf{grad}\ g$,
(b) If ω(1-form) $\leftrightarrow$ $\mathbf{g}$, then $D\omega$ $\leftrightarrow$ $\mathbf{curl\ g}$,
(c) If ω(2-form) $\leftrightarrow$ $\mathbf{g}$, then $D\omega$ $\leftrightarrow$ div $\mathbf{g}$.

For a proof of (b) let

$$\omega = g_1\,dx + g_2\,dy + g_3\,dz \quad \text{and} \quad \mathbf{g} = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j} + \bar{g}_3\mathbf{k}.$$

Then,

$$D\omega = [(g_3)_y - (g_2)_z]dydz + [(g_1)_z - (g_3)_x]dzdx + [(g_2)_x - (g_1)_y]dxdy,$$

and hence, $D\omega \leftrightarrow \mathbf{curl\ g}$.

From (VI) we obtain new properties of **grad**, **curl**, and div. The linearity property of D yields the following correspondences.

(VII) (a) $\mathbf{grad}\ (cg) = c\ \mathbf{grad}\ g$,
$\mathbf{grad}\ (g + g') = \mathbf{grad}\ g + \mathbf{grad}\ g'$;
(b) $\mathbf{curl}\ c\mathbf{g} = c\ \mathbf{curl\ g}$,
$\mathbf{curl}\ (\mathbf{g} + \mathbf{g}') = \mathbf{curl\ g} + \mathbf{curl\ g}'$;
(c) div $c\mathbf{g} = c$ div $\mathbf{g}$, and
div $(\mathbf{g} + \mathbf{g}') =$ div $\mathbf{g}$ + div $\mathbf{g}'$.

The property $D(D\omega) = 0$ yields

(VIII) (a) $\mathbf{curl\ grad}\ g = \mathbf{0}$ (all g),
(b) div $\mathbf{curl\ g} = 0$ (all $\mathbf{g}$).

In order to show (a), let $g \leftrightarrow g$ denote the correspondence of a 0-form and scalar function. Then $Dg \leftrightarrow \mathbf{grad}\ g$ is the correspondence of a 1-form and vector function by VI(a). Hence, $D(Dg) \leftrightarrow \mathbf{curl\ grad}\ g$ follows by VI(b).

However, $D(Dg) = 0$ by Proposition 3.4, and this proves (a). The argument may be symbolized

(0) $g \leftrightarrow g$,
(1) $Dg \leftrightarrow \mathbf{grad}\ g$,
(2) $D(Dg) \leftrightarrow \mathbf{curl\ grad}\ g$,
(2) $0 \leftrightarrow \mathbf{curl\ grad}\ g$,

where the prefix numbers denote form size.

The property $D(\omega \cdot \omega') = (-1)^k \omega \cdot D\omega' + D\omega \cdot \omega'$ yields the following corresponding properties.

(IX) (a) $\mathbf{grad}\ (gg') = g\ \mathbf{grad}\ g' + (\mathbf{grad}\ g)g'$,
(b) $\text{div}\ (g\mathbf{g}') = g\ \text{div}\ \mathbf{g}' + \mathbf{grad}\ g \cdot \mathbf{g}'$,
(c) $\mathbf{curl}\ (g\mathbf{g}') = g\ \mathbf{curl}\ \mathbf{g}' + (\mathbf{grad}\ g) \times \mathbf{g}'$,
(d) $\text{div}\ (\mathbf{g} \times \mathbf{g}') = -\mathbf{g} \cdot \mathbf{curl}\ \mathbf{g}' + \mathbf{curl}\ \mathbf{g} \cdot \mathbf{g}'$.

For the proof of (c) let ω be a 1-form corresponding to $\mathbf{g}'$; then

(1) $\omega \leftrightarrow \mathbf{g}'$;
(0) $g \leftrightarrow g$;
(2) $D(g\omega) \leftrightarrow \mathbf{curl}\ g\mathbf{g}'$,
$D(g\omega) = gD\omega + Dg \cdot \omega$;
(2) $D\omega \leftrightarrow \mathbf{curl}\ \mathbf{g}'$;
(1) $Dg \leftrightarrow \mathbf{grad}\ g$;
(2) $Dg \cdot \omega \leftrightarrow (\mathbf{grad}\ g) \times \mathbf{g}'$;
$\mathbf{curl}\ g\mathbf{g}' = g\ \mathbf{curl}\ \mathbf{g}' + (\mathbf{grad}\ g) \times \mathbf{g}'$.

The final equality follows by substitution into $D(g\omega) = gD\omega + Dg \cdot \omega$.

The concepts of **curl** and divergence have been treated here in a purely formal way. Actually they first arose in connection with problems in physics. For example, we consider a fluid in motion with $\mathbf{g}(\mathbf{u})$ describing the velocity at position $\mathbf{u}$. Then div $\mathbf{g}(\mathbf{u})$ may be interpreted as the net rate at which fluid is being moved from the vicinity of $\mathbf{u}$. Also $\mathbf{curl}\ \mathbf{g}(\mathbf{u})$ may be regarded as the angular velocity of the fluid at $\mathbf{u}$. The equation div $\mathbf{g} = 0$ applies when the fluid is *incompressible*; the equation $\mathbf{curl}\ \mathbf{g} = \mathbf{0}$ holds when the flow is *irrotational.*

Questions

1. The product of two 1-forms corresponds to the ______ operation on vector fields.
2. The three operators on scalar and vector functions are ______, ______, and ______.
3. The differential operator on 2-forms corresponds to the ______ operator on vector fields.
4. The three operators on scalar and vector functions are all ______.
   (a) commutative,
   (b) associative,
   (c) linear.

Exercises

1. Find **curl g** and div **g** if $\mathbf{g}(x, y, z) =$
   (a) $(x - z)\mathbf{i} + (2y + z)\mathbf{j} + (x + y)\mathbf{k}$,
   (b) $x^2y\mathbf{i} + xz^2\mathbf{j} + yz^3\mathbf{k}$,
   (c) $yz\mathbf{i} + xz\mathbf{j} + z\mathbf{k}$,
   (d) $(x - z)^2\mathbf{i} + yz\mathbf{j} + (x - y)\mathbf{k}$.

Proofs

1. Prove the exterior product of two 1-forms corresponds to the cross product of vector fields.
2. Prove div **curl g** $= 0$ using the corresponding property $D(D\omega) = 0$.
3. Prove **grad** $(gg') = g$ **grad** $g' +$ (**grad** $g)g'$ using the corresponding property for $D(\omega \cdot \omega')$.

5. Integral Theorems

Thus far we have observed various correspondences between differential forms and scalar and vector functions. There has been no correspondence noted for integrals, since the integral of a differential form has not yet been defined. A general definition (indicated in the next section) implies that the defining integral formula for a differential form in the $\mathbf{R}^2$ and $\mathbf{R}^3$ cases is the same as that of its corresponding scalar or vector function. Thus,

(I) If $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, $a \le r \le b$, then

$$\int_{\mathscr{C}^\circ} g_1\, dx + g_2\, dy = \int_a^b \left[(g_1 \circ \mathbf{f})\frac{d\bar{x}}{dr} + (g_2 \circ \mathbf{f})\frac{d\bar{y}}{dr}\right] dr.$$

(II) If $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}$, $a \le r \le b$, then

$$\int_{\mathscr{C}^\circ} g_1\, dx + g_2\, dy + g_3\, dz = \int_a^b \left[(g_1 \circ \mathbf{f})\frac{d\bar{x}}{dr} + (g_2 \circ \mathbf{f})\frac{d\bar{y}}{dr} + (g_3 \circ \mathbf{f})\frac{d\bar{z}}{dr}\right] dr.$$

(III) If $\mathscr{R}^\circ$ is represented by $\{\mathbf{f}, \mathbf{I}^2\}$, then

$$\int_{\mathscr{R}^\circ} g\, dx\, dy = \int_{\mathbf{I}^2} (g \circ \mathbf{f}) \det J_{\mathbf{f}}\, dA.$$

(IV) If $\mathscr{S}^\circ$ is represented by $\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}$, then

$$\int_{\mathscr{S}^\circ} g_1\, dydz + g_2\, dzdx + g_3\, dxdy =$$

$$\int_{\mathbf{I}^2} \left[(g_1 \circ \mathbf{f})\det \begin{bmatrix} \bar{y}_r & \bar{y}_s \\ \bar{z}_r & \bar{z}_s \end{bmatrix} + (g_2 \circ \mathbf{f})\det \begin{bmatrix} \bar{z}_r & \bar{z}_s \\ \bar{x}_r & \bar{x}_s \end{bmatrix} + (g_3 \circ \mathbf{f})\det \begin{bmatrix} \bar{x}_r & \bar{x}_s \\ \bar{y}_r & \bar{y}_s \end{bmatrix}\right] dA.$$

(V) If $\mathscr{T}^\circ$ is represented by $\{\mathbf{f}, \mathbf{I}^3\}$, then

$$\int_{\mathscr{T}^\circ} g\, dx\, dy\, dz = \int_{\mathbf{I}^3} (g \circ \mathbf{f}) \det J_{\mathbf{f}}\, dV.$$

It should be noted that the integral of a k-form is defined only on a k-dimensional set. We go now to the fundamental integral equality. We let $(\mathscr{S}^n)^0$ denote an oriented n-surface in $\mathbf{R}^m$ whose boundary $(\partial\mathscr{S}^n)^0$ either is empty or is an oriented $(n-1)$-surface in $\mathbf{R}^m$, $1 \leq n \leq m \leq 3$. A precise definition of such a boundary will not be attempted here, although boundaries will be later discussed in various geometric instances. It must also be assumed that the orientations of $(\mathscr{S}^n)^0$ and $(\partial\mathscr{S}^n)^0$ are compatible; this too will be explained in geometric cases. The requirement that $(\partial\mathscr{S}^n)^0$ be an $(n-1)$-surface is severe, and we shall later adapt our conclusions to more general situations. Our basic result asserts that if ω is an $(n-1)$-form on $(\partial\mathscr{S}^n)^0$, then we have

Stokes Differential Form Equality

$$\int_{(\mathscr{S}^n)^0} D\omega = \int_{(\partial\mathscr{S}^n)^0} \omega.$$

This equality may be regarded as an extension of the fundamental theorem of calculus, and has several very important corresponding equalities for scalar and vector functions. We first consider the case in which $(\mathscr{S}^n)^0$ is an oriented curve $\mathscr{C}^\circ$ in $\mathbf{R}^2$. Then $\partial\mathscr{C}^\circ$ consists of the end points of $\mathscr{C}^\circ$, or is empty if $\mathscr{C}^\circ$ is a closed curve. There will be no attempt to justify $\partial\mathscr{C}^\circ$ as a 0-surface or the meaning of an integral on an oriented domain of two points. Instead, we proceed to state the corresponding equality in this case and then give a proof.

Fundamental Theorem for Line Integrals

Given g of class C^∞ on the oriented curve $\mathscr{C}^\circ$ represented by $\mathbf{f}(r)$, $a \leq r \leq b$, then

$$\int_{\mathscr{C}^\circ} \nabla g \cdot d\mathbf{f} = g(\mathbf{f}(b)) - g(\mathbf{f}(a)).$$

This theorem is valid for oriented curves in $\mathbf{R}^2$ and $\mathbf{R}^3$. It corresponds to Stokes' equality by letting $\omega \leftrightarrow g$, from which $D\omega \leftrightarrow \nabla g$, and then defining the integral of g on $\{\mathbf{f}(b), \mathbf{f}(a)\}$ to be $g(\mathbf{f}(b)) - g(\mathbf{f}(a))$. Letting $\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}$, a proof for the $\mathbf{R}^2$ case is

$$\begin{aligned}
\int_{\mathscr{C}^\circ} \nabla g \cdot d\mathbf{f} &= \int_a^b (\nabla g \circ \mathbf{f}) \cdot \frac{d\mathbf{f}}{dr}\, dr \\
&= \int_a^b \left[(g_x \circ \mathbf{f}) \frac{d\bar{x}}{dr} + (g_y \circ \mathbf{f}) \frac{d\bar{y}}{dr} \right] dr \\
&= \int_a^b \frac{d(g \circ \mathbf{f})}{dr}\, dr \qquad \text{(chain rule)} \\
&= g \circ \mathbf{f}(b) - g \circ \mathbf{f}(a) \qquad \text{(fundamental theorem of calculus)} \\
&= g(\mathbf{f}(b)) - g(\mathbf{f}(a)).
\end{aligned}$$

Example 5.1 If $g(x, y, z) = xy^2z$, then

$$\nabla g = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k},$$

and if $\mathscr{C}^\circ$ is any curve oriented from (2, 1, 3) to (3, 0, 1), then

$$\int_{\mathscr{C}^\circ} (y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}) \cdot d\mathbf{f} = g(3, 0, 1) - g(2, 1, 3) = -6.$$

Before moving on to other correspondences of Stokes' equality, we shall consider some consequences of the fundamental theorem for line integrals. If $\mathscr{C}^\circ$ is a closed curve, then $\mathbf{f}(a) = \mathbf{f}(b)$ and we have the grounds for our next proposition.

Proposition 5.1 If $\mathscr{C}^\circ$ is a closed curve, then $\int_{\mathscr{C}^\circ} \nabla g \cdot d\mathbf{f} = 0$.

A vector function which is the gradient of a scalar function is called *exact*. Thus, by Proposition 5.1 the integral of an exact function about any simple closed curve is zero. We next investigate methods for determining whether or not a given vector function $\mathbf{h}$ is exact. For the $\mathbf{R}^2$ case, if

$$\mathbf{h} = \bar{h}_1\mathbf{i} + \bar{h}_2\mathbf{j} = \nabla g(x, y) = g_x\mathbf{i} + g_y\mathbf{j},$$

then

$$(\bar{h}_2)_x - (\bar{h}_1)_y = g_{yx} - g_{xy} = 0.$$

Therefore, the equality $(\bar{h}_2)_x = (\bar{h}_1)_y$ is necessary in order for $\mathbf{h}$ to be exact. It can be shown that this condition is not always sufficient. We shall not pursue that point further, however, but instead consider a method for finding a suitable function g in those cases that it does exist.

Example 5.2 Let $\mathbf{h}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$; then it may be verified that

$$(\bar{h}_2)_x - (\bar{h}_1)_y = 6xy^2 - 6xy^2 = 0.$$

If $\nabla g = g_x\mathbf{i} + g_y\mathbf{j} = \mathbf{h}$, then $g_x = 2xy^3$ and $g_y = 3x^2y^2$. Integrating $g_x = 2xy^3$ with respect to x, keeping y constant, gives $g = x^2y^3 + H(y)$, where $H(y)$ is an arbitrary function of y to be determined. Differentiating the resulting equality with respect to y gives $g_y = 3x^2y^2 + dH/dy$. Comparison with the previous equation $g_y = 3x^2y^2$ shows $dH/dy = 0$, and hence $H = 0$ is a solution; thus $\nabla(x^2y^3) = \mathbf{h}$.

For the $\mathbf{R}^3$ case, if $\mathbf{h} = \nabla g$, then **curl h** $=$ **curl** $\nabla g = \mathbf{0}$ by Formula VIII(a) in Section 4. The method for then finding g such that $\nabla g = \mathbf{h}$ is similar to that for the $\mathbf{R}^2$ case.

We now consider additional correspondences of Stokes' equality. Let$(\mathscr{S}^n)^0$ be the oriented region $\mathscr{R}^o$ in $\mathbf{R}^2$; then $\partial\,\mathscr{R}^o$ consists of its perimeter and if $\omega \leftrightarrow \bar{g}_1\mathbf{i} + \bar{g}_2\,\mathbf{j}$, then $D\omega \leftrightarrow (\bar{g}_2)_x - (\bar{g}_1)_y$. Thus Stokes' equality gives for $\mathbf{g} = \bar{g}_1\mathbf{i} + \bar{g}_2\,\mathbf{j}$ the theorem that follows.

Green's Theorem

$$\int_{\mathscr{R}^0} [(\bar{g}_2)_x - (\bar{g}_1)_y]dA = \int_{\partial\mathscr{R}^0} \mathbf{g} \cdot d\mathbf{f}.$$

If $\partial\,\mathscr{R}^o$ is a simple closed curve, then the orientation compatibility conditions for Green's theorem requires that $\mathscr{R}^o$ has positive orientation if and only if $\partial\,\mathscr{R}^o$ has counterclockwise orientation.

Example 5.3 We seek to verify Green's theorem for $\mathbf{g} = xy^2\mathbf{i} + x\mathbf{j}$ on the disk $\mathscr{R}^o$ described by $x^2 + y^2 \leq 1$ and having positive orientation. Then $\mathscr{R}^o$ is represented by

$$\mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}, \qquad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi,$$

and $\partial\,\mathscr{R}^o$ by

$$\mathbf{f}'(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad 0 \leq \theta \leq 2\pi.$$

Thus

$$\int_{\mathscr{R}^0} [(\bar{g}_2)_x - (\bar{g}_1)_y]dA = \int_0^1 dr \int_0^{2\pi} (1 - 2r^2\cos\theta\sin\theta)r\,d\theta = \pi;$$

$$\int_{\partial\mathscr{R}^0} \mathbf{g} \cdot d\mathbf{f}' = \int_0^{2\pi} [\cos\theta\sin^2\theta(-\sin\theta) + \cos\theta\,(\cos\theta)]d\theta = \pi.$$

If $\mathbf{g}$ is exact, then

$$\int_{\mathscr{R}^0} [(\bar{g}_2)_x - (\bar{g}_1)_y] dA = \int_{\mathscr{R}^0} 0 \, dA = 0,$$

and hence by Green's theorem, we have the next property.

Proposition 5.2 If $\mathbf{g}$ is exact on a region $\mathscr{R}^o$, then $\int_{\partial \mathscr{R}^0} \mathbf{g} \cdot d\mathbf{f} = 0$.

If $(\mathscr{S}^n)^0$ is an oriented surface $\mathscr{S}^o$ in space, then $\partial \mathscr{S}^o$ may assume many geometric possibilities. If the representing function of $\mathscr{S}^o$ is injective, then $\partial \mathscr{S}^o$ is a closed curve. If $\mathscr{S}^o$ is a closed surface, such as a spherical surface, then $\partial \mathscr{S}^o$ is the empty set. Given the 1-form correspondence $\omega \leftrightarrow \mathbf{g}$ on $\mathscr{S}^o$, then $D\omega \leftrightarrow \mathbf{curl\ g}$ and Stokes' equality in this case gives the following theorem.

Stokes' Theorem

$$\int_{\mathscr{S}^0} \mathbf{curl\ g} \cdot \mathbf{n} \, dA = \int_{\partial \mathscr{S}^0} \mathbf{g} \cdot d\mathbf{f}.$$

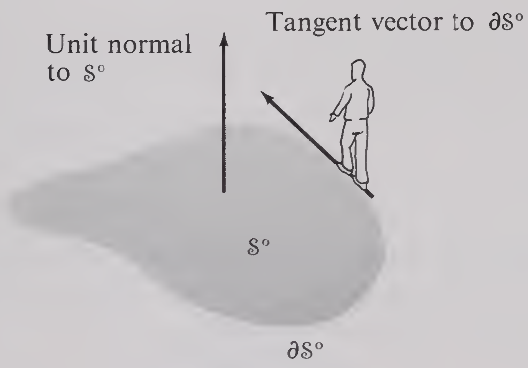


Figure 15.5

If $\mathscr{S}^o$ has an injective representation, then the compatibility condition on $\mathscr{S}^o$ and $\partial \mathscr{S}^o$ requires that an object moving along $\partial \mathscr{S}^o$ in the direction of a tangent vector to $\partial \mathscr{S}^o$ and on the unit normal side of $\mathscr{S}^o$ would have $\mathscr{S}^o$ on its left (see Figure 15.5). If $\mathscr{S}$ is closed, then $\partial \mathscr{S}^o$ is empty, and this implies our next result.

Proposition 5.3 If $\mathscr{S}^o$ is a closed surface, then $\int_{\mathscr{S}^o} \mathbf{curl\ g} \cdot \mathbf{n} \, dA = 0$.

Example 5.4 Let $\mathscr{S}^{\circ}$ be the portion of the plane $z = x + y$ defined on the disk $x^2 + y^2 \le 1$. Then $\mathscr{S}^{\circ}$ is represented by

$$\mathbf{f}(r, \theta) = r \cos \theta\, \mathbf{i} + r \sin \theta\, \mathbf{j} + r(\cos \theta + \sin \theta)\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Also $\partial \mathscr{S}^{\circ}$ is represented by

$$\mathbf{f}'(\theta) = \cos \theta\, \mathbf{i} + \sin \theta\, \mathbf{j} + (\cos \theta + \sin \theta)\mathbf{k}, \quad 0 \le \theta \le 2\pi.$$

If $\mathbf{g}(x, y, z) = z\mathbf{i}$, then $\mathbf{curl}\ \mathbf{g} = \mathbf{j}$ and hence

$$\int_{\mathscr{S}^{\circ}} \mathbf{curl}\ \mathbf{g} \cdot \mathbf{n}\, dA = \int_0^1 dr \int_0^{2\pi} \det \begin{bmatrix} \cos \theta + \sin \theta & r(-\sin \theta + \cos \theta) \\ \cos \theta & -r \sin \theta \end{bmatrix} d\theta$$
$$= -\pi;$$

$$\int_{\partial \mathscr{S}^{\circ}} \mathbf{g} \cdot d\mathbf{f}' = \int_0^{2\pi} (\cos \theta + \sin \theta)(-\sin \theta) d\theta = -\pi.$$

Thus Stokes' theorem is verified.

Our final corresponding formula for Stokes' equality is the case in which $(\mathscr{S}^n)^{\circ}$ is the oriented solid $\mathscr{T}^{\circ}$ in space. Then $\partial \mathscr{T}^{\circ}$ consists of the surface which forms the cover of $\mathscr{T}^{\circ}$. Given the 2-form correspondence $\omega \leftrightarrow \mathbf{g}$ on $\mathscr{T}^{\circ}$, then $D\omega \leftrightarrow \operatorname{div} \mathbf{g}$, and this gives another special case of Stokes' equality.

Divergence Theorem

$$\int_{\mathscr{T}^{\circ}} \operatorname{div} \mathbf{g}\, dV = \int_{\partial \mathscr{T}^{\circ}} \mathbf{g} \cdot \mathbf{n}\, dA.$$

In this case the compatibility condition of the representing functions requires that if $\mathscr{T}^{\circ}$ has positive orientation, then the representation of $\partial \mathscr{T}^{\circ}$ assigns an outwardly directed unit normal.

Example 5.5 We wish to verify the divergence theorem for $\mathbf{g}(x, y, z) = x\mathbf{i}$ on the positively oriented solid sphere $\mathscr{T}^{\circ}$ described by $x^2 + y^2 + z^2 \le 1$. Then $\mathscr{T}^{\circ}$ is represented by

$$\mathbf{f}(r, \phi, \theta) = r \sin \phi \cos \theta \mathbf{i} + r \sin \phi \sin \theta\, \mathbf{j} + r \cos \phi\, \mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi,$$

and $\partial \mathscr{T}^{\circ}$ by

$$\mathbf{f}'(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta\, \mathbf{j} + \cos \phi\, \mathbf{k}, \qquad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Thus,

$$\int_{\mathscr{T}^0} \operatorname{div} \mathbf{g}\, dV = \int_{\mathscr{T}^0} 1\, dV = \operatorname{Vol} \mathscr{T} = \frac{4\pi}{3},$$

$$\begin{aligned}\int_{\partial\mathscr{T}^0} \mathbf{g} \cdot \mathbf{n}\, dA &= \int_0^{\pi} d\phi \int_0^{2\pi} \sin\phi \cos\theta \det \begin{bmatrix} \cos\phi \sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{bmatrix} d\theta \\ &= \frac{4\pi}{3}.\end{aligned}$$

The requirement that $(\mathscr{S}^n)^0$ and $(\partial\, \mathscr{S}^n)^0$ be n-surfaces and $(n-1)$-surfaces respectively is too restrictive for most applications of Stokes' equality, which is also valid for sets which can be obtained by "pasting" at the boundaries finitely many oriented n-surfaces, each of which can be represented by an injective function. This pasting must be done so that representing functions give opposite orientations at the pasted boundaries.

Example 5.6 Let $\mathscr{C}^{\circ}$ be the set which is the graph of $y = |x|$, $-1 \leq x \leq 1$, oriented from $\langle -1, 1\rangle$ to $\langle 1, 1\rangle$. Then, $\mathscr{C}^{\circ}$ is composed of the 1-surfaces $\mathscr{C}_1{}^{\circ}$ from $\langle -1, 1\rangle$ to $\langle 0, 0\rangle$ and $\mathscr{C}_2{}^{\circ}$ from $\langle 0, 0\rangle$ to $\langle 1, 1\rangle$. The pasting point $\langle 0, 0\rangle$ serves as a terminal point of $\mathscr{C}_1{}^{\circ}$ and an initial point of $\mathscr{C}_2{}^{\circ}$. The oriented boundary set is $\partial\mathscr{C}^{\circ} = \{\langle -1, 1\rangle, \langle 1, 1\rangle\}$.

Example 5.7 Let $\mathscr{R}^{\circ}$ be the annulus bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and having positive orientation. Then $\mathscr{R}^{\circ}$ is composed of two oriented 2-surfaces $\mathscr{R}_1{}^{\circ}$ and $\mathscr{R}_2{}^{\circ}$ respectively represented by (see Figure 15.6)

$$\mathbf{f}_1(r, \theta) = r\cos\theta\, \mathbf{i} + r\sin\theta\mathbf{j}, \quad 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi;$$

$$\mathbf{f}_2(r, \theta) = r\cos\theta\, \mathbf{i} + r\sin\theta\, \mathbf{j}, \quad 1 \leq r \leq 2, \quad \pi \leq \theta \leq 2\pi.$$

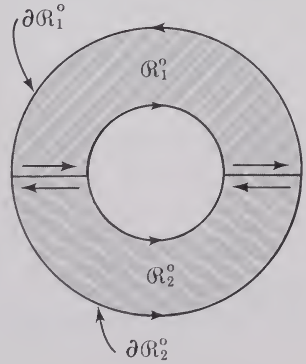


Figure 15.6

The counterclockwise boundaries $\partial\mathscr{R}_1°$ and $\partial\mathscr{R}_2°$ are each composed of four 1-surfaces, two of which are common to both $\partial\mathscr{R}_1°$ and $\partial\mathscr{R}_2°$. Along these two the orientations of $\partial\mathscr{R}_1°$ and $\partial\mathscr{R}_2°$ are opposite. The boundary $\partial\mathscr{R}°$ consists of two circles, the outer circle having counterclockwise orientation and the inner circle clockwise orientation.

The integral of an n-form ω on a pasted domain is defined to be the sum of the integrals of ω on the oriented n-surfaces which combine to give the domain. The opposite orientations on the pasted boundaries give a cancellation of integrals; this is justified by the next result (see Proofs, exercises 1,2 in Section 2).

Proposition 5.4 If $(\mathscr{S}_1{}^n)^0$, $(\mathscr{S}_2{}^n)^0$ are the same n-surface, with opposite orientations, then

$$\int_{(\mathscr{S}_1{}^n)^0} \omega = -\int_{(\mathscr{S}_2{}^n)^0} \omega.$$

Example 5.8 We seek to verify Green's theorem for $\mathbf{g} = g\mathbf{i} + 0\mathbf{j}$ on $\mathscr{R}° = [0, 1] \times [0, 1]$ with positive orientation. Then $\partial\mathscr{R}°$ is composed of four segments $\mathscr{C}_1°$, $\mathscr{C}_2°$, $\mathscr{C}_3°$, and $\mathscr{C}_4°$ (see Figure 15.7).

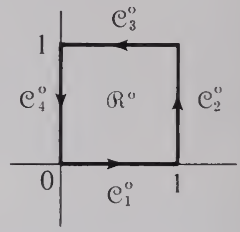


Figure 15.7

We use respective representing functions $\mathbf{f}(x, y) =$

$$x\mathbf{i} + 0\mathbf{j}, \qquad \mathbf{i} + y\mathbf{j}, \qquad x\mathbf{i} + \mathbf{j}, \qquad 0\mathbf{i} + y\mathbf{j},$$

where $0 \le x \le 1$, $0 \le y \le 1$. The latter two representations give opposite orientations to that of $\mathscr{C}_3°$ and $\mathscr{C}_4°$. Thus

$$\begin{aligned}\int_{\partial\mathscr{R}^0} \mathbf{g} \cdot d\mathbf{f} &= \int_0^1 g(x, 0)1\,dx + \int_0^1 g(1, y)0\,dy - \int_0^1 g(x, 1)1\,dx \\ &\quad - \int_0^1 g(0, y)0\,dy \\ &= \int_0^1 [g(x, 0) - g(x, 1)]dx.\end{aligned}$$

Since

$$\begin{aligned}\int_{\mathscr{R}^0} (0 - g_y)dA = \int_0^1 dx \int_0^1 -g_y\,dy &= \int_0^1 -g\Big|_{y=0}^{y=1} dx \\ &= \int_0^1 [-g(x, 1) + g(x, 0)]dx,\end{aligned}$$

the desired equality is obtained.

Questions

1. The correspondence of Stokes' equality for integrals on oriented curves is called ________.
2. $\int_{\mathscr{C}^o} \nabla g \cdot d\mathbf{f} = 0$ provided $\mathscr{C}^o$ is a ________ curve.
3. If $\mathbf{h}(x, y, z)$ is exact, then ________ $\mathbf{h} = \mathbf{0}$.
4. The correspondence of Stokes' equality for plane regions is called ________.
5. The correspondence of Stokes' equality for solids is called ________.

Problems

1. Do Problem Set C at the end of the chapter.

Exercises

1. Verify Stokes' theorem (choose either of two possible orientations for $\mathscr{S}^o$ and a compatible orientation for $\mathscr{C}^o$).
   (a) $\mathbf{g}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$ and $\mathscr{S}^o$ is the surface of revolution about the x axis of the upper half of the circle $x^2 + y^2 = 1$,
   (b) $\mathbf{g}(x, y, z) = y\mathbf{i} + x\mathbf{k}$ and $\mathscr{S}^o$ is the surface of revolution about the y axis of $y = x^2$, $0 \le x \le 1$,
   (c) $g(x, y, z) = xy\mathbf{i} - y\mathbf{j} + xz\mathbf{k}$ and $\mathscr{S}^o$ is the portion of the plane $z = x - 3y$ on the domain $[0, 1] \times [0, 1]$.
2. Verify the divergence theorem (use positive orientation for $\mathscr{T}$):
   (a) $\mathbf{g}(x, y, z) = 2y\mathbf{j} - z\mathbf{k}$ and $\mathscr{T}$ is the sphere $x^2 + y^2 + z^2 \le 1$,
   (b) $\mathbf{g}(x, y, z) = x^2\mathbf{i} + z\mathbf{k}$ and $\mathscr{T} = [0, 1] \times [0, 1] \times [0, 1]$,
   (c) $\mathbf{g}(x, y, z) = y\mathbf{i} + z^2\mathbf{k}$ and $\mathscr{T}$ is the cylinder $x^2 + y^2 \le 1$, $0 \le z \le 1$.

Proofs

1. Prove that if $\mathbf{g}(x, y) = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j}$ satisfies $(\bar{g}_2)_x - (\bar{g}_1)_y = 1$ on a region $\mathscr{R}$ consisting of a counterclockwise-oriented smooth, simple closed curve $\mathscr{C}^o$ together with its interior, then $A(\mathscr{R}) = \int_{\mathscr{C}^o} \mathbf{g} \cdot d\mathbf{f}$. Thus, show
$$A(\mathscr{R}) = \int_{\mathscr{C}^o} x\, dy = -\int_{\mathscr{C}^o} y\, dx = \tfrac{1}{2}\int_{\mathscr{C}^o} (x\, dy - y\, dx).$$

6. *Differential Forms* ($\mathbf{R}^n$)

In this section we consider the elements of the structure theory of differential forms. For $0 \le k \le n$ an *elementary k-form* in n variables $x_1, \ldots, x_n$ is symbolized by
$$g(x_1, \ldots, x_n)\, dx_{i_1} \cdots dx_{i_k},$$

where the coefficient function g is of class C^∞ on an open set $\mathbf{U}$ in $\mathbf{R}^n$ and $1 \le i_1 < \cdots < i_k \le n$. A *k-form* ω is written

$$\omega = \omega_1 + \cdots + \omega_n,$$

where $\omega_1, \ldots, \omega_n$ are elementary k-forms. The symbol $\Omega(\mathbf{U})$ will denote the set of all differential forms having $\mathbf{U}$ as the domain of the coefficient functions. The three fundamental operations on $\Omega(\mathbf{U})$ are

(I) sum of two k-forms,
(II) multiplication of a k-form by a real number, and
(III) exterior product of a k-form and an l-form.

Addition is defined by adding coefficients of like terms, and scalar multiplication by c is accomplished by multiplying each coefficient function by c. These operations are natural, and were illustrated in previous sections. The next result is easily verified.

Proposition 6.1 The set of all k-forms in $\Omega(\mathbf{U})$ with the operations of addition and scalar multiplication is a vector space.

The exterior product is more complicated, and its definition requires an investigation of permutations. We first introduce the notation

$$g\,d[Q] = g\,dx_{i_1} \cdots dx_{i_k},$$

where $[Q]$ is the ordered set $[i_1, \ldots, i_k]$. Then $[Q_1, Q_2]$ denotes the ordered set of numbers in Q_1 followed by those in Q_2. Letting $[Q_1 \cup Q_2]$ denote those numbers in either Q_1 or Q_2 with the natural order, and $Q_1 \cap Q_2$ those numbers in both Q_1 and Q_2, the *signum* of $[Q_1, Q_2]$ is defined by the rule

$$\operatorname{sgn}[Q_1, Q_2] = \begin{cases} 0 & \text{if } Q_1 \cap Q_2 \text{ is nonempty,} \\ 1 & \text{if } [Q_1, Q_2] \text{ is an even permutation} \\ & \quad \text{of } [Q_1 \cup Q_2] \text{ and } Q_1 \cap Q_2 \text{ is empty,} \\ -1 & \text{if } [Q_1, Q_2] \text{ is an odd permutation} \\ & \quad \text{of } [Q_1 \cup Q_2] \text{ and } Q_1 \cap Q_2 \text{ is empty.} \end{cases}$$

Example 6.1 (a) Let $Q_1 = [1, 3, 7]$ and $Q_2 = [2, 3, 6]$. Then $Q_1 \cap Q_2 = \{3\}$ is nonempty, and hence, $\operatorname{sgn}[Q_1, Q_2] = 0$. (b) Let $Q_1 = [1, 4, 7]$ and $Q_2 = [2, 3, 6, 9]$. Then

$$[Q_1, Q_2] = [1, 4, 7, 2, 3, 6, 9] \quad \text{and} \quad [Q_1 \cup Q_2] = [1, 2, 3, 4, 6, 7, 9].$$

It may be verified that $[Q_1 \cup Q_2]$ can be obtained from $[Q_1, Q_2]$ by an odd number of interchanges, and hence, $\operatorname{sgn}[Q_1, Q_2] = -1$.

We are now ready for our product definition.

Definition of Exterior Product

The exterior product *on* $\Omega(\mathbf{U})$ *is the operation which associates with each pair of differential forms* ω_1 *and* ω_2 *in* $\Omega(\mathbf{U})$ *a differential form* $\omega_1 \cdot \omega_2$ *such that*

(a) *if* $\omega_1 = g_1\, dQ_1$ *and* $\omega_2 = g_2\, dQ_2$ *are elementary forms, then* $\omega_1 \cdot \omega_2 = \operatorname{sgn}\,[Q_1, Q_2] g_1 g_2 d[Q_1 \cup Q_2]$;
(b) $\omega_1 \cdot (\omega_2 + \omega_3) = \omega_1 \cdot \omega_2 + \omega_1 \cdot \omega_3$;
(c) $(\omega_1 + \omega_2) \cdot \omega_3 = \omega_1 \cdot \omega_3 + \omega_2 \cdot \omega_3$;
(d) $(\omega_1 \cdot \omega_2) \cdot \omega_3 = \omega_1 \cdot (\omega_2 \cdot \omega_3)$;
(e) $\omega_1 \cdot (c\omega_2) = (c\omega_1) \cdot \omega_2 = c(\omega_1 \cdot \omega_2)$.

It follows that if ω_1 is a k_1-form and ω_2 a k_2-form, then either $\omega_1 \cdot \omega_2 = 0$ (the zero differential form) or $\omega_1 \cdot \omega_2$ is a $(k_1 + k_2)$-form. Properties (b)–(e) are familiar properties of real-number algebra.

Example 6.2

(a) $g_1 dx_1 dx_3\, dx_7 \cdot g_2\, dx_2\, dx_3\, dx_6 = 0$ (see Example 6.1(a)),
(b) $g_1 dx_1 dx_4\, dx_7 \cdot g_2\, dx_2\, dx_3\, dx_6\, dx_9$
$$= -g_1 g_2\, dx_1 dx_2\, dx_3\, dx_4\, dx_6\, dx_7\, dx_9$$
(see Example 6.1(b)).

We now consider the differential operator on $\Omega(\mathbf{U})$.

Definition of Exterior Differential

The exterior differential D *is the operator on* $\Omega(\mathbf{U})$ *which satisfies the conditions*

(a) $Dg = g_{x_1} dx_1 + \cdots + g_{x_n}\, dx_n$,
(b) $D(g\, dx_{i_1} \cdots dx_{i_k}) = (Dg) \cdot dx_{i_1} \cdots dx_{i_k}$, and
(c) D is linear.

Example 6.3 $D(x_1{}^2 x_2\, dx_1 + x_2 x_3{}^2\, dx_3) = D(x_1{}^2 x_2) \cdot dx_1 + D(x_2 x_3{}^2) \cdot dx_3 = (2x_1 x_2\, dx_1 + x_1{}^2\, dx_2 + 0\, dx_3) \cdot dx_1 + (0\, dx_1 + x_3{}^2\, dx_2 + 2x_2 x_3\, dx_3) \cdot dx_3 = -x_1{}^2\, dx_1\, dx_2 + x_3{}^2\, dx_2\, dx_3$.

If ω is a k-form, then either $D\omega = 0$ or $D\omega$ is a $(k+1)$-form. Further properties which are extensions of those in Section 3 are found in the following proposition.

Proposition 6.2

(a) $D(\omega_1 \cdot \omega_2) = (-1)^k \omega_1 \cdot D\omega_2 + D\omega_1 \cdot \omega_2$, where ω_1 is a k-form,
(b) $D(D\omega) = 0$ (ω arbitrary).

If the n-surface $\mathscr{S}^n$ is represented by $\{\mathbf{f}, \mathbf{I}^n\}$, then the formula

$$\int_{\mathscr{S}^n} g\, dV = \int_{\mathbf{I}^n} (g \circ \mathbf{f}) |J_{\mathbf{f}}|\, dV$$

determines the correspondence between the functions

$$g \leftrightarrow (g \circ \mathbf{f}) |J_{\mathbf{f}}|.$$

The integral of differential forms involves a similar correspondence. Let $\mathbf{U}$ and $\mathbf{W}$ be respective open sets in $\mathbf{R}^n$ and $\mathbf{R}^m$ and $\Omega(\mathbf{U})$, $\Omega(\mathbf{W})$ their spaces of differential forms. The exterior differential on each of these will be denoted D. Coordinate variables of $\mathbf{R}^n$ and $\mathbf{R}^m$ will be $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, respectively. If $\mathbf{f}$ is of class C^∞ from $\mathbf{U}$ to $\mathbf{W}$, then we have the following characterizing definition (see Figure 15.8).

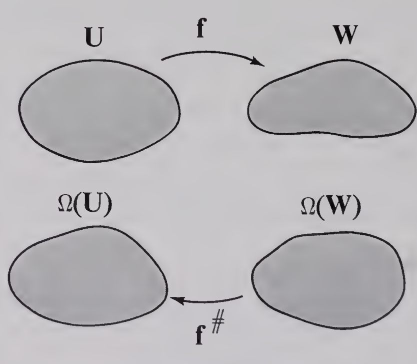


Figure 15.8

Definition of Induced Transformation

The $\mathbf{f}$ induced transformation *is the linear function* $\mathbf{f}^{\#}$ *from* $\Omega(\mathbf{W})$ *to* $\Omega(\mathbf{U})$ *which satisfies the conditions,*

(a) $\mathbf{f}^{\#}(g) = g \circ \mathbf{f}$, and
(b) $\mathbf{f}^{\#}(g\, dy_{i_1} \cdots dy_{i_k}) = (g \circ \mathbf{f}) \cdot (D\bar{f}_{i_1} \cdots D\bar{f}_{i_k})$, *where* $\bar{f}_{i_1}, \cdots, \bar{f}_{i_k}$ *are the coordinate functions of* $\mathbf{f}$.

If ω is a k-form in $\Omega(\mathbf{W})$, then $\mathbf{f}^{\#}(\omega)$ is a k-form in $\Omega(\mathbf{U})$. Since $\mathbf{f}^{\#}$ is linear, it suffices to establish rules for finding the $\mathbf{f}^{\#}$-image of elementary k-forms. The next result asserts that $\mathbf{f}^{\#}$-images can be obtained using the Jacobian matrix of $\mathbf{f}$.

Proposition 6.3 If $\omega = g\, dy_{i_1} \cdots dy_{i_k}$, then $\mathbf{f}^{\#}(\omega)$ has for the coefficient of $dx_{j_1} \cdots dx_{j_k}$ the product of $g \circ \mathbf{f}$ and

$$\det \begin{bmatrix} (\bar{f}_{i_1})x_{j_1} & \cdots & (\bar{f}_{i_1})x_{j_k} \\ \vdots & & \vdots \\ (\bar{f}_{i_k})x_{j_1} & \cdots & (\bar{f}_{i_k})x_{j_k} \end{bmatrix}$$

It may be observed that the matrix in Proposition 6.3 is a submatrix of $J_{\mathbf{f}}$.

Example 6.4 Given $\omega = y_1 y_2\, dy_1\, dy_2$ and $\mathbf{f}(x_1, x_2, x_3) = \langle x_1{}^2 x_2, x_2 x_3{}^2, x_1 x_3 \rangle$, the Jacobian matrix of $\mathbf{f}$ is

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 2x_1x_2 & x_1{}^2 & 0 \\ 0 & x_3{}^2 & 2x_2x_3 \\ x_3 & 0 & x_1 \end{bmatrix} \end{array}.$$

The coefficient of $dx_1 dx_2$ in $\mathbf{f}^{\#}(\omega)$ is

$$(x_1{}^2 x_2)(x_2 x_3{}^2)\det \begin{bmatrix} 2x_1x_2 & x_1{}^2 \\ 0 & x_3{}^2 \end{bmatrix} = 2x_1{}^3 x_2{}^3 x_3{}^4.$$

Proceeding similarly with the coefficients of $dx_1 dx_3$ and $dx_2\, dx_3$ gives

$$\mathbf{f}^{\#}(\omega) = 2x_1{}^3 x_2{}^3 x_3{}^4 dx_1 dx_2 + 4x_1{}^3 x_2{}^4 x_3{}^3 dx_1 dx_3 + 2x_1{}^4 x_2{}^3 x_3{}^3 dx_2 dx_3.$$

Properties of the induced transformation are given by the next result (see Proofs, exercises 2, 3, 4).

Proposition 6.4

(a) $(\mathbf{f} \circ \mathbf{h})^{\#} = \mathbf{h}^{\#} \circ \mathbf{f}^{\#}$,
(b) $\mathbf{f}^{\#}(\omega_1 \cdot \omega_2) = \mathbf{f}^{\#}(\omega_1) \cdot \mathbf{f}^{\#}(\omega_2)$,
(c) $D\mathbf{f}^{\#} = \mathbf{f}^{\#} D$.

Associated with each n-surface $\mathscr{S}^n$ are two orientation classes of representations. Let $\{\mathbf{f}, \mathbf{I}^n\}$ and $\{\mathbf{f}', (\mathbf{I}^n)'\}$ represent $\mathscr{S}^n$ with $\mathbf{f} = \mathbf{f}' \circ \mathbf{h}$. Then $\mathbf{f}$ and $\mathbf{f}'$ are said to assign the *same orientation* to $\mathscr{S}^n$ if $\det J_{\mathbf{h}} \geq 0$. They assign *opposite orientation* if $\det J_{\mathbf{h}} \leq 0$. More generally this condition on the relating function $\mathbf{h}$ is defined locally in terms of its Jacobian determinant.

Definition of Orientations

If $\{\mathbf{f}, \mathbf{I}^n\}$ *and* $\{\mathbf{f}', (\mathbf{I}^n)'\}$ *represent an n-surface in* $\mathbf{R}^m$, *then they assign the* same orientation *if there exist* $\mathbf{u}$ *and* $\mathbf{u}'$ *respectively interior to* $\mathbf{I}^n$ *and* $(\mathbf{I}^n)'$ *and an* $n \times n$ *matrix* A *such that* $\mathbf{f}(\mathbf{u}) = \mathbf{f}'(\mathbf{u}')$, $\det A > 0$, *and* $J_{\mathbf{f}}(\mathbf{u}) = J_{\mathbf{f}'}(\mathbf{u}')A$. *Otherwise* $\mathbf{f}$ *and* $\mathbf{f}'$ *have* opposite orientation.

An *oriented n-surface* $(\mathscr{S}^n)^0$ is an oriented surface with a representation from one of the two representing classes. We are now ready to define the integral of a differential form on an oriented n-surface $(\mathscr{S}^n)^0$ in $\mathbf{R}^m$. We first define the integral of $g\,dx_1 \cdots dx_n$ on an n-interval $(\mathbf{I}^n)^+$ with positive orientation as

$$\int_{(\mathbf{I}^n)^+} g\,dx_1 \cdots dx_n = \int_{\mathbf{I}^n} g\,dV.$$

This definition offers very little that is new. It becomes significant when used with the following definition (see Figure 15.9).

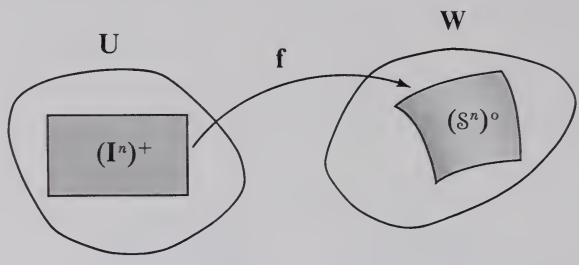


Figure 15.9

Definition of Integral of Differential Form

Let $\mathbf{I}^n$ *be an n-interval in* $\mathbf{U}$ *and* $(\mathscr{S}^n)^0$ *an oriented n-surface contained in* $\mathbf{W}$ *and represented by* $\{\mathbf{f}, \mathbf{I}^n\}$. *If* ω *is an n-form in* $\Omega(\mathbf{W})$, *then the integral of* ω *on* $(\mathscr{S}^n)^0$ *is*

$$\int_{(\mathscr{S}^n)^0} \omega = \int_{(\mathbf{I}^n)^+} \mathbf{f}^{\#}(\omega).$$

Integrals are defined only for k-forms on n-surfaces where $k = n$. The roles of $\mathbf{U}$ and $\mathbf{W}$ are necessary in the construction of the integral, though their presence is usually ignored in specific problems. It is usually stated that ω is an n-form on $(\mathscr{S}^n)^0$, meaning that ω is in $\Omega(\mathbf{W})$ for some open set $\mathbf{W}$ in $\mathbf{R}^m$ containing $(\mathscr{S}^n)^0$.

Example 6.5 Let $\omega = y_1\, dy_4$ on $(\mathscr{S}^1)^0$ represented by $\mathbf{f}(x) = \langle x, 0, x^2, x^3 \rangle$, $0 \le x \le 1$. Then

$$\mathbf{f}^{\#}(\omega) = x(3x^2)dx = 3x^3\, dx \quad \text{and} \quad \int_{(\mathscr{S}^1)^0} \omega = \int_0^1 3x^3\, dx = \tfrac{3}{4}.$$

Some properties of the integral will now be observed.

Proposition 6.5

(a) $\displaystyle\int_{(\mathscr{S}^n)^0} \omega_1 + \omega_2 = \int_{(\mathscr{S}^n)^0} \omega_1 + \int_{(\mathscr{S}^n)^0} \omega_2,$

(b) $\displaystyle\int_{(\mathscr{S}^n)^0} c\omega = c \int_{(\mathscr{S}^n)^0} \omega.$

Proposition 6.6 If $(\mathscr{S}_1{}^n)^0$ and $(\mathscr{S}_2{}^n)^0$ are the same n-surface with opposite orientation, then

$$\int_{(\mathscr{S}_1{}^n)^0} \omega = -\int_{(\mathscr{S}_2{}^n)^0} \omega.$$

The proof of Stokes' differential form equality,

$$\int_{(\mathscr{S}^n)^0} D\omega = \int_{(\partial\mathscr{S}^n)^0} \omega,$$

is not difficult when $(\mathscr{S}^n)^0$ is an oriented n-interval (see Proofs, exercise 1). We shall now indicate a method of extension to an n-surface $(\mathscr{S}^n)^0$ in $\mathbf{R}^m$ which is the $\mathbf{f}$-image of an n-interval $\mathbf{I}^n$, where $\mathbf{f}$ is injective. In this case, the boundary $(\partial\, \mathscr{S}^n)^0$ is the $\mathbf{f}$-image of the boundary of $\mathbf{I}^n$; using Proposition 6.4(c),

$$\begin{aligned} \int_{(\mathscr{S}^n)^0} D\omega &= \int_{(\mathbf{I}^n)^+} \mathbf{f}^{\#}(D\omega) = \int_{(\mathbf{I}^n)^+} D(\mathbf{f}^{\#}\omega) \\ &= \int_{(\partial\mathbf{I}^n)^0} \mathbf{f}^{\#}(\omega) = \int_{(\partial\mathscr{S}^n)^0} \omega. \end{aligned}$$

Therefore, Stokes' equality is valid for n-surfaces in $\mathbf{R}^m$ which are represented by injective functions. With a suitably defined pasting of n-surfaces, this equality can be extended to a much wider class of n-dimensional sets in $\mathbf{R}^m$.

Questions

1. If ω is a k-form, then $\mathbf{f}^{\#}(\omega)$ is either 0 or a ________ -form.
   (a) $k - 1$,
   (b) k,
   (c) $k + 1$.

2. Corresponding to each n-surface are ______ oriented n-surfaces.
   (a) 2,
   (b) 3,
   (c) infinitely many.

3. $\int_{(\mathscr{S}^n)^0}\omega$ is defined only if ω is an ______ -form.
   (a) $n - 1$,
   (b) n,
   (c) $n + 1$.

4. The induced transformation operator (#) does not commute with the ______.
   (a) composition operation,
   (b) differential operator,
   (c) exterior product.

Exercises

1. Find sgn $[Q_1, Q_2]$ if
   (a) $Q_1 = [1, 2, 5, 9]$, $Q_2 = [3, 6]$;
   (b) $Q_1 = [1, 2, 8]$, $Q_2 = [2, 3, 7]$;
   (c) $Q_1 = [1, 3, 5, 7, 11]$, $Q_2 = [2, 4, 6, 8, 9, 10]$.

2. Find:
   (a) $x_1{}^3 dx_1 dx_5\, dx_8 \cdot 2x_1 x_7\, dx_2\, dx_4\, dx_7$,
   (b) $e^{x_1 x_2}\, dx_1 dx_3\, dx_9 \cdot x_4\, dx_5$.

3. Find:
   (a) $D(x_1{}^2 x_4\, dx_2 + x_2\, dx_4)$, (b) $D(x_1{}^3 x_2\, dx_2\, dx_4 + x_3\, dx_3\, dx_4)$.

4. Find $\mathbf{f}^{\#}(\omega)$ for each case below.
   (a) $\omega = y_1 y_3\, dy_1 dy_2$ and $\mathbf{f}(x_1, x_2, x_3) = \langle x_1 - x_2, x_1{}^2, x_2 + x_3\rangle$,
   (b) $\omega = y_2\, dy_1 - y_1 dy_2$ and $\mathbf{f}(x_1, x_2, x_3) = \langle x_1 - x_3, x_2 x_3\rangle$.

5. Find $\int_{(\mathscr{S}^2)^0} \omega$ given that $\omega = y_1 y_2{}^2 dy_1 dy_4$ and $(\mathscr{S}^2)^0$ is represented by

$$\mathbf{f}(x_1, x_2) = \langle x_1 - x_2, x_1 x_2, x_2{}^2, x_1 + 2x_2\rangle, \qquad [0, 1] \times [0, 1].$$

Proofs

1. Prove Stokes' differential-form equality for $\omega = g(y_1, y_2, y_3) dy_1 dy_2$ and $(\mathscr{S}^3)^0$ the positively oriented 3-interval $[0, 1] \times [0, 1] \times [0, 1]$.

2. Prove that $(\mathbf{f} \circ \mathbf{h})^{\#}(\omega) = \mathbf{h}^{\#} \circ \mathbf{f}^{\#}(\omega)$ when all spaces are $\mathbf{R}^2$ and $\omega = dy_1 dy_2$. (*Hint*: Use Proposition 6.3 and $J_{\mathbf{f} \circ \mathbf{h}} = (J_{\mathbf{f}} \circ \mathbf{h})J_{\mathbf{h}}$.)

3. Prove $\mathbf{f}^{\#}(\omega_1 \cdot \omega_2) = \mathbf{f}^{\#}(\omega_1) \cdot \mathbf{f}^{\#}(\omega_2)$ when ω_1 and ω_2 are elementary forms.

4. Prove $D\mathbf{f}^{\#}(\omega) = \mathbf{f}^{\#} D(\omega)$ if $\omega =$
   (a) g,
   (b) dy_j.

Problems

A. *Line Integrals and Surface Integrals*

The *line integral* of a vector function $\mathbf{g}(x, y) = \bar{g}_1(x, y)\mathbf{i} + \bar{g}_2(x, y)\mathbf{j}$ on an oriented curve $\mathscr{C}^\circ$ represented by

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j}, \qquad a \le r \le b,$$

is given by the formula

A.1 $$\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f} = \int_a^b (\mathbf{g} \circ \mathbf{f}) \cdot \frac{d\mathbf{f}}{dr}\, dr,$$

where

$$\mathbf{g} \circ \mathbf{f}(r) = \bar{g}_1(\bar{x}(r), \bar{y}(r))\mathbf{i} + \bar{g}_2(\bar{x}(r), \bar{y}(r))\mathbf{j}$$

and

$$\frac{d\mathbf{f}}{dr} = \frac{d\bar{x}}{dr}\mathbf{i} + \frac{d\bar{y}}{dr}\mathbf{j}.$$

1. Find $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ if $\mathbf{g}(x, y) = xy\mathbf{i} - y\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j}$, $0 \le r \le 2$, as follows:
   (a) Find $\mathbf{g} \circ \mathbf{f}$.
   (b) Find $d\mathbf{f}/dr$.
   (c) Evaluate $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f} = \int_0^2 (\mathbf{g} \circ \mathbf{f}) \cdot \frac{d\mathbf{f}}{dr}\, dr$.

2. Find $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ given the following conditions.
   (a) $\mathbf{g}(x, y) = x^2y\mathbf{i} + y^2\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = r^2\mathbf{i} + r^3\mathbf{j}$, $0 \le r \le 1$;
   (b) $\mathbf{g}(x, y) = (x - y)\mathbf{i} + (3x + 2y)\mathbf{j}$ and $\mathscr{C}^\circ$ is the ellipse $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$.

A similar formula holds for the line integral of a vector function

$$\mathbf{g}(x, y, z) = \bar{g}_1(x, y, z)\mathbf{i} + \bar{g}_2(x, y, z)\mathbf{j} + \bar{g}_3(x, y, z)\mathbf{k}$$

on an oriented curve $\mathscr{C}^o$ in space represented by

$$\mathbf{f}(r) = \bar{x}(r)\mathbf{i} + \bar{y}(r)\mathbf{j} + \bar{z}(r)\mathbf{k}, \qquad a \le r \le b.$$

Again we have

A.2 $$\int_{\mathscr{C}^o} \mathbf{g} \cdot d\mathbf{f} = \int_a^b (\mathbf{g} \circ \mathbf{f}) \cdot \frac{d\mathbf{f}}{dr}\, dr.$$

In A.2,

$$\mathbf{g} \circ \mathbf{f}(r) = \bar{g}_1(\bar{x}(r), \bar{y}(r), \bar{z}(r))\mathbf{i} + \bar{g}_2(\bar{x}(r), \bar{y}(r), \bar{z}(r))\mathbf{j} + \bar{g}_3(\bar{x}(r), \bar{y}(r), \bar{z}(r))\mathbf{k}$$

and

$$\frac{d\mathbf{f}}{dr} = \frac{d\bar{x}}{dr}\mathbf{i} + \frac{d\bar{y}}{dr}\mathbf{j} + \frac{d\bar{z}}{dr}\mathbf{k}.$$

3. Find $\int_{\mathscr{C}^o} \mathbf{g} \cdot d\mathbf{f}$ given that
   (a) $\mathbf{g}(x, y, z) = xz\mathbf{i} + x^2y\mathbf{j} + yz\mathbf{k}$ and $\mathscr{C}^o$ is represented by
   $$\mathbf{f}(r) = r\mathbf{i} + r^2\mathbf{j} + r^3\mathbf{k}, \quad 0 \le r \le 1;$$
   (b) $\mathbf{g}(x, y, z) = xy\mathbf{i} - z\mathbf{k}$ and $\mathscr{C}^o$ is represented by
   $$\mathbf{f}(r) = r\mathbf{i} - r^2\mathbf{j} + (3r - 1)\mathbf{k}, \quad 0 \le r \le 1.$$

The *surface integral* of a vector function

$$\mathbf{g}(x, y, z) = \bar{g}_1(x, y, z)\mathbf{i} + \bar{g}_2(x, y, z)\mathbf{j} + \bar{g}_3(x, y, z)\mathbf{k}$$

on an oriented surface $\mathscr{S}^o$ represented by

$$\mathbf{f}(r, s) = \bar{x}(r, s)\mathbf{i} + \bar{y}(r, s)\mathbf{j} + \bar{z}(r, s)\mathbf{k}, \mathbf{I}^2,$$

is given by the formula

A.3 $$\int_{\mathscr{S}^o} \mathbf{g} \cdot \mathbf{n}\, dA = \int_{\mathbf{I}^2} (\mathbf{g} \circ \mathbf{f}) \cdot (\mathbf{f}_r \times \mathbf{f}_s)\, dA.$$

4. Given $\mathbf{g}(x, y, z) = x\mathbf{i} + (3y + z)\mathbf{j} + (2z - y)\mathbf{k}$ on the oriented surface $\mathscr{S}^o$ represented by $\mathbf{f}(r, s) = rs\mathbf{i} + (3r - s)\mathbf{j} + (r + 2s)\mathbf{k}$, $[0, 1] \times [0, 2]$,
   (a) find $\mathbf{g} \circ \mathbf{f}$,
   (b) find $\mathbf{f}_r \times \mathbf{f}_s$,
   (c) evaluate $\int_{\mathscr{S}^o} \mathbf{g} \cdot \mathbf{n}\, dA = \int_0^1 dr \int_0^2 (\mathbf{g} \circ \mathbf{f}) \cdot (\mathbf{f}_r \times \mathbf{f}_s)\, ds$.
5. Find $\int_{\mathscr{S}^o} \mathbf{g} \cdot \mathbf{n}\, dA$ for each case below.
   (a) $g(x, y, z) = z\mathbf{i} + (x^2 + 2y)\mathbf{j}$ and $\mathscr{S}^o$ is represented by
   $$\mathbf{f}(r, s) = rs\mathbf{i} + r\mathbf{j} + s^2\mathbf{k}, \quad [0, 1] \times [0, 1].$$
   (b) $g(x, y, z) = (x + y)\mathbf{i} + z\mathbf{k}$ and $\mathscr{S}^o$ is represented by
   $$\mathbf{f}(r, s) = r\mathbf{i} + (r - s)\mathbf{j} + rs\mathbf{k}, \quad [0, 1] \times [0, 2].$$

6. Find $\int_{\mathscr{C}^\circ} \mathbf{g} \cdot d\mathbf{f}$ when
   (a) $\mathbf{g}(x, y) = xy\mathbf{i} + y^2\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = (r^2 + 1)\mathbf{i} + 2r\mathbf{j}$, $0 \le r \le 1$;
   (b) $g(x, y) = x\mathbf{i} + y^2\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by
   $$\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad 0 \le \theta \le \pi/2;$$
   (c) $\mathbf{g}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} - xz\mathbf{k}$ and $\mathscr{C}^\circ$ is represented by
   $$\mathbf{f}(r) = (r + 2)\mathbf{i} + 3r^2\mathbf{j} - r\mathbf{k}, \quad 0 \le r \le 1.$$
7. Find $\int_{\mathscr{S}^\circ} \mathbf{g} \cdot \mathbf{n}\, dA$ given each of the following sets of conditions.
   (a) $\mathbf{g}(x, y, z) = (x + y)\mathbf{i} + z\mathbf{j} + \mathbf{k}$ and $\mathscr{S}^\circ$ is represented by
   $$\mathbf{f}(r, s) = (r - s)\mathbf{i} + (r + 3s)\mathbf{j} + rs\mathbf{k}, \quad [0, 1] \times [0, 2].$$
   (b) $\mathbf{g}(x, y, z) = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$ and $\mathscr{S}^\circ$ is represented by
   $$\mathbf{f}(r, s) = r\mathbf{i} + r^2\mathbf{j} + s^2\mathbf{k}, \quad [0, 1] \times [0, 2].$$

B. *Algebra of Differential Forms; Differential Operator*

Differential forms in $\mathbf{R}^2$ are written

$g(x, y)$	0-form
$g_1(x, y)\, dx + g_2(x, y)\, dy$	1-form
$g(x, y)\, dxdy$	2-form

The 0-forms are manipulated as real functions; in other cases addition is defined for the same size forms in a natural way. Thus,

B.1 (a) $(g_1dx + g_2\, dy) + (g_1'dx + g_2'dy) = (g_1 + g_1')\, dx + (g_2 + g_2')\, dy$,
(b) $g\, dxdy + g'dxdy = (g + g')\, dxdy$.

Scalar multiplication is also defined in the expected manner.

B.2 (a) $c(g_1dx + g_2\, dy) = (cg_1)\, dx + (cg_2)\, dy$,
(b) $c(g\, dxdy) = (cg)\, dxdy$.

1. Perform the operations indicated below.
   (a) $(xy\, dx + x^3y\, dy) + (xy^2dx - xy^2dy)$,
   (b) $xy^3\, dxdy + x^2y\, dxdy$,
   (c) $4(xy\, dx + y^2dy)$,
   (d) $-2(e^{xy}dxdy)$,
   (e) $3(x\, dx + y\, dy) - 2(x^2dx - y^2dy)$,
   (f) $3(x^2dxdy) - x^3dxdy$.

Differential forms in $\mathbf{R}^3$ are written

$g(x, y, z)$	0-form,
$g_1(x, y, z)dx + g_2(x, y, z)dy + g_3(x, y, z)dz$	1-form,
$g_1(x, y, z)dydz + g_2(x, y, z)dzdx + g_3(x, y, z)dxdy$	2-form,
$g(x, y, z)dxdydz$	3-form.

Operations of addition and scalar multiplication are similar to the $\mathbf{R}^2$ case.

2. Perform the indicated operations.

   (a) $3(x\,dx - xz\,dy + yz\,dz) + (z\,dx + yz\,dy - xz\,dz)$,
   (b) $(x\,dydz - dzdx + xz\,dxdy) + 2(y\,dydz + dzdx - x^2dxdy)$,
   (c) $3(2xz\,dxdydz) + 4(xy\,dxdydz)$.

The (*exterior*) *product operation* $(\cdot)$ on differential forms satisfies the following conditions.

B.3 (a) The exterior product is associative and distributive about addition,

(b) $dx \cdot dx = dy \cdot dy = dz \cdot dz = 0$;
(c) $dy \cdot dz = -dz \cdot dy = dydz$,
$dz \cdot dx = -dx \cdot dz = dzdx$,
$dx \cdot dy = -dy \cdot dx = dxdy$;
(d) $dx \cdot (dy \cdot dz) = -dy \cdot (dx \cdot dz) = dy \cdot (dz \cdot dx) =$
$-dz \cdot (dy \cdot dx) = dz \cdot (dx \cdot dy) = -dx \cdot (dz \cdot dy) = dxdydz$.

Thus the product of dx, dy, and dz in any order is $dxdydz$ or $-dxdydz$, depending on whether the order is an even or odd permutation of dx, dy, dz. Coefficient functions of differential forms are combined with the ordinary product when the exterior product of forms is obtained.

3. Perform indicated operations.

   (a) $xy\,dx \cdot y^2dy$, (b) $x^2dy \cdot xy\,dx$,
   (c) $x\,dx \cdot (y^2dx + xy\,dy)$, (d) $xy \cdot (x\,dx + y\,dy)$,
   (e) $x\,dx \cdot y^2dxdy$, (f) $xy \cdot y^3dxdy$,
   (g) $(xy\,dx - z^2dz) \cdot xz\,dxdy$, (h) $xy^2dxdz \cdot (z\,dx + y^3z\,dy)$,
   (i) $(x\,dx - y\,dz) \cdot (y^2dx + x^3y\,dy)$,
   (j) $x^2dxdy \cdot yz\,dxdz$, (k) $xz \cdot (xy\,dxdydz)$.

The (*exterior*) *differential operator* D on differential forms assigns to each k-form either the 0-form or a $(k + 1)$-form. For the $\mathbf{R}^2$ case it satisfies the following properties.

B.4 (a) $Dg = g_x\,dx + g_y\,dy$,
(b) $D(g_1\,dx + g_2\,dy) = Dg_1 \cdot dx + Dg_2 \cdot dy$,
(c) $D(g\,dxdydz) = 0$.

4. Find $D\omega$ if $\omega =$

   (a) x^3y, (b) $x^3y\,dx$, (c) $xy^3dx + x\,dy$,
   (d) xe^ydxdy.

The operator D satisfies for the $\mathbf{R}^3$ case,

B.5 (a) $Dg = g_x\,dx + g_y\,dy + g_z\,dz$,
(b) $D(g_1 dx + g_2\,dy + g_3\,dz) = Dg_1 \cdot dx + Dg_2 \cdot dy + Dg_3 \cdot dz$,
(c) $D(g_1 dydz + g_2\,dzdx + g_3\,dxdy) = Dg_1 \cdot dydz + Dg_2 \cdot dzdx + Dg_3 \cdot dxdy$.

5. Find $D\omega$ if $\omega =$

(a) xyz^2, (b) $xy\,dx + yz\,dz$,
(c) $x^2 dxdy + yz\,dxdz$, (d) $xe^{yz}dxdydz$.

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6. Perform the indicated operations.

(a) $x^2 \cdot (xy\,dx + y\,dy)$,
(b) $x\,dx \cdot [(x\,dx + y\,dy) + (x\,dx - y^2 dy)]$,
(c) $(x\,dxdy + y\,dxdz) \cdot (x^3 dxdy - y\,dydz)$,
(d) $3(x\,dy \cdot y\,dx) + x^2 dxdy$,
(e) $[x\,dy \cdot (y\,dxdz)] \cdot x^2$,
(f) $(x\,dx - y\,dz) \cdot (y\,dxdz + z\,dxdy)$.

7. For the $\mathbf{R}^2$ case, find $D\omega$ if $\omega =$

(a) $\cos x\,dx + xy\,dy$, (b) $x \cos y$,
(c) $x \ln y\,dxdy$.

8. For the $\mathbf{R}^3$ case find $D\omega$ if $\omega =$

(a) x^2yz^3, (b) $y^2 dx + xy\,dz$,
(c) $y\,dxdz + z^2 dydz$, (d) $xe^z dxdydz$.

C. *Fundamental Theorem for Line Integrals; Green's Theorem*

The *fundamental theorem for line integrals* says that for a function $g(x, y)$ on an oriented curve $\mathscr{C}^o$ represented by $\mathbf{f}(r)$, $a \le r \le b$,

C.1 $\int_{\mathscr{C}^o} \nabla g \cdot d\mathbf{f} = g(\mathbf{f}(b)) - g(\mathbf{f}(a))$.

1. Verify C.1 for

(a) $g(x, y) = x^2y$ on the parabolic arc $y = x^2$ oriented from $(1, 1)$ to $(2, 4)$, and
(b) $g(x, y) = x$ on the upper half-plane portion of the circle $x^2 + y^2 = 4$ oriented from $(2, 0)$ to $(-2, 0)$.

A vector function $\mathbf{h}$ is *exact* if $\mathbf{h} = \nabla g$ for some g. Thus, C.1 is applicable to finding the line integral of an exact function $\mathbf{h}$ provided a suitable g can be found. The following property is pertinent.

C.2 If $\mathbf{h}(x, y) = \bar{h}_1(x, y)\mathbf{i} + \bar{h}_2(x, y)\mathbf{j}$ is exact, then $(\bar{h}_2)_x - (\bar{h}_1)_y = 0$.

If $(\bar{h}_2)_x - (\bar{h}_1)_y = 0$ and $\mathbf{h}(x, y)$ is exact, then the following procedure will frequently yield a function $g(x, y)$ such that $g_x\mathbf{i} + g_y\mathbf{j} = \nabla g = \mathbf{h} = \bar{h}_1\mathbf{i} + \bar{h}_2\mathbf{j}$.

C.3 (I) From $g_x = \bar{h}_1$, set $g(x, y) = \int \bar{h}_1(x, y)\,dx$ and by partial integration with respect to x obtain an equation $g(x, y) = G(x, y) + H(y)$, where $G(x, y)$ is known and $H(y)$ is to be determined.

(II) From $g_y = \bar{h}_2$ set $G_y + dH/dy = \bar{h}_2$ and by integration obtain $H(y)$.

2. Given $\mathbf{h}(x, y) = 2xye^{x^2y}\mathbf{i} + x^2e^{x^2y}\mathbf{j}$, find $g(x, y)$ such that $\nabla g = \mathbf{h}$.
3. Verify that $(\bar{h}_2)_x = (\bar{h}_1)_y$ and find g such that $\nabla g = \mathbf{h}$ in each case.
   (a) $\mathbf{h}(x, y) = (2xy + 3)\mathbf{i} + (x^2 + 4)\mathbf{j}$,
   (b) $\mathbf{h}(x, y) = \sin y\,\mathbf{i} + (2y + x\cos y)\mathbf{j}$.
4. Find $\int_{\mathscr{C}^o} \mathbf{h} \cdot d\mathbf{f}$ by C.1 given the following conditions.
   (a) $\mathbf{h}(x, y) = x^3\mathbf{i} + y^3\mathbf{j}$ and $\mathscr{C}^o$ is the line segment from $(2, 3)$ to $(1, -2)$.
   (b) $\mathbf{h}(x, y) = y\sin xy\,\mathbf{i} + x\sin xy\,\mathbf{j}$ and $\mathscr{C}^o$ is any curve from $(1, 0)$ to $(3, \pi/2)$.

Next let $\mathscr{R}$ be a plane region consisting of a simple closed oriented curve $\mathscr{C}^o$ together with its interior. If $\mathbf{f}(r)$, $a \le r \le b$, represents $\mathscr{C}^o$ so that increasing values of r cause $\mathbf{f}(r)$ to make a counterclockwise turn as it moves along $\mathscr{C}^o$, then Green's theorem says that for a function $\mathbf{g}(x, y) = \bar{g}_1\mathbf{i} + \bar{g}_2\mathbf{j}$ with domain $\mathscr{R}$,

C.4 $\int_{\mathscr{R}} [(\bar{g}_2)_x - (\bar{g}_1)_y]\,dA = \int_{\mathscr{C}} \mathbf{g} \cdot d\mathbf{f}$.

The boundary of many elementary regions is not a smooth curve, but is instead composed of finitely many smooth curves joined end to end. Green's theorem is valid for such regions, and the line integral on the boundary may be found by adding the integrals on the separate curves. All representations must, however, maintain a consistent, counterclockwise oriented direction.

5. Verify Green's theorem for each of the specific cases below.
   (a) $\mathbf{g}(x, y) = xy\mathbf{i} - x\mathbf{j}$ and $\mathscr{R}$ is the disk $x^2 + y^2 \le 1$,
   (b) $\mathbf{g}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ and $\mathscr{R} = [0, 1] \times [0, 1]$,
   (c) $\mathbf{g}(x, y) = xy\mathbf{i} - y^3\mathbf{j}$ and $\mathscr{R}$ is the triangular region with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$.

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6. Verify $\int_{\mathscr{C}^o} \nabla g \cdot d\mathbf{f} = g(\mathbf{f}(b)) - g(\mathbf{f}(a))$ in both cases given.
   (a) $g(x, y) = x - y$ and $\mathscr{C}^o$ is represented by $\mathbf{f}(r) = r^2\mathbf{i} - r^3\mathbf{j}$, $0 \le r \le 1$.
   (b) $g(x, y) = x^2y$ and $\mathscr{C}^o$ is the semicircle $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le \pi$.

7. Evaluate $\int_{\mathscr{C}^\circ} \mathbf{h} \cdot d\mathbf{f}$ using the fundamental theorem for line integrals, given that

   (a) $\mathbf{h}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = (r^2 - 3)\mathbf{i} + (2r - 1)\mathbf{j}$, $0 \leq r \leq 2$;

   (b) $\mathbf{h}(x, y) = (e^{y^2} + 2)\mathbf{i} + (2xye^{y^2} - 14y)\mathbf{j}$ and $\mathscr{C}^\circ$ is represented by $\mathbf{f}(r) = (r - 1)\mathbf{i} + 4r^3\mathbf{j}$, $-1 \leq r \leq 0$.

8. Verify Green's theorem given that

   (a) $\mathbf{g}(x, y) = y\mathbf{i} + 3x\mathbf{j}$ and $\mathscr{R}$ is the region $x^2/1 + y^2/4 \leq 1$, or

   (b) $\mathbf{g}(x, y) = xy\mathbf{i} + y^2\mathbf{j}$ and $\mathscr{R}$ is the region bounded by $y = x^2$ and $y = x$.

Appendix of Proofs (Vector Algebra)

Exchange Theorem (Chapter II)

Let **B** *be a basis of a finite dimensional space* **V** *and* **T** *an independent set in* **V**. *Then* **T** *has no more elements than* **B**, *and may be substituted into* **B** *for some equal size set to obtain a new basis of* **V**.

The idea of the proof will be presented for the case

$$\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \quad \text{and} \quad \mathbf{T} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}.$$

The technique we use involves successive substitutions of one element at a time from **T** into **B**. After each substitution, it will be shown that the result is an independent set which spans **V**, and hence, is a basis of **V**. For the first substitution we observe that Sp $\mathbf{B} = \mathbf{V}$ and $\mathbf{v}_1 \neq \mathbf{0}$, since **T** is independent; hence, there are constants c_1, c_2, c_3, and c_4, not all zero, such that

$$\mathbf{v}_1 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4.$$

We assume the case $c_1 \neq 0$, which then implies that

$$\mathbf{u}_1 = \frac{1}{c_1}(\mathbf{v}_1 - c_2\mathbf{u}_2 - c_3\mathbf{u}_3 - c_4\mathbf{u}_4)$$

is a linear combination of $\mathbf{B}_1 = \{\mathbf{v}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. It will now be shown that $\mathbf{B}_1$ is a basis of $\mathbf{V}$. If

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$$

is an arbitrary element of $\mathbf{V}$, then substitution of $\mathbf{u}_1$ for its linear combination of $\mathbf{B}_1$ gives $\mathbf{v}$ as a linear combination of $\mathbf{B}_1$, and hence, Sp $\mathbf{B}_1 = \mathbf{V}$. In order to show that $\mathbf{B}_1$ is independent, we assume

$$b_1\mathbf{v}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + b_4\mathbf{u}_4 = \mathbf{0},$$

and we intend to show that each $b_i = 0$. Substitution of $\mathbf{v}_1$ as a linear combination of $\mathbf{B}$ gives

$$(b_1c_1)\mathbf{u}_1 + (b_1c_2 + b_2)\mathbf{u}_2 + (b_1c_3 + b_3)\mathbf{u}_3 + (b_1c_4 + b_4)\mathbf{u}_4 = \mathbf{0}.$$

Since $\mathbf{B}$ is independent, all coefficients of the $\mathbf{u}_i$ are zero. From $b_1c_1 = 0$ and $c_1 \neq 0$ we obtain $b_1 = 0$. It then easily follows that $b_2 = b_3 = b_4 = 0$, and hence, $\mathbf{B}_1$ is independent. Therefore, one element of $\mathbf{T}$ has been substituted into $\mathbf{B}$ to obtain a basis $\mathbf{B}_1$ of $\mathbf{V}$. The argument for subsequent substitutions is essentially the same. For example, we next let

$$\mathbf{v}_2 = c_1'\mathbf{v}_1 + c_2'\mathbf{u}_2 + c_3'\mathbf{u}_3 + c_4'\mathbf{u}_4$$

and observe that $c_2' = c_3' = c_4' = 0$ is impossible, since $\mathbf{v}_2 = c_1'\mathbf{v}_1$ contradicts the independence of $\mathbf{T}$. Proceeding as in the first stage, $\mathbf{v}_2$ then may be substituted for some element of $\mathbf{B}_1$ (other than $\mathbf{v}_1$) to obtain a new basis $\mathbf{B}_2$. The case in which m is greater than four cannot occur, for the foregoing process would yield after four stages a four-element proper subset $\mathbf{T}_4$ of $\mathbf{T}$ which is a basis of $\mathbf{V}$. Then a remaining element of $\mathbf{T}$ is a linear combination of $\mathbf{T}_4$, and this contradicts the independence of $\mathbf{T}$ (by Proposition 2.1 in Chapter II).

> **Proposition 3.5** (Chapter VII) The augmenting of an $m \times n$ matrix A, $n < m$, by a column vector $\mathbf{u}$ causes $|A|$ to be multiplied by $|\mathbf{u}^\perp|$, where $\mathbf{u}^\perp$ is the component of $\mathbf{u}$ orthogonal to the span space of the column vectors of A.

A proof will be indicated for the case $n = 2$, $m = 4$, and $A = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$. Letting $A_1 = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u} \rangle$, we must show that $|A_1| = |\mathbf{u}^\perp|\,|A|$, where $\mathbf{u}^\perp$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{S} = \mathrm{Sp}(\{\mathbf{u}_1, \mathbf{u}_2\})$. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is dependent or $\mathbf{u} = \mathbf{0}$, then each side of the desired equality is 0. Otherwise we extend $\{\mathbf{u}^\perp\}$ to an orthogonal basis $\{\mathbf{u}^\perp, \mathbf{v}\}$ of $\mathbf{S}^\perp$, where $|\mathbf{v}| = 1$. Letting

$$B = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}, \mathbf{v} \rangle \quad \text{and} \quad C = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}^\perp, \mathbf{v} \rangle,$$

then $\mathbf{u}^\perp \cdot \mathbf{u}_1 = \mathbf{u}^\perp \cdot \mathbf{u}_2 = \mathbf{v} \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{u}_2 = \mathbf{u}^\perp \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{v} = 1$ implies the following matrix forms.

$$B^*B = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u} & 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u} & 0 \\ \mathbf{u} \cdot \mathbf{u}_1 & \mathbf{u} \cdot \mathbf{u}_2 & \mathbf{u} \cdot \mathbf{u} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1{}^*A_1 = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u} \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u} \\ \mathbf{u} \cdot \mathbf{u}_1 & \mathbf{u} \cdot \mathbf{u}_2 & \mathbf{u} \cdot \mathbf{u} \end{bmatrix},$$

$$C^*C = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & 0 & 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & 0 & 0 \\ 0 & 0 & |\mathbf{u}^\perp|^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A^*A = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 \end{bmatrix}.$$

From the definition of determinant we can easily see that the following equations are true.

$$\det B^*B = \det A_1{}^*A_1,$$
$$\det C^*C = |\mathbf{u}^\perp|^2 \det A^*A.$$

Also, since $\mathbf{u}' = \mathbf{u} - \mathbf{u}^\perp$ is in $\mathbf{S}$ (by Proposition 5.2 in Chapter III), it follows using elementary column operations of type II that

$$\begin{aligned} \det B &= \det \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}^\perp + \mathbf{u}', \mathbf{v} \rangle \\ &= \det \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}^\perp, \mathbf{v} \rangle \\ &= \det C. \end{aligned}$$

Therefore, we have, as desired,

$$\begin{aligned} |A_1| &= \sqrt{\det A_1{}^*A_1} = \sqrt{\det B^*B} = |\det B| = |\det C| \\ &= |\mathbf{u}^\perp| \sqrt{\det A^*A} = |\mathbf{u}^\perp|\,|A|. \end{aligned}$$

Proposition 2.1 (Chapter VIII) If A is a symmetric $n \times n$ matrix, then all roots of the characteristic polynomial of A are real.

Preliminary to the proof we establish the following result.

Lemma If b and c are real numbers and λ_o is a real or complex root of $\det(A - \lambda I) = 0$, then $\lambda_o{}^2 + b\lambda_o + c$ is a root of

$$\det[(A^2 + bA + cI) - \lambda I] = 0.$$

Here I denotes the identity $n \times n$ matrix. Our proof assumes an extension of the determinant definition to matrices with complex number entries and the retention of the property that det $BA =$ det B det A. The desired conclusion follows from the equality chain,

$$\begin{aligned}
&\det[(A^2 + bA + cI) - (\lambda_o^2 + b\lambda_o + c)I] \\
&\quad = \det[(A^2 - \lambda_o^2 I) + b(A - \lambda_o I)] \\
&\quad = \det[(A - \lambda_o I)(A + \lambda_o I + bI)] \\
&\quad = \det(A - \lambda_o I) \det(A + \lambda_o I + bI) \\
&\quad = 0.
\end{aligned}$$

For the proof of Proposition 2.1, let λ_o be a root of the characteristic polynomial of A. Then λ_o satisfies an equation $\lambda_o^2 + b\lambda_o + c = 0$, where b and c are real numbers. In order to show λ_o is real, it suffices to prove that $b^2 - 4c \geq 0$. By the lemma, $0 = \lambda_o^2 + b\lambda_o + c$ is a root of

$$\det[(A^2 + bA + cI) - \lambda I] = 0,$$

and hence,

$$\det(A^2 + bA + cI) = 0.$$

We now let Mat $f = A$; since A is symmetric, it follows that f is self-adjoint. Letting id denote the identity function on $\mathbf{R}^n$ and $f^2 = f \circ f$, we have

$$\text{Mat}(f^2 + bf + c\,\text{id}) = A^2 + bA + cI.$$

Since det $(A^2 + bA + cI) = 0$, there exists a nonzero vector $\mathbf{u}$ such that

$$(f^2 + bf + c\,\text{id})(\mathbf{u}) = \mathbf{0}.$$

Taking the dot product of both sides of this equality with $\mathbf{u}$ and applying $f^2(\mathbf{u}) \cdot \mathbf{u} = f(\mathbf{u}) \cdot f(\mathbf{u})$ (since f is self-adjoint), give

$$f(\mathbf{u}) \cdot f(\mathbf{u}) + bf(\mathbf{u}) \cdot \mathbf{u} = -(c\mathbf{u}) \cdot \mathbf{u}.$$

Completing the square on the left side yields

$$\left|f(\mathbf{u}) + \frac{b}{2}\mathbf{u}\right| = \frac{(b^2 - 4c)}{4}|\mathbf{u}|^2.$$

The desired equality $b^2 - 4c \geq 0$ follows.

Proposition 2.3 (Chapter VIII) If A is a symmetric matrix, then the eigenvectors of A contain an orthonormal basis of $\mathbf{R}^n$.

From Proposition 1.1 in Chapter VIII we may associate with each eigenvalue λ of A its subspace $\mathbf{S}_\lambda$ of eigenvectors. This subspace $\mathbf{S}_\lambda$ has an orthonormal basis $\mathbf{B}_\lambda$ by the Gram-Schmidt theorem, and the union $\mathbf{B}$ of the $\mathbf{B}_\lambda$'s for all eigenvalues λ of A yields an orthonormal set by Proposition 2.2 in Chapter VIII.

We let $\mathbf{S} = \mathrm{Sp}\,\mathbf{B}$, and seek to show that $\mathbf{S} = \mathbf{R}^n$. If not, then there exists a nonzero vector $\mathbf{w}$ in $\mathbf{S}^\perp$. Setting Mat $f = A$, it will now also be shown that $f(\mathbf{w})$ is in $\mathbf{S}^\perp$. By Proposition 5.2 in Chapter III, there are vectors $\mathbf{u}$ in $\mathbf{S}$ and $\mathbf{v}$ in $\mathbf{S}^\perp$ such that $f(\mathbf{w}) = \mathbf{u} + \mathbf{v}$. Since f is self-adjoint and $\mathbf{w}$ is in $\mathbf{S}^\perp$, it is seen for each $\mathbf{u}'$ in $\mathbf{B}$ that

$$f(\mathbf{w}) \cdot \mathbf{u}' = f(\mathbf{u}') \cdot \mathbf{w} = 0.$$

Since $\mathbf{u}$ is a linear combination of $\mathbf{B}$, it follows that $0 = f(\mathbf{w}) \cdot \mathbf{u} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u}$ and hence $\mathbf{u} = \mathbf{0}$. Therefore, $f(\mathbf{w}) = \mathbf{v}$ is in $\mathbf{S}^\perp$. This implies that we may consider the restriction of f to $\mathbf{S}^\perp$ as a self-adjoint function from $\mathbf{S}^\perp$ to $\mathbf{S}^\perp$. Using the correspondence between self-adjoint operators and symmetric matrices, it may be deduced from Proposition 2.1 that there is a real number λ_0 and associated eigenvector $\mathbf{u}_0$ in $\mathbf{S}^\perp$, $\mathbf{u}_0 \neq \mathbf{0}$, such that $f(\mathbf{u}_0) = \lambda_0 \mathbf{u}_0$. This implies $\mathbf{u}_0$ is in $\mathbf{S}_{\lambda_0}$, which has the basis $\mathbf{B}_{\lambda_0}$, where $\mathbf{B}_{\lambda_0}$ is a subset of $\mathbf{B}$. Therefore, $\mathbf{u}_0$ is in both $\mathbf{S}$ and $\mathbf{S}^\perp$, and hence, $\mathbf{u}_0 = \mathbf{0}$. This gives a contradiction, and the supposition $\mathbf{S} \neq \mathbf{R}^n$ is proved false.

Appendix of Proofs (Vector Calculus)

Approximation Theorem for Scalar Functions (Chapter X)

If f is of class C^1, then

$$\lim_{\mathbf{u}\to\mathbf{u}_0} \frac{f(\mathbf{u}) - f(\mathbf{u}_o) - f_{\mathbf{u}-\mathbf{u}_0}(\mathbf{u}_o)}{|\mathbf{u} - \mathbf{u}_o|} = 0.$$

A proof will be made for $f(x, y)$. If $\varepsilon > 0$, then by the continuity of f_x and f_y there exist δ_1 and δ_2 such that

(I) if $|\mathbf{u} - \mathbf{u}_o| < \delta_1$, then $|f_x(\mathbf{u}) - f_x(\mathbf{u}_o)| < \dfrac{\varepsilon}{2}$,

(II) if $|\mathbf{u} - \mathbf{u}_o| < \delta_2$, then $|f_y(\mathbf{u}) - f_y(\mathbf{u}_o)| < \dfrac{\varepsilon}{2}$.

Let δ be the minimum of δ_1, δ_2, and let $\mathbf{u}_1$ satisfy $|\mathbf{u}_1 - \mathbf{u}_o| < \delta$. By the n-variable mean value theorem, there exists $\mathbf{u}_o{}^*$ in $[\mathbf{u}_o, \mathbf{u}_1]$ such that $f_{\mathbf{u}_1-\mathbf{u}_0}(\mathbf{u}_o{}^*) = f(\mathbf{u}_1) - f(\mathbf{u}_o)$. Therefore,

$$\begin{aligned}
&|f(\mathbf{u}_1) - f(\mathbf{u}_o) - f_{\mathbf{u}_1-\mathbf{u}_0}(\mathbf{u}_o)| \\
&\quad = |f_{\mathbf{u}_1-\mathbf{u}_0}(\mathbf{u}_o{}^*) - f_{\mathbf{u}_1-\mathbf{u}_0}(\mathbf{u}_o)| \\
&\quad = |\nabla f(\mathbf{u}_o{}^*) \cdot (\mathbf{u}_1 - \mathbf{u}_o) - \nabla f(\mathbf{u}_o) \cdot (\mathbf{u}_1 - \mathbf{u}_o)| \\
&\quad = |(\nabla f(\mathbf{u}_o{}^*) - \nabla f(\mathbf{u}_o)) \cdot (\mathbf{u}_1 - \mathbf{u}_o)| \\
&\quad \leq |\nabla f(\mathbf{u}_o{}^*) - \nabla f(\mathbf{u}_o)|\,|\mathbf{u}_1 - \mathbf{u}_o| \\
&\quad = |\langle f_x(\mathbf{u}_o{}^*) - f_x(\mathbf{u}_o), f_y(\mathbf{u}_o{}^*) - f_y(\mathbf{u}_o)\rangle|\,|\mathbf{u}_1 - \mathbf{u}_o| \\
&\quad < \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right)|\mathbf{u}_1 - \mathbf{u}_o| \\
&\quad = \varepsilon|\mathbf{u}_1 - \mathbf{u}_o|.
\end{aligned}$$

This gives the desired equality for $|\mathbf{u}_1 - \mathbf{u}_o| < \delta$,

$$\frac{|f(\mathbf{u}_1) - f(\mathbf{u}_o) - f_{\mathbf{u}_1-\mathbf{u}_o}(\mathbf{u}_o)|}{|\mathbf{u}_1 - \mathbf{u}_o|} < \varepsilon.$$

Chain Rule for Vector Functions (Chapter XI)

$(\mathbf{g} \circ \mathbf{f})_{\mathbf{v}_0}(\mathbf{u}_o) = \mathbf{g}_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o))$, *where* $\mathbf{w}_o = \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$.

We first prove the case (where g is a scalar function) for

$$\mathbf{f}(r, s) = \langle \bar{x}(r, s), \bar{y}(r, s)\rangle$$

and $g(x, y)$. Then using the chain rule theorem,

$$\begin{aligned}
(g \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) &= \mathbf{v}_o \cdot \nabla(g \circ \mathbf{f})(\mathbf{u}_o) \\
&= \mathbf{v}_o \cdot \langle (g \circ \mathbf{f})_r(\mathbf{u}_o), (g \circ \mathbf{f})_s(\mathbf{u}_o)\rangle \\
&= \mathbf{v}_o \cdot \langle (g_x \circ \mathbf{f})\bar{x}_r + (g_y \circ \mathbf{f})\bar{y}_r, (g_x \circ \mathbf{f})\bar{x}_s + (g_y \circ \mathbf{f})\bar{y}_s\rangle|_{\mathbf{u}_o} \\
&= \mathbf{v}_o \cdot [(g_x \circ \mathbf{f})\langle \bar{x}_r, \bar{x}_s\rangle|_{\mathbf{u}_o}] + \mathbf{v}_o \cdot [(g_y \circ \mathbf{f})\langle \bar{y}_r, \bar{y}_s\rangle|_{\mathbf{u}_o}] \\
&= (\mathbf{v}_o \cdot \nabla\bar{x})(g_x \circ \mathbf{f})|_{\mathbf{u}_o} + (\mathbf{v}_o \cdot \nabla\bar{y})(g_y \circ \mathbf{f})|_{\mathbf{u}_o} \\
&= \langle \mathbf{v}_o \cdot \nabla\bar{x}, \mathbf{v}_o \cdot \nabla\bar{y}\rangle \cdot \nabla g \circ \mathbf{f}|_{\mathbf{u}_o} \\
&= \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o) \cdot \nabla g(\mathbf{f}(\mathbf{u}_o)) \\
&= g_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)),
\end{aligned}$$

where $\mathbf{w}_o = \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$. This completes the proof for the case $\mathbf{g} = g$, a scalar function. The proof for a vector function $\mathbf{g}$ follows from applying this conclusion to each coordinate function of $\mathbf{g}$. Thus, if $\mathbf{g} = \langle \bar{g}_1, \bar{g}_2\rangle$ and $\mathbf{w}_o = \mathbf{f}_{\mathbf{v}_o}(\mathbf{u}_o)$, then

$$\begin{aligned}
(\mathbf{g} \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o) &= \langle (\bar{g}_1 \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o), (\bar{g}_2 \circ \mathbf{f})_{\mathbf{v}_o}(\mathbf{u}_o)\rangle \\
&= \langle (\bar{g}_1)_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)), (\bar{g}_2)_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o))\rangle \\
&= \mathbf{g}_{\mathbf{w}_o}(\mathbf{f}(\mathbf{u}_o)).
\end{aligned}$$

This completes the proof.

Proposition 6.2 (Chapter XI) Let $\mathbf{f}(\mathbf{u}_o) = \mathbf{g}(\mathbf{v}_o)$ and

$$\mathbf{f}(x_1, \ldots, x_n) \sim \mathbf{g}(y_1, \ldots, y_n).$$

If $D_{\mathbf{u}_o}\mathbf{f}$ is injective, then $D_{\mathbf{v}_o}\mathbf{g}$ is also injective and

$$\mathbf{f}(\mathbf{u}_o) + \mathrm{Sp}\{\mathbf{f}_{x_1}(\mathbf{u}_o), \ldots, \mathbf{f}_{x_n}(\mathbf{u}_o)\} = \mathbf{g}(\mathbf{v}_o) + \mathrm{Sp}\{\mathbf{g}_{y_1}(\mathbf{v}_o), \ldots, g_{y_n}(\mathbf{v}_o)\}.$$

A proof will be given for the case in which $n = 2$, $m = 3$. We let $\mathbf{f}(r, s) = \langle \bar{x}, \bar{y}, \bar{z}\rangle$ and $\mathbf{g}(r', s') = \langle \bar{x}', \bar{y}', \bar{z}'\rangle$. Since $\mathbf{f}$ and $\mathbf{g}$ are equivalent, there exists $\mathbf{h}(r, s) = \langle \bar{r}', \bar{s}'\rangle$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$. Hence,

$$\mathbf{v}_o = \mathbf{h}(\mathbf{u}_o) \quad \text{and} \quad \bar{x} = \bar{x}' \circ \mathbf{h},\ \bar{y} = \bar{y}' \circ \mathbf{h},\ \bar{z} = \bar{z}' \circ \mathbf{h}$$

by Proposition 1.2 in Chapter IX. Therefore, by the chain rule theorem

$$\begin{aligned}
\mathbf{f}_r(\mathbf{u}_o) &= \langle \bar{x}_r, \bar{y}_r, \bar{z}_r\rangle|_{\mathbf{u}_o} \\
&= \langle (\bar{x}' \circ \mathbf{h})_r, (\bar{y}' \circ \mathbf{h})_r, (\bar{z}' \circ \mathbf{h})_r\rangle|_{\mathbf{u}_o} \\
&= \langle (\bar{x}'_{r'} \circ \mathbf{h})\bar{r}'_r + (\bar{x}'_{s'} \circ \mathbf{h})\bar{s}'_r, (\bar{y}'_{r'} \circ \mathbf{h})\bar{r}'_r \\
&\qquad + (\bar{y}'_{s'} \circ \mathbf{h})\bar{s}'_r, (\bar{z}'_{r'} \circ \mathbf{h})\bar{r}'_r + (\bar{z}'_{s'} \circ \mathbf{h})\bar{s}'_r\rangle|_{\mathbf{u}_o} \\
&= \bar{r}'_r\langle \bar{x}'_{r'} \circ \mathbf{h}, \bar{y}'_{r'} \circ \mathbf{h}, \bar{z}'_{r'} \circ \mathbf{h}\rangle|_{\mathbf{u}_o} \\
&\qquad + \bar{s}'_r\langle \bar{x}'_{s'} \circ \mathbf{h}, \bar{y}'_{s'} \circ \mathbf{h}, \bar{z}'_{s'} \circ \mathbf{h}\rangle|_{\mathbf{u}_o} \\
&= \bar{r}'_r(\mathbf{u}_o)\mathbf{g}_{r'}(\mathbf{v}_o) + \bar{s}'_r(\mathbf{u}_o)\mathbf{g}_{s'}(\mathbf{v}_o).
\end{aligned}$$

This proves $\mathbf{f}_r(\mathbf{u}_o)$ is in $\mathrm{Sp}\{\mathbf{g}_{r'}(\mathbf{v}_o), \mathbf{g}_{s'}(\mathbf{v}_o)\}$. In a similar manner we may obtain the like conclusion for $\mathbf{f}_s(\mathbf{u}_o)$, and hence, $\mathrm{Sp}\{\mathbf{f}_r(\mathbf{u}_o), \mathbf{f}_s(\mathbf{u}_o)\}$ is a subset of

$$\mathrm{Sp}\{\mathbf{g}_{r'}(\mathbf{v}_o), \mathbf{g}_{s'}(\mathbf{v}_o)\}.$$

Since $D_{\mathbf{u}_o}\mathbf{f}$ is injective, its matrix has rank 2, and this implies

$$2 = \dim \mathrm{Sp}\{\mathbf{f}_r(\mathbf{u}_o), \mathbf{f}_s(\mathbf{u}_o)\} \le \dim \mathrm{Sp}\{\mathbf{g}_{r'}(\mathbf{v}_o), \mathbf{g}_{s'}(\mathbf{v}_o)\} \le 2.$$

Therefore, $D_{\mathbf{v}_o}\mathbf{g}$ is injective and

$$\mathrm{Sp}\{\mathbf{f}_r(\mathbf{u}_o), \mathbf{f}_s(\mathbf{u}_o)\} = \mathrm{Sp}\{\mathbf{g}_r'(\mathbf{v}_o), \mathbf{g}_{s'}(\mathbf{v}_o)\},$$

as desired.

Theorem on Commutativity of Partial Differentiation (Chapter XII)

If $\mathbf{f}$ *is a class* C^2 *function from* $\mathbf{U}$ *in* $\mathbf{R}^n$ *to* $\mathbf{R}^m$, *then*

$$\mathbf{f}_{x_i x_j} = \mathbf{f}_{x_j x_i}.$$

We consider a proof for $f(x, y)$ at $\mathbf{u}_o = \langle 0, 0\rangle = \mathbf{0}$. It is convenient to introduce the *second difference quotient function* $\Delta_{12}{}^2(s, t)$ defined by

$$\Delta_{12}{}^2(s, t) = \frac{1}{st}\,[f(s, t) - f(s, 0) - f(0, t) + f(0, 0)]$$

wherever meaningful. We first prove the following preliminary result.

Lemma If $a > 0$, $b > 0$, then there exist a_1 and b_1, such that $0 < a_1 < a$, $0 < b_1 < b$, and

$$f_{xy}(a_1, b_1) = \Delta_{12}{}^2(a, b).$$

The proof of this lemma requires two applications of the mean value theorem. Let $g(x)$ be the real function of one variable defined by

$$g(x) = f(x, b) - f(x, 0), \; 0 \leq x \leq a.$$

By the mean value theorem there exists a_1, $0 < a_1 < a$, such that

$$\frac{dg}{dx}(a_1) = \frac{g(a) - g(0)}{a}$$

Next let the real function $h(y)$ of one variable be defined by

$$h(y) = f_x(a_1, y), \; 0 \leq y \leq b.$$

Again using the mean value theorem, we choose b_1, $0 < b_1 < b$ such that

$$\frac{dh}{dy}(b_1) = \frac{h(b) - h(0)}{b}.$$

Then

$$\begin{aligned} f_{xy}(a_1, b_1) &= \frac{dh}{dy}(b_1) \\ &= \frac{1}{b}[f_x(a_1, b) - f_x(a_1, 0)] \\ &= \frac{1}{b}\frac{dg}{dx}(a_1) \\ &= \frac{g(a) - g(0)}{ab} \\ &= \frac{1}{ab}[f(a, b) - f(a, 0) - f(0, b) + f(0, 0)] \\ &= \Delta_{12}{}^2(a, b). \end{aligned}$$

This proves the lemma.

The assumption that a and b are positive in the lemma was for simplicity of notation. A similar conclusion may be obtained for any nonzero a, b. Using the class C^2 property of f, we obtain from the lemma

$$f_{xy}(\mathbf{u}_o) = \lim_{<s, t> \to <0, 0>} \Delta_{12}{}^2(s, t).$$

We can obtain a corresponding equality for f_{yx} by reversing the roles of x and y.

Interchanging s, t in the formula for $\Delta_{12}{}^2(s, t)$ produces

$$\Delta_{21}{}^2(s, t) = \frac{1}{ts}[f(t, s) - f(t, 0) - f(0, s) + f(0, 0)].$$

It is evident from the symmetry that

$$\lim_{\langle s, t\rangle \to \langle 0, 0\rangle} \Delta_{12}{}^2(s, t) = \lim_{\langle s, t\rangle \to \langle 0, 0\rangle} \Delta_{21}{}^2(s, t).$$

Thus it can be seen that

$$f_{yx}(\mathbf{u}_o) = \lim_{\langle s, t\rangle \to \langle 0, 0\rangle} \Delta_{21}{}^2(s, t),$$

and hence $f_{yx}(\mathbf{u}_o) = f_{xy}(\mathbf{u}_o)$.

Theorem on Critical Vectors (Chapter XIII)

Let $\mathbf{u}_o$ be a critical vector of a class C^3 function f having as its domain an open set in $\mathbf{R}^n$.
(a) *If $H_f(\mathbf{u}_o)$ is negative-definite, then f has a local maximum at $\mathbf{u}_o$.*
(b) *If $H_f(\mathbf{u}_o)$ is positive-definite, then f has a local minimum at $\mathbf{u}_o$.*
(c) *If $H_f(\mathbf{u}_o)$ is indefinite, then f has a saddle point at $\mathbf{u}_o$.*

The proof for (b) will be given first. We set

$$Q(\mathbf{v}) = \frac{1}{2!}\langle \mathbf{v}\rangle^* H_f(\mathbf{u}_o)\langle \mathbf{v}\rangle;$$

then $Q(\mathbf{v})$ attains a minimum value $M > 0$ on the set $\{\mathbf{v}: |\mathbf{v}| = 1\}$. This follows since $H_f(\mathbf{u}_o)$ is positive-definite, $Q(\mathbf{v})$ is continuous, and $\{\mathbf{v}: |\mathbf{v}| = 1\}$ is closed and bounded (see introductory paragraph in Section 5, Chapter XII). Using matrix multiplication properties, it may be verified that for any $\mathbf{v} \neq \mathbf{0}$

$$Q\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{1}{|\mathbf{v}|^2} Q(\mathbf{v}).$$

Since $\mathbf{v}/|\mathbf{v}|$ has norm 1, we obtain by substituting $\mathbf{v} = \mathbf{u} - \mathbf{u}_o$, where $\mathbf{u} \neq \mathbf{u}_o$ is assumed arbitrary,

$$0 < M \le Q\left(\frac{\mathbf{u} - \mathbf{u}_o}{|\mathbf{u} - \mathbf{u}_o|}\right) = \frac{1}{|\mathbf{u} - \mathbf{u}_o|^2} Q(\mathbf{u} - \mathbf{u}_o).$$

Therefore, $Q(\mathbf{u} - \mathbf{u}_o) \ge M|\mathbf{u} - \mathbf{u}_o|^2$ for all $\mathbf{u} \neq \mathbf{u}_o$. Since f has a critical vector at $\mathbf{u}_o$, the Taylor polynomial $P(\mathbf{u})$ of f of degree 2 about $\mathbf{u}_o$ is

$$P(\mathbf{u}) = f(\mathbf{u}_o) + Q(\mathbf{u} - \mathbf{u}_o).$$

Applying the second-degree approximation theorem, there exists a $\delta > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta$, then

$$|f(\mathbf{u}) - P(\mathbf{u})| < M|\mathbf{u} - \mathbf{u}_o|^2 \leq Q(\mathbf{u} - \mathbf{u}_o).$$

Hence, for $|\mathbf{u} - \mathbf{u}_o| < \delta$,

$$\begin{aligned} f(\mathbf{u}) - f(\mathbf{u}_o) &= Q(\mathbf{u} - \mathbf{u}_o) + [f(\mathbf{u}) - f(\mathbf{u}_o) - Q(\mathbf{u} - \mathbf{u}_o)] \\ &= Q(\mathbf{u} - \mathbf{u}_o) + [f(\mathbf{u}) - P(\mathbf{u})] \\ &> 0. \end{aligned}$$

This proves (b); the proof of (a) is similar. The proof of (c) is a consequence of the following lemma. Notation from the previous theorem and proof will be retained.

Lemma

(a) If $\mathbf{u}_o$ is a local maximum of f, then $H_f(\mathbf{u}_o)$ is negative-semidefinite (or negative-definite).
(b) If $\mathbf{u}_o$ is a local minimum of f, then $H_f(\mathbf{u}_o)$ is positive-semidefinite (or positive-definite).

We prove (a) using the contrapositive form, which asserts that if $H_f(\mathbf{u}_o)$ is not negative-semidefinite, then $\mathbf{u}_o$ does not give a local maximum. Supposing that $H_f(\mathbf{u}_o)$ is not negative-semidefinite, there exists $\mathbf{v}_1$ such that $|\mathbf{v}_1| = 1$ and $Q(\mathbf{v}_1) = N > 0$. By the second-degree approximation theorem there exists a $\delta > 0$ such that if $|\mathbf{u} - \mathbf{u}_o| < \delta$, then $|f(\mathbf{u}) - P(\mathbf{u})| < N|\mathbf{u} - \mathbf{u}_o|^2$. Hence, if t satisfies $0 < t < \delta$, then $|t\mathbf{v}_1| = |t| < \delta$, and substituting $\mathbf{u} = \mathbf{u}_o + t\mathbf{v}_1$, we have

$$\begin{aligned} Q((\mathbf{u}_o + t\mathbf{v}_1) - \mathbf{u}_o) &= Q(t\mathbf{v}_1) \\ &= t^2 Q(\mathbf{v}_1) \\ &= t^2 N \\ &> t^2 \frac{|f(\mathbf{u}_o + t\mathbf{v}_1) - P(\mathbf{u}_o + t\mathbf{v}_1)|}{|\mathbf{u}_o + t\mathbf{v}_1 - \mathbf{u}_o|^2} \\ &= f(\mathbf{u}_o + t\mathbf{v}_1) - P(\mathbf{u}_o + t\mathbf{v}_1). \end{aligned}$$

This implies that $Q(\mathbf{u} - \mathbf{u}_o) > f(\mathbf{u}) - P(\mathbf{u})$ for certain $\mathbf{u}$ which can be chosen arbitrarily close to $\mathbf{u}_o$. Hence, for these values of $\mathbf{u}$,

$$f(\mathbf{u}) - f(\mathbf{u}_o) = Q(\mathbf{u} - \mathbf{u}_o) + [f(\mathbf{u}) - P(\mathbf{u})] > 0.$$

This shows f cannot have a local maximum at $\mathbf{u}_o$, as desired. The proof of (b) is similar.

Iterated Integral Theorem (Chapter XIII)

Let $g(x, y)$ be continuous on $I^2 = [a_1, b_1] \times [a_2, b_2]$, and for each x in $[a_1, b_1]$ let $g^x(y)$ be the function on $[a_2, b_2]$ defined by $g^x(y) = g(x, y)$. If $G(x)$ is the function on $[a_1, b_1]$ defined by

$$G(x) = \int_{a_2}^{b_2} g^x \, dy,$$

then

$$\int_{I^2} g \, dA = \int_{a_1}^{b_1} G \, dx.$$

For the proof it must first be shown that G is continuous at an arbitrary c in $[a_1, b_1]$. Given $\varepsilon > 0$, let $\delta > 0$ satisfy the definition of uniform continuity for g and $\varepsilon/(b_2 - a_2)$. If $|x - c| < \delta$, then

$$\begin{aligned}
|G(x) - G(c)| &= \left| \int_{a_2}^{b_2} g^x \, dy - \int_{a_2}^{b_2} g^c \, dy \right| \\
&= \left| \int_{a_2}^{b_2} (g^x - g^c) \, dy \right| \\
&\leq \int_{a_2}^{b_2} |g^x - g^c| \, dy \\
&< \int_{a_2}^{b_2} \frac{\varepsilon}{b_2 - a_2} \, dy \\
&= \varepsilon.
\end{aligned}$$

This proves that G is continuous at c. We next establish the following result.

Lemma If p_1 is a partition of $[a_1, b_1]$ and p_2 of $[a_2, b_2]$, then
(a) $U_G(p_1) \leq U_g(p_1 \times p_2)$, (b) $L_G(p_1) \geq L_g(p_1 \times p_2)$.

For the proof of (a) we let $p_1 = \{[c_0, c_1, \ldots, c_k]\}$, $p_2 = \{[d_0, d_1, \ldots, d_l]\}$, $I_{ij}^2 = [c_{i-1}, c_i] \times [d_{j-1}, d_j]$, and M_{ij} be the maximum of g on I_{ij}^2, $i = 1, \ldots, k$ and $j = 1, \ldots, l$. If $c_{i-1} \leq x \leq c_i$, then

$$G(x) = \int_{a_2}^{b_2} g^x \, dy \leq U_{g^x}(p_2) \leq \sum_{j=1}^{l} M_{ij}(d_j - d_{j-1}).$$

The last inequality follows because the maximum of g on I_{ij}^2 is greater than or equal to the maximum of g on a line segment subset of I_{ij}^2. Therefore, if M_i denotes the maximum of G on $[c_{i-1}, c_i]$, then

$$M_i \leq \sum_{j=1}^{l} M_{ij}(d_j - d_{j-1})$$

and consequently

$$\begin{aligned} U_G(p_1) &= \sum_{i=1}^{k} M_i(c_i - c_{i-1}) \\ &\le \sum_{i=1}^{k} \sum_{j=1}^{l} M_{ij}(c_i - c_{i-1})(d_j - d_{j-1}) \\ &= U_g(p_1 \times p_2). \end{aligned}$$

This proves (a); part (b) is shown similarly.

In order to complete the proof of our theorem, it suffices to show that for each $\varepsilon > 0$ the inequality

$$\int_{\mathbf{I}^2} g\, dA + \varepsilon \ge \int_{a_1}^{b_1} G\, dx_1 \ge \int_{\mathbf{I}^2} g\, dA - \varepsilon$$

is satisfied. We choose a partition $p = p_1 \times p_2$ such that $U_g(p) - L_g(p) < \varepsilon$. Then

$$\begin{aligned} \int_{\mathbf{I}^2} g\, dA + \varepsilon &\ge L_g(p) + \varepsilon \\ &\ge U_g(p) \\ &\ge U_G(p_1) \\ &\ge \int_{a_1}^{b_1} G\, dx_1 \\ &\ge L_G(p_1) \\ &\ge L_g(p) \\ &\ge U_g(p) - \varepsilon. \\ &\ge \int_{\mathbf{I}^2} g\, dA - \varepsilon. \end{aligned}$$

This concludes the proof.

Solutions

Exercises

Chapter I

Introduction

1. 513; arctan .042.
2. arcsin .2 west of north.
3. (a) 49.5; arctan .81 with respect to plane,
   (b) 17.3 and 10 for the 20 lb. force; 21.21 and 21.21 for the 30 lb. force.

Section 3

1. 60.7; 88.
2. (a) $\langle 8, -8, 4\rangle$, (b) $\langle 4, -4, 2\rangle$, (c) $\langle 4t, -4t, 2t\rangle$,
   (d) $\langle 3 + 4t, 2 - 4t, 5 + 2t\rangle$.

Section 5

1. (b) $(1, -1), (-1,1), (3,-3), (-29,29), \ldots$, etc.)
   $\{\langle x, 5x, 2x\rangle : x \text{ is a real number}\}$.

Section 6

3. (a) and (b) Not closed under addition or scalar multiplication; no.
   (c) Closed under both addition and scalar multiplication; yes.
   (d) Closed under scalar multiplication but not closed under addition; no.

Section 7

1. (a) Closure under addition is expressed by "If f and g are polynomials of degree 5, then $f+g$ is a polynomial of degree 5." Closure under scalar multiplication is expressed by "If f is a polynomial of degree 5 and c is a real number then cf is a polynomial of degree 5." This set is not a subspace, since closure fails in both cases (e.g., consider $f(x) = x^5$, $g(x) = -x^5$, $c = 0$).

Chapter II

Section 1

1. (a) The line in the Euclidean plane through the origin and (2,1),
   (b) the entire Euclidean plane,
   (c) the line in the Euclidean plane through the origin and (1,3).
2. $2/7\ \mathbf{u} + 3/7\ \mathbf{v}$.
3. $1\mathbf{u} + 3\mathbf{v} + 5\mathbf{w}$.
4. (a) $(-1)\mathbf{u} + 4(\mathbf{u} + \mathbf{v})$, (b) $1(\mathbf{u} - \mathbf{v}) + 1(\mathbf{u} + \mathbf{w}) + 2(\mathbf{v} - \mathbf{w})$.
5. (a) $-7/5\ (x - 2) + 11/5\ (2x + 1)$,
   (b) $ax + b = (a - 2b)/5\ (x - 2) + (2a + b)/5\ (2x + 1)$.
6. $ax^3 + bx^2 + cx + d = ax^3 + b(x^2 - 2) + cx + (2b + d)$.

Section 2

1. (a) Independent, (b) Dependent, (c) Dependent,
   (d) Independent, (e) Dependent.

Section 4

1. (a) $a(1) + b(x - 2) + c(x^2 + x) = 0$ for all x implies that $a = b = c = 0$.
2. $\{x - 1, 2x + 1, x^2\}$.
3. (a) $a\mathbf{u} + b(2\mathbf{u} - \mathbf{w}) + c(\mathbf{v} + \mathbf{w}) = \mathbf{0}$ implies $a = b = c = 0$,
   (b) $1(\mathbf{u} - \mathbf{w}) + 1(\mathbf{v} + \mathbf{w}) + (-1)(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.
4. (a) yes, (b) yes, (c) no.

Chapter III

Section 1

1. 250.

Section 2

1. (a) $1/4$, (b) $(e^3 - 1)/3$.
2. (a) $\sqrt{1/3}$, (b) $\sqrt{(e^2 - 1)/2}$.

Section 4

1. (a) -6, (b) $-15/2$, (c) 4.
2. (a) $\pm 1/\sqrt{5}\,\langle 2,-1\rangle$, (b) $\pm\langle 0,1,0\rangle$.

Section 5

1. (a) origin, (b) Euclidean plane, (c) line $y = x$, (d) line $y = -x/3$, (e) plane perpendicular to $\langle 1,2, -1\rangle$, (f) line perpendicular to the plane through $(1, 0, 3)$ and $(2, -1, 0)$.
2. (a) $\langle 26/5, 13/5\rangle + \langle -1/5, 2/5\rangle$, (b) $\langle 3/2, 0, 3/2\rangle + \langle -1/2, 0, 1/2\rangle$, (c) $\langle -1/3, 7/3, 8/3\rangle + \langle 7/3, -7/3, 7/3\rangle$.

Chapter IV

Section 2

1. $17a + 6b - 8c = 3$.
2. (a) $\langle -8/11, -1/11, 0\rangle + r\langle 1, 3, 1\rangle$, (b) $(-5/11, 8/11, 3/11), (-4/11, 1, 4/11)$, (c) $1/\sqrt{11}$.
3. $\langle 7, 6, 3\rangle + r\langle 1, 0, 3\rangle + s\langle 2, -1, 5\rangle$.

Section 4

1. (a) 0, (b) 1, (c) 2, (d) 2, (e) 2.
2. (a) $1/13\,\langle 9, 11, -10, 7\rangle$, (b) $\langle -3, -3/2, 3/2, 3\rangle$, (c) $\langle 0, 0, -2, 2\rangle$.
3. $\langle 1, 1, 0, 2\rangle + r_1\langle 2, -1, 1, 2\rangle + r_2\langle 1, -2, 3, 0\rangle$.
4. (a) $\{\mathbf{v}: \langle 1, -1, 1, 1\rangle\cdot\mathbf{v} = 4\}$, (b) $\langle 1, -1, 1, 1\rangle + \{\langle 1, -1, 1, 1\rangle\}^{\perp}$.
5. $-w + z = 2$.

Chapter V

Section 2

1. (a) yes, (b) no.
2. $\langle 5, 9\rangle$.
3. $\langle 1, 1\rangle$,

Section 4

1. (a) $\langle -1 - 5x_2, 2 + 3x_1 + 4x_2\rangle$.

Section 5

1. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Section 6

2. (a) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & 9 \\ 0 & 3 \\ 6 & 12 \end{bmatrix}$.
3. (a) $2f^* + 3g^*$, (b) $f^* \circ g^* \circ h$, (c) $f^* \circ f$.
4. $\langle 5, 5, 2 \rangle$.

Section 7

1. (a) $a = 5/13$, $b = -12/13$, reflection; $a = -5/13$, $b = 12/13$, rotation.
   (b) $a = 2\sqrt{5}/5$, $b = -4\sqrt{5}/15$, $c = 2\sqrt{5}/15$, $d = -\sqrt{5}/3$, rotation;
   $a = 2\sqrt{5}/5$, $b = 4\sqrt{5}/15$, $c = -2\sqrt{5}/15$, $d = \sqrt{5}/3$, reflection.
2. (a) $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Section 8

1. (a) $\begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 7 & 1 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$.
2. (a) $\begin{bmatrix} 4 & 6 \\ -5 & -9 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
3. $\{\langle 2, -2 \rangle, \langle 9, -19/3 \rangle\}$.
4. $\{\langle 3, 1 \rangle, \langle 12, 5 \rangle\}$.
5. (a) $\langle 2, 1/2 \rangle$, (b) $\langle -\sqrt{3} + 5/2, -1 - 5\sqrt{3}/2 \rangle$.
6. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Chapter VI

Section 1

1. (a) $\langle 1, -2, 0 \rangle + r\langle 1, 1, 0 \rangle$, (b) $\langle -5, 4, -3 \rangle + r\langle 2, -1, 3 \rangle$.

Section 2

1. (a) $\mathrm{Sp}\{\langle 0, -2, -7 \rangle, \langle 1, 0, -16 \rangle, \langle 1, 14, 0 \rangle\}$,
   (b) $\mathrm{Sp}\{\langle 0, 0, 13, 10 \rangle, \langle 0, 1, 28, 22 \rangle, \langle 0, 1, 2, 0 \rangle\}$.

Section 3

1. (a) yes, (b) no, (c) yes, (d) no.

Section 4

1. (a) 2, (b) 1, (c) 2.
2. 3.

Section 6

1. (Answers not unique) (a) $E_{1+(1)2}E_{2+(1)1}$,
   (b) $E_{1+(2)2}E_{1+(-1)3}E_{(3)3}E_{3+(-1)2}E_{3+(1)1}$.
2. $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Section 7

1. (a) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$; $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 4 & 3 & 0 \end{bmatrix}$; $\begin{aligned} x - 2y + z &= 0. \\ 2x + 4y + 3z &= 0. \end{aligned}$
   (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & -3 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 & 16 \\ 1 & -1 & 4 \\ 2 & -3 & 8 \end{bmatrix}$; $\begin{aligned} x + y &= 0 \\ x - y &= 4 \\ 2x - 3y &= 8. \end{aligned}$
2. (Answers are not unique) (a) (1) $x = 2$, $y = -1$, $z = 0$,
   (2) $x = 3c$, $y = -2c$, $z = c$, (3) $\langle 2, -1, 0\rangle + c\langle 3, -2, 1\rangle$.
   (b) (1) $w = 1/2$, $x = 1/2$, $y = 0$, $z = 0$, (2) $w = 0$, $x = c_1 + c_2$, $y = c_2$, $z = c_1$,
   (3) $\langle 1/2, 1/2, 0, 0\rangle + r_1\langle 0, 1, 0, 1\rangle + r_2\langle 0, 1, 1, 0\rangle$.

Chapter VII

Section 1

1. [2, 1, 3, 4], [3, 2, 1, 4], [4, 2, 3, 1], [1, 3, 2, 4], [1, 4, 3, 2], [1, 2, 4, 3].
2. (a) even, (b) even.
3. (a) [1, 5, 4, 6, 3, 2], (b) [4, 7, 1, 8, 6, 5, 3, 2].
4. [3, 2, 1, 4]; [1, 2, 4, 3].
5. (a) 105, (b) 0.

Section 2

1. (a) $-a$, (b) 0, (c) $4a$, (d) $3a$.
2. c^3a.

Section 3

1. (a) a, (b) $|c|a$, (c) 0, (d) $|c|^3a$.

Section 4

1. (a) a, (b) $3a$.

Chapter VIII

Section 1

1. (a) 1, 2, (b) $\mathrm{Sp}\{\langle 1, 0, 1\rangle\}$, $\mathrm{Sp}\{\langle 1, 1, 0\rangle, \langle 0, 0, 1\rangle\}$.

Section 2

1. The columns of C are an orthonormal basis of eigenvectors of A

$$\left(\text{e.g., (a)} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\right).$$

2. One choice for B is any orthonormal basis of eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (e.g., $B = \{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle, \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle\}$).

Section 3

1. (a) indefinite, (b) indefinite.

Section 4

1. $b = 0$ gives a circle; $|b| < 1$, $b \neq 0$ gives an ellipse; $|b| = 1$ gives a parabola; $|b| > 1$ gives a hyperbola.

Chapter IX

Section 2

1. $\mathbf{f}(\theta, \phi) = (3 + 2\sin\theta)\sin\phi\,\mathbf{i} + (3 + 2\sin\theta)\cos\phi\,\mathbf{j} + (2 + 2\cos\theta)\mathbf{k}$.
2. $\mathbf{f}(r, \theta) = (\cos\theta + r)\mathbf{i} + (\sin\theta + 2r)\mathbf{j} + 3r\mathbf{k}$.
3. $\mathbf{f}(r, \theta) = [(1 - r) + r\cos\theta]\mathbf{i} + [2(1 - r) + r\sin\theta]\mathbf{j} + 3(1 - r)\mathbf{k}$.
4. (a) $\mathbf{f}(r) = r\mathbf{i} + 3r^2\mathbf{j} + r^2(1 + 36r^2)\mathbf{k}$,
   (b) $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + 6(\sin^2\theta + 2)\mathbf{k}$.

Section 3

1. (a) $\langle 2, 3, 10\rangle$ (b) $\mathbf{i} + \mathbf{j}$, (c) $\langle 1, -3\rangle$, (d) $\langle -2, -1, -1\rangle$.

Section 4

1. (a) $\{\langle x, y\rangle: x^2 + y^2 < 1\}$; yes. (b) $\{\langle x, y\rangle: x + y < 1\}$; no.
   (c) empty set; no. (d) $\{\langle x, y\rangle: x > 0\}$; yes.
   (e) $\{\langle x, y\rangle: y < 0\}$; no. (f) $\{\langle x, y\rangle: x < y\}$; yes.
2. (a) $\delta \le 1/2$, (b) $\delta \le 1 - \sqrt{29}/6$, (c) $\delta \le 1 - r$.

Section 5

1. (a) $f = h \circ g_1 g_2{}^2$ where $g_1(r, s) = r$, $g_2(r, s) = s$, and $h(x) = \cos x$.
   (b) $f = g_1(h \circ g_2)$ where $g_1(r, s) = r$, $g_2(r, s) = s$, and $h(x) = e^x$.
   (c) $f = \dfrac{g_1}{g_2(h \circ g_3)}$ where $g_1(r, s, t) = r$, $g_2(r, s, t) = s$, $g_3(r, s, t) = t$, and $h(x) = \ln x$.

Chapter X

Section 1

1. -6.
2. (a) 5, (b) -2.
3. 28/5.

Section 2

1. (a) $\langle 1/2, 3/2\rangle$, (b) $(\langle 2 + \sqrt{31})/3, (-8 + 2\sqrt{31})/3\rangle$.

Section 4

2. (a) $25b_1 + 10b_2 = 4$, (b) $b_1(2 - 3b_2)^2 + 2b_2(1 - 3b_1)(2 - 3b_2) = 2$.

Chapter XI

Section 2

1. $(5\mathbf{i} + 7\mathbf{j}) + r(\mathbf{i} + \mathbf{j})$; $(5\mathbf{i} + 7\mathbf{j}) + r(\mathbf{i} - \mathbf{j})$.

Section 3

1. (a) $\langle -22, 0, 19\rangle$, (b) $\langle 6, -4\rangle$.

Section 4

1. (a) $\begin{bmatrix} 9 & 6 \\ -1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 \\ -1 & 9 \end{bmatrix}$, (c) $\langle -14, 30\rangle$.

2. (a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, (c) $\langle 0, 3\rangle$.

Section 5

1. (a) $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ for $\mathbf{A_f}$; $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ for $A_{\mathbf{g}}$; $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 \\ 2 & 6 \end{bmatrix}$ for $\mathbf{A_{f+g}}$; $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$ for $\mathbf{A_{3f}}$.

   (b) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 1 & 2 \end{bmatrix}$ for $\mathbf{A_f}$; $\begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 1 & -4 \end{bmatrix}$ for $\mathbf{A_g}$; $\begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & -2 \end{bmatrix}$ for $\mathbf{A_{f+g}}$; $\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & -6 \\ 3 & 6 \end{bmatrix}$ for $A_{3\mathbf{f}}$.

2. $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ for $\mathbf{A_f}$; $\begin{bmatrix} 6 \\ 0 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 3 & -2 \\ 1 & -1 \\ -4 & 1 \end{bmatrix}$ for $\mathbf{A_g}$; $\begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} -3 & -5 \\ -2 & -2 \\ -1 & 5 \end{bmatrix}$ for $\mathbf{A_{g \circ f}}$.

3. $\langle 1, 0, 0, 1\rangle + \mathrm{Sp}\{\langle 1, 1, 0, 1\rangle, \langle 0, -1, 1, 1\rangle, \langle 1, -1, 0, 0\rangle\}$.

Section 6

1. (a) tangent 1-plane is $\langle 1/2, 1/4\rangle + \mathrm{Sp}\{\langle 1, 1\rangle\}$,
   (b) $h(r) = r^2$.

2. (a) tangent 1-plane is $\langle \sqrt{3}/2, -1/2\rangle + \mathrm{Sp}\{\langle 1, \sqrt{3}\rangle\}$,
   (b) $h(\theta) = -\cos\theta$.

3. (a) arc of parabola $y - 1 = (x-1)^2$ from $(0, 2)$ to $(2, 2)$ for both **f** and **g**,
   (b) $\mathbf{h}(r) = 1 + r^{1/3}$,
   (c) $d\mathbf{f}/dr\,(0)$ does not exist; $d\mathbf{g}/ds(1) = \langle 1, 0\rangle$.

Chapter XII

Section 1

1. $-4rt$.

2. $$[0 \quad 2t \quad 2]\begin{bmatrix} 2t^6(r+2t) & 6(r-s)t^4(r+2t) & 2(r-s)t^6 \\ 6(r-s)t^4(r+2t) & 6(r-s)^2t^2(r+2t) & 3(r-s)^2t^4 \\ 2(r-s)t^6 & 3(r-s)^2t^4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Section 2

1. (a) $-1 - x - x^2/2 + (y - \pi)^2/2$, (b) .002.
2. (a) $1 + y^2$, (b) .003.

Section 3

1. $-\sqrt{8} < c < \sqrt{8}$.
2. (c) saddle point.
3. (c) local minimum.

Section 5

1. 1, $-9/5$.
2. $3\sqrt{2}/2$.

Chapter XIII

Section 1

1. (a) 5, (b) 15, (c) 66, (d) 24.
2. (c) $\sqrt{.58}$.
3. (0, 1), (0, 4), (2, 1), (2, 4).
4. norm $= \sqrt{5}$.
5. k any number $> 10^7$.
6. (a) (0, 1, 0), (0, 1, 4), (0, 3, 0), (0, 3, 4), (2, 1, 0), (2, 1, 4), (2, 3, 0), (2, 3, 4),
   (b) $2\sqrt{6}$.

Section 2

1. (a) 40 and 170, (b) 5/6 and 3, (c) -3 and 9.
2. Answer should be between .79 and .99.

Section 3

1. (a) 4, (b) 3.
3. $(5k^2 + 1)/6k^3$ $(= U_g(p_k) - L_g(p_k))$.

Section 4

1. (a) $\left|\int_{\mathbf{I}^n} g_1 g_2 \, dV\right| \le \sqrt{\int_{\mathbf{I}^n} g_1{}^2 \, dV} \sqrt{\int_{\mathbf{I}^n} g_2{}^2 \, dV}$,
   (b) $\sqrt{\int_{\mathbf{I}^n} (g_1 + g_2)^2 \, dV} \le \sqrt{\int_{\mathbf{I}^n} g_1{}^2 \, dV} + \sqrt{\int_{\mathbf{I}^n} g_2{}^2 \, dV}$.

Chapter XIV

Section 2

1. (a) $\langle 1, 1\rangle + r\langle 1, 2\rangle + s\langle 6, 1\rangle + rs\langle -2, 0\rangle$, $0 \le r, s \le 1$,
   (b) 125/12.

Section 3

1. $2\pi/3[(1 + e^4)^{3/2} - (1 + e^2)^{3/2}]$.
2. $32\pi(2 - \sqrt{3})$.
3. 0.

Section 4

1. 2π.
2. $2\pi/3$.
3. π.
4. $\pi/3$.

Section 5

1. $-\sqrt{42}$.
2. $2\sqrt{2/3}$ using both $\mathbf{f}$ and $\mathbf{f}'$; no.
3. 1/8 using both $\mathbf{f}$ and $\mathbf{f}'$; yes.

Section 6

1. 14/3.
2. $\pi/2$.
3. $7\sqrt{10}/2$.
4. $2\sqrt{2}\pi^2$.
5. $5\pi/6$.

Section 7

1. $\pi(1 + 2\sqrt{2/3})$.

Chapter XV

Section 1

1. (a) opposite, (b) same, (c) same.
2. (a) positive, (b) positive.
3. (a) $\mathbf{f}'(r) = r^2\mathbf{i} - r\mathbf{j}$, $-2 \leq r \leq -1$,
   (b) $\mathbf{f}'(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$, $-\pi \leq \theta \leq 0$.

Section 2

1. (a) $-1/15$, (b) $1/15$.
2. 3.
3. 4π.
4. $-1/2$.
5. 0.

Section 4

1. (a) $-2\mathbf{j}$ and 3,
   (b) $z(z^2 - 2x)\mathbf{i} + (z^2 - x^2)\mathbf{k}$ and $y(2x + 3z^2)$,
   (c) $-x\mathbf{i} + y\mathbf{j}$ and 1,
   (d) $-(1 + y)\mathbf{i} - (1 + 2x - 2z)\mathbf{j}$ and $2x - z$.

Section 5

1. (a) 0, (b) π (or $-\pi$), (c) $5/2$ (or $-5/2$).
2. (a) $4\pi/3$, (b) 2, (c) π.

Section 6

1. (a) -1, (b) 0, (c) $+1$.
2. (a) $-2x_1{}^4x_7\,dx_1\,dx_2\,dx_4\,dx_5\,dx_7\,dx_8$,
   (b) $-x_4 e^{x_1x_2}\,dx_1\,dx_3\,dx_5\,dx_9$.
3. (a) $2x_1x_4\,dx_1\,dx_2 + (1 - x_1{}^2)dx_2\,dx_4$,
   (b) $3x_1{}^2x_2\,dx_1\,dx_2\,dx_4$.
4. (a) $2x_1(x_1 - x_2)(x_2 + x_3)\,dx_1\,dx_2$,
   (b) $x_2x_3\,dx_1 + x_3(x_3 - x_1)\,dx_2 - x_1x_2\,dx_3$.
5. 0.

Solutions

Problems

Chapter I

I A

1. (a) $=\langle 1+3, 2-4\rangle = \langle 4, -2\rangle$, (b) $=\langle 2(1), 2(3)\rangle = \langle 2, 6\rangle$,
   (c) $\langle -1, 1, 7\rangle$,
   (d) $=\langle 3, 3, 0\rangle - \langle 6, 0, 2\rangle = \langle -3, 3, -2\rangle$.
2. (a) $-(1+3)\mathbf{i} + (1-1)\mathbf{j} = 4\mathbf{i}$, (b) $8\mathbf{i} + 12\mathbf{j}$,
   (c) $4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, (d) $4\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$,
   (e) $14\mathbf{i} - 6\mathbf{j} + 26\mathbf{k}$.
3. (a) $\langle 6, 0, 8, 8\rangle$, (b) $\langle 6, 3, 0, 21\rangle$,
   (c) $\langle 3, 1, 6, 10, 7\rangle$, (d) $\langle 5, -2, 10, 5, 24\rangle$.

I B

1. (a) $=\langle 2-1, 3-1\rangle = \langle 1, 2\rangle$.
   (b) Set $Q=(x, y)$; then $3\mathbf{i} - \mathbf{j} = \mathbf{PQ} = (x-1)\mathbf{i} + (y-2)\mathbf{j}$ gives $Q=(4, 1)$.
   (c) Set $S=(x, y)$; then $\langle -2, 1\rangle = \mathbf{PQ} = \mathbf{RS} = \langle x-2, y-1\rangle$ gives $S=(0, 2)$.
2. (a) $=\langle 3-2, -2-0, 6-1\rangle = \langle 1, -2, 5\rangle$.
   (b) Set $P=(x, y, z)$; then $\langle 4, 2, -6\rangle = \langle 1-x, -y, 3-z\rangle$ gives $P=(-3, -2, 9)$.
   (c) Set $R=(x, y, z)$; then $\langle -2, 3, 3\rangle = \langle 3-x, -y, 7-z\rangle$ gives $R=(5, -3, 4)$.
3. Set $S=(x, y)$; then $\langle 1, 3\rangle = \mathbf{PQ} = \mathbf{RS} = \langle x-6, y-5\rangle$ gives $S=(7, 8)$.
4. Set $S=(x, y, z)$; then $\langle -5, -1, 1\rangle = \mathbf{PQ} = \mathbf{RS} = \langle x-2, y-1, z-5\rangle$ gives $S=(-3, 0, 6)$.

5. From $\langle x-2, y-0, z-5\rangle = \mathbf{PQ} = \langle -1, 1, 1\rangle$, one vertex is $(1, 1, 6)$. From $\langle x-2, y-3, z-1\rangle = \mathbf{PR} = \langle 1, 0, 3\rangle$ and $\langle x-2, y-3, z-1\rangle = \mathbf{PQ} = \langle -1, 1, 1\rangle$ two other vertices are $(3, 3, 4)$, $(1, 4, 2)$. From $\mathbf{PS} = \langle x-1, y-1, z-6\rangle$ the final vertex is $(2, 4, 5)$.
6. Let $S = (x, 0)$; then $\mathbf{PQ} = \langle 1, -6\rangle$, $\mathbf{RS} = \langle x-7, -5\rangle$ give $(x-7)/1 = -5/-6$, and hence, $S = (47/6, 0)$.
7. Let $S = (0, y)$; then $(y-4)/5 = -6/-1$ gives $S = (0, 34)$.
8. Let $S = (x, y, 0)$; then $\mathbf{PQ} = \langle 4, 3, 4\rangle$, $\mathbf{RS} = \langle x+1, y-7, 2\rangle$, and $(x+1)/4 = (y-7)/3 = 2/4$ gives $S = (1, 17/2, 0)$.
9. Let $S = (0, y, z)$; then $\mathbf{PQ} = \langle -1, -1, 12\rangle$, $\mathbf{RS} = \langle -4, y-1, z-3\rangle$ and $-4/-1 = (y-1)/-1 = (z-3)/12$ gives $S = (0, -3, 51)$.

Chapter II

II A

1. (a) 0, 2, 1; no. (b) 1, 4, 3; no. (c) 0, 1, 3, 3; yes.

2. (a) $\begin{bmatrix} 0 & 12 \\ 1 & 2 \\ -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 12 \\ -1 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 0 & 12 \\ 0 & 6 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 6 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 \\ 3 & 1 & 2 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 4 & 0 \\ 0 & -11 & 2 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & -5 \\ 5 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 4 \\ 0 & -5 \\ 0 & -20 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 4 \\ 0 & -5 \\ 0 & 0 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 5 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -13 & -8 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 6 & 9 \\ 0 & 1 \\ -6 & -8 \end{bmatrix}$,

(c) $\begin{bmatrix} 6 & 9 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 6 & 9 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7. (a) $\to \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 3 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 4 \\ 0 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$; $r = 2$.

(b) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 4 \\ 0 & -4 & -8 & 4 \\ 0 & 4 & 8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $r = 2$.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$,

(b) $\to \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 7 \\ 0 & -33 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -33 \end{bmatrix}$,

(c) $\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -4 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -12 & -24 \\ 0 & -12 & -27 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -12 & -24 \\ 0 & 0 & -3 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -3 \end{bmatrix}$,

(d) $\to \begin{bmatrix} 6 & 1 & 2 \\ 6 & 8 & 0 \\ 6 & 9 & 15 \end{bmatrix} \to \begin{bmatrix} 6 & 1 & 2 \\ 0 & 7 & -2 \\ 0 & 8 & 13 \end{bmatrix} \to \begin{bmatrix} 6 & 1 & 2 \\ 0 & 56 & -16 \\ 0 & 56 & 91 \end{bmatrix}$

$\to \begin{bmatrix} 6 & 1 & 2 \\ 0 & 56 & -16 \\ 0 & 0 & 107 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 0 \\ 0 & 56 & 0 \\ 0 & 0 & 107 \end{bmatrix}$.

II B

1. (a) $\begin{bmatrix} 1 & 2 & 5 \\ 3 & -4 & -1 \end{bmatrix}$,

(b) $\to \begin{bmatrix} 1 & 2 & 5 \\ 0 & -10 & -16 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9/5 \\ 0 & -10 & -16 \end{bmatrix}$,

(c) $\begin{aligned} x \qquad &= 9/5 \\ -10y &= -16 \end{aligned}$ gives $x = 9/5$, $y = 8/5$.

2. (a) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & 3 & -1 & 10 \\ 3 & -1 & 2 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

(c) $x = 2$, $y = 3$, $z = 1$.

3. (a) $\begin{bmatrix} 4 & -1 & -11 \\ 3 & 5 & 9 \end{bmatrix} \to \begin{bmatrix} 12 & -3 & -33 \\ 12 & 20 & 36 \end{bmatrix} \to \begin{bmatrix} 12 & -3 & -33 \\ 0 & 23 & 69 \end{bmatrix}$

$\to \begin{bmatrix} 4 & -1 & -11 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$; $x = -2$, $y = 3$.

(b) $\begin{bmatrix} 2 & -4 & 1 & 19 \\ 3 & 5 & -1 & -6 \\ 1 & 6 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & -16 & -5 & 7 \\ 0 & -13 & -10 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & -16 & -5 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$; $x = 3, y = -2, z = 5.$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & -1 & -2 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$: $w = 2, x = 0, y = 1, z = 3.$

4. Set $y = c$; $2x + 3c = 6$ gives $x = 3 - 3c/2$.
5. Set $z = c$; $y + 2c = 6$ gives $y = 6 - 2c$; $x + 2(6 - 2c) - c = 4$ gives $x = -8 + 5c$.
6. (a) $z = 2$; set $y = c$; $x - c + 6 = 8$ gives $x = 2 + c$.
   (b) $z = 2$; $y + 4 = 4$ gives $y = 0$; $x - 0 - 6 = 5$ gives $x = 11$.
   (c) Set $z = c_1, y = c_2$; $x + c_2 - 3c_1 = 5$ gives $x = 5 + 3c_1 - c_2$.
   (d) Set $z = c_1, y = c_2$; then $x = -c_1/2 - 3c_2/2$.
7. (a) $\begin{bmatrix} 1 & 1 & -2 & 4 \\ 3 & -1 & -1 & 2 \\ 5 & 1 & -5 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & -4 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ gives $\begin{aligned} x + y - 2z &= 4 \\ -4y + 5z &= -10. \end{aligned}$

   We ignore the third equation, since it is satisfied by all x, y, z. A solution is $z = c, y = 5/2 + 5c/4, x = 3/2 + 3c/4$.

   (b) $\begin{bmatrix} 1 & 1 & -2 & 4 \\ 3 & -1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & -4 & 5 & -10 \end{bmatrix}$ gives $\begin{aligned} x + y - 2z &= 4 \\ -4y + 5z &= -10, \end{aligned}$

   which has solution $z = c, y = 5/2 + 5c/4, x = 3/2 + 3c/4$.

   (c) $\begin{bmatrix} 2 & 1 & 10 \\ 1 & -1 & 5 \\ 5 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ gives $y = 0, x = 5$.

   (d) $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 1 & 6 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ gives $y = 1, x = 5$.

   (e) $\begin{bmatrix} 2 & 1 & -1 & 4 \\ 3 & -1 & 1 & 5 \\ 1 & 2 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & -5 & -3 \end{bmatrix}$ gives $z = 3/5, y = 1, x = 9/5$.
8. (a) consistent: $4 - 3 = 1$. (b) consistent; $4 - 2 = 2$.
   (c), (d) inconsistent. (e) consistent; $5 - 4 = 1$.

II C

1. (a) $\langle 0, -12\rangle = x\langle 2, 1\rangle + y\langle 1, 3\rangle$ gives $\begin{aligned} 2x + y &= 0 \\ x + 3y &= -12, \end{aligned}$

   (b) The augmented matrix of the system in (a) is $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -12 \end{bmatrix}$, which has

echelon form $\begin{bmatrix} 1 & 3 & -12 \\ 0 & -5 & 24 \end{bmatrix}$. The system is, therefore, consistent.

2. (a) The system $\begin{aligned} x + 2y &= 1 \\ y &= -1 \\ 3x + 5y &= 4 \end{aligned}$ has the augmented matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 5 & 4 \end{bmatrix}$, which has echelon form $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$; it is therefore consistent.

(b) The system $\begin{aligned} x + 2y &= 3 \\ y &= 1 \\ 3x + 5y &= 4 \end{aligned}$ has augmented matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}$ with echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$; thus the system is inconsistent.

3. (a) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 7 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ implies yes,

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ implies no,

(c) $\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 7 & -7 \\ 2 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ implies yes.

4. The system $\begin{aligned} x + 2y &= a \\ 2x &= b \\ 3x + 4y &= c \end{aligned}$ has as its augmented matrix $\begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 4 & c \end{bmatrix}$ with echelon form $\begin{bmatrix} 1 & 2 & a \\ 0 & -2 & -3a + c \\ 0 & 0 & 4a + b - 2c \end{bmatrix}$. The system is consistent if $4a + b - 2c = 0$.

5. $\begin{bmatrix} 1 & 3 & a \\ 2 & 1 & b \\ 0 & 0 & c \\ 5 & 6 & d \end{bmatrix}$ has echelon form $\begin{bmatrix} 1 & 3 & a \\ 0 & -5 & -2a + b \\ 0 & 0 & c \\ 0 & 0 & -7a - 9b + 5d \end{bmatrix}$.

An answer is $c = 0$, $-7a - 9b + 5d = 0$.

6. (a) $\begin{bmatrix} 2 & 1 & -1 & a \\ 1 & -1 & -1 & b \\ 3 & 3 & -1 & c \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & b \\ 0 & 3 & 1 & a - 2b \\ 0 & 0 & 0 & -2a + b + c \end{bmatrix}$

gives $-2a + b + c = 0$.

(b) $\begin{bmatrix} 1 & 2 & a \\ 3 & -1 & b \\ 1 & -1 & c \end{bmatrix} \to \begin{bmatrix} 1 & 2 & a \\ 0 & -7 & -3a + b \\ 0 & 0 & 2a - 3b + 7c \end{bmatrix}$ gives $2a - 3b + 7c = 0$.

(c) $\begin{bmatrix} 1 & 2 & a \\ 0 & 0 & b \\ 2 & 4 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & 0 & b \\ 0 & 0 & -2a+c \end{bmatrix}$ gives $b = 0$, $-2a + c = 0$.

(d) $\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -1 & -2 & -2a+b \\ 0 & 0 & 1 & c \end{bmatrix}$ gives all 3-tuples.

II D

1. $\begin{aligned} 2x + 4y &= 0 \\ 3x + 6y &= 0 \end{aligned}$ has solution $y = c$, $x = -2c$, c arbitrary.

2. The system $\begin{aligned} 3x + 2y + z &= 0 \\ x \phantom{{}+2y} + z &= 0 \\ 4x + 5y - z &= 0 \end{aligned}$ has for the echelon form of its augmented matrix

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution has 1 arbitrary constant.

3. The system $\begin{aligned} x + 3y \phantom{{}+5z} &= 0 \\ 2x + y + z &= 0 \\ -y + 5z &= 0 \end{aligned}$ has as echelon form of its augmented matrix

$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & -24 & 0 \end{bmatrix}$. Thus the only solution is $x = y = z = 0$.

4. (a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix}$ gives dependence;

(b) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

gives dependence;

(c) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \\ 5 & 4 & 7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -20 & 0 \end{bmatrix}$ gives independence;

(d) $\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 7 & 5 & 0 \\ 5 & 4 & 16 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ gives dependence;

(e) $\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ gives dependence.

5. (a) $\begin{bmatrix} 1 & 3 & 0 \\ a & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -3a+4 & 0 \end{bmatrix}$ gives $-3a + 4 = 0$, $a = 4/3$;

(b) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & a & 0 \\ 7 & 5 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -1 & a & 0 \\ 0 & -16 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -1 & a & 0 \\ 0 & 0 & -16a+2 & 0 \end{bmatrix}$

gives $-16a + 2 = 0$, $a = 1/8$.

II E

1. (a) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ gives independence; yes.

   (b) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ gives dependence; no.

   (c) $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$; yes.

   (d) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 7 & 3 & 18 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; no.

   (e) $\begin{bmatrix} 1 & 3 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 4 & 7 & -2 & 0 \\ 2 & 5 & 6 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; no.

   (f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -3 & -5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; no.

2. (a) From $x\langle 1, 3\rangle = \langle a, b\rangle$, the span set of $\{\langle 1, 3\rangle\}$ is described by $-3a + b = 0$.

   (b) From $x\langle 1, 1\rangle + y\langle 2, 0\rangle + z\langle 3, -4\rangle = \langle 0, 0\rangle$, there is the augmented matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -10 & 0 \end{bmatrix}$ which shows that the set is dependent.

   (c) The augmented matrix $\begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 4 & 1 & 1 & 0 \\ 2 & 0 & -3 & 2 & 0 \end{bmatrix}$ has echelon form $\begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & -2 & -7 & -4 & 0 \\ 0 & 0 & -13 & -7 & 0 \end{bmatrix}$; the set is dependent.

   (d) From $\begin{bmatrix} 1 & 2 & a \\ 1 & 0 & b \\ 2 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & 2 & a \\ 0 & -2 & -a+b \\ 0 & 0 & -a-3b+2c \end{bmatrix}$ the span set is described by $-a - 3b + 2c = 0$.

Chapter III

III A

1. (a) $=2(1) + 3(5) = 17$, (b) 0,
   (c) $=3(2) + 6(1) + 0(7) = 12$, (d) $=1(3) + 2(-5) = -7$
   (e) 6, (f) 18, (g) 40, (h) 41.

2. (a) $=\sqrt{3^2+5^2}=\sqrt{34}$, (b) $\sqrt{5}$,
(c) $=\sqrt{3^2+2^2+4^2}=\sqrt{29}$, (d) $=\sqrt{2^2+(-1)^2}=\sqrt{5}$,
(e) $\sqrt{11}$, (f) $\sqrt{14}$, (g) $\sqrt{22}$, (h) $\sqrt{31}$.

3. (a) $=\sqrt{(3-1)^2+(-2-4)^2}=2\sqrt{10}$, $=\arctan(-2-4)/(3-1)=\arctan(-3)$.
(b) $=\sqrt{3^2+(-10)^2}=\sqrt{109}$, $=\arctan -10/3$.

4. (by trigonometry) $= 6\cos\pi/4\,\mathbf{i}+6\sin\pi/4\,\mathbf{j}=3\sqrt{2}(\mathbf{i}+\mathbf{j})$.

5. $\mathbf{PQ}=2\cos 3\pi/4\mathbf{i}+2\sin 3\pi/4\,\mathbf{j}=\sqrt{2}(-\mathbf{i}+\mathbf{j})$; hence, $Q=(3-\sqrt{2},\sqrt{2})$.

6. (a) From $\mathbf{PQ}=\langle 1,-5,1\rangle$, $|\overline{PQ}|=3\sqrt{3}$ the direction cosines are $1/3\sqrt{3}$, $-5/3\sqrt{3}$, $1/3\sqrt{3}$;
(b) $\mathbf{PQ}=\langle 2,-1,-4\rangle$, $|\overline{PQ}|=\sqrt{21}$; direction cosines are $2/\sqrt{21}$, $-1/\sqrt{21}$, $-4/\sqrt{21}$.

7. (a) $=4\langle 1/2,-1/2,\sqrt{2}/2\rangle=\langle 2,-2,2\sqrt{2}\rangle$,
(b) $=6\langle 1/3,3/3,-2/3\rangle=\langle 2,4,-4\rangle$.

8. (a) $\mathbf{PQ}=\langle 2/3,4/3,4/3\rangle$; hence, $Q=(2+2/3,1+4/3,5+4/3)=(8/3,7/3,19/3)$,
(b) $=(2+3,1+0,5-4)=(5,1,1)$.

9. (a) $=\dfrac{\langle 1,1\rangle\cdot\langle -1,3\rangle}{\sqrt{2}\,\sqrt{10}}=1/\sqrt{5}$, (b) $=\dfrac{\langle 1,5\rangle\cdot\langle 6,3\rangle}{\sqrt{26}\,\sqrt{45}}=7/\sqrt{130}$,
(c) $\dfrac{\langle 1,2,-2\rangle\cdot\langle 5,-2,0\rangle}{\sqrt{9}\sqrt{29}}=1/3\sqrt{29}$.

10. (a) $\langle -1,5\rangle\cdot\langle a-2,4\rangle=0$ gives $a=22$;
(b) $\langle 2,3-a\rangle\cdot\langle -1,4-a\rangle=0$ gives $a=2$ or $a=5$.

11. (a) Set $R=(x,0)$; then $\langle -3,4\rangle\cdot\langle x-7,-1\rangle=0$ gives $R=(17/3,0)$.
(b) Set $R=(0,y)$; then $\langle -3,4\rangle\cdot\langle -7,y-1\rangle=0$ gives $R=(0,-17/4)$.

12. (a) Let (x,y) be on the line; then $\langle -1,5\rangle\cdot\langle x-2,y-3\rangle=0$ gives $x-5y+13=0$;
(b) $\langle 1,-6\rangle\cdot\langle x-1,y-1\rangle=0$ gives $x-6y+5=0$.

13. (a) Let (x,y,z) be on the plane; then $\langle -1,5,-1\rangle\cdot\langle x-1,y-1,z-4\rangle=0$ gives $x-5y+z=0$.
(b) $\langle -1,-8,2\rangle\cdot\langle x-2,y-1,z-1\rangle=0$ gives $x+8y-2z-8=0$.

III B

1. (a) $=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & -2\\ 4 & 2 & -1\end{bmatrix}=\mathbf{i}-7\mathbf{j}-10\mathbf{k}$,

(b) $=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 4\\ 2 & -5 & 1\end{bmatrix}=21\mathbf{i}+5\mathbf{j}-17\mathbf{k}$.

2. (a) $\mathbf{PQ}\times\mathbf{PR}=\langle 1,1,1\rangle\times\langle 3,2,2\rangle=\langle 0,1,-1\rangle$; $\mathbf{PS}=\langle a-1,b,1\rangle$.
From $\mathbf{PS}=r(\mathbf{PQ}\times\mathbf{PR})$ obtain $r=-1$, and hence $a=1$, $b=-1$.
(b) $\langle -3,a-1,b-2\rangle=\mathbf{PS}=r(\mathbf{PQ}\times\mathbf{PR})=r\langle -3,7,-1\rangle$ gives $r=1$, $a=8$, $b=1$.

3. (a) $\mathbf{PQ} \times \mathbf{PR} = \langle 0, -2, 2\rangle$; hence $\mathbf{PS} = \pm 1/\sqrt{2}\,\langle 0, -1, 1\rangle$ and $S = (1, -1/\sqrt{2}, 2 + 1/\sqrt{2})$ or $S = (1, 1/\sqrt{2}, 2 - 1/\sqrt{2})$,
   (b) $\mathbf{PQ} \times \mathbf{PR} = \langle 1, -3, -4\rangle$, $\mathbf{PS} = \pm 1/\sqrt{26}\,\langle 1, -3, -4\rangle$ gives $S = \langle 2 + 1/\sqrt{26}, -3/\sqrt{26}, 1 - 4/\sqrt{26}\rangle$ or $S = \langle 2 - 1/\sqrt{26}, 3/\sqrt{26}, 1 + 4/\sqrt{26}\rangle$.
4. (a) Let $S = (x, y, z)$ be on the line; then $\mathbf{PS} = r(\mathbf{PQ} \times \mathbf{PR})$ gives $\langle x - 1, y, z - 3\rangle = r\langle -1, -9, 4\rangle$, and hence $(x - 1)/(-1) = y/(-9) = (z - 3)/4$.
   (b) $\langle x - 2, y + 1, z\rangle = r\langle 23, 12, -7\rangle$ gives $(x - 2)/23 = (y + 1)/12 = z/(-7)$.
5. (a) Let $R = (x, y, z)$; then $\mathbf{PQ} \times \mathbf{PR} = \mathbf{0}$ gives $3y + z - 6 = 0$, $3x + z - 9 = 0$, $x - y - 1 = 0$. Any two of these three provides an answer.
   (b) $\langle -4, 3, 2\rangle \times \langle x - 4, y - 2, z - 1\rangle = \mathbf{0}$ gives $2y - 3z - 1 = 0$, $2x + 4z - 12 = 0$, $3x + 4y - 20 = 0$. Any two of these three gives an answer.

III C

1. (a) $\mathbf{v}_1' = \langle 3, 0, 1\rangle$,
   (b) $(a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{u}_1 = 0$ is satisfied by $a = 7$, $b = -10$; this gives $\mathbf{v}_2' = \langle 1, -10, -3\rangle$ (the solution for a and b is not unique),
   (c) $\mathbf{v}_1 = 1/\sqrt{10}\,\langle 3, 0, 1\rangle$, $\mathbf{v}_2 = 1/\sqrt{110}\,\langle 1, -10, -3\rangle$.
2. Set $\mathbf{v}_1' = \langle 2, 0, -1\rangle$ and $\mathbf{v}_2' = a\langle 2, 0, -1\rangle + b\langle 1, 1, 3\rangle$. Then $\mathbf{v}_2' \cdot \mathbf{v}_1' = 0$ is satisfied by $a = 1$, $b = 5$, which gives $\mathbf{v}_2' = \langle 7, 5, 14\rangle$. An orthonormal basis is $\{1/\sqrt{5}\,\langle 2, 0, -1\rangle, 1/3\sqrt{30}\,\langle 7, 5, 14\rangle\}$. This answer is not unique since there are many solutions for a and b.
3. (a), (b) Set $\mathbf{v}_1' = \langle 1, 0, 0, 1\rangle$, $\mathbf{v}_2' = a\mathbf{u}_1 + b\mathbf{u}_2$. Then $\mathbf{v}_2' \cdot \mathbf{v}_1' = 0$ has solution $a = 1$, $b = -1$, which gives $\mathbf{v}_2' = \langle 1, -1, 0, -1\rangle$.
   (c) $\mathbf{v}_3' \cdot \mathbf{v}_1' = 2c + 2d + 2e = 0$, $\mathbf{v}_3' \cdot \mathbf{v}_2' = -3d - e = 0$ are satisfied by $c = 2$, $d = 1$, $e = -3$; this gives $\mathbf{v}_3' = \langle -1, -2, -6, 1\rangle$. An orthonormal basis is $\{1/\sqrt{2}\,\langle 1, 0, 0, 1\rangle, 1/\sqrt{3}\,\langle 1, -1, 0, -1\rangle, 1/\sqrt{42}\,\langle -1, -2, -6, 1\rangle\}$.
4. Set $\mathbf{v}_1' = \langle 1, 0, 2, 3\rangle$ and $\mathbf{v}_2' = a\langle 1, 0, 2, 3\rangle + b\langle -1, 0, 0, 1\rangle$. Then $\mathbf{v}_2' \cdot \mathbf{v}_1' = 0$ has solution $a = 1$, $b = -7$ which gives $\mathbf{v}_2' = \langle 8, 0, 2, -4\rangle$.
   Set $\mathbf{v}_3' = c\langle 1, 0, 2, 3\rangle + d\langle -1, 0, 0, 1\rangle + e\langle 0, 1, 1, 1\rangle$;
   then $\mathbf{v}_3' \cdot \mathbf{v}_1' = 14c + 2d + 5e = 0$, $\mathbf{v}_3' \cdot \mathbf{v}_2' = -12d - 2e = 0$ is satisfied by $c = 2$, $d = 1$, $e = -6$ which gives $\mathbf{v}_3' = \langle 1, -6, -2, 1\rangle$. An orthonormal basis is $\{1/\sqrt{14}\,\langle 1, 0, 2, 3\rangle, 1/\sqrt{21}\,\langle 4, 0, 1, -2\rangle, 1/\sqrt{42}\,\langle 1, -6, -2, 1\rangle\}$.

Chapter IV

IV A

1. (a) $x = 2 + r$, $y = 3 - r$,   (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + r\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,
   (c) $y = -x + 5$,   (d) $x + y = 5$.
2. (a) $x = 2 + 4r$, $y = 3 + 5r$,   (b) $x\mathbf{i} + y\mathbf{j} = (2\mathbf{i} + 3\mathbf{j}) + r(4\mathbf{i} + 5\mathbf{j})$.

3. (a) $y = 3x - 6$, (b) $x = 0 + r,\ y = -6 + 3r$,
(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} + r\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, (d) $x\mathbf{i} + y\mathbf{j} = -6\mathbf{j} + r(\mathbf{i} + 3\mathbf{j})$.

4. (a) $(3\mathbf{i} + 5\mathbf{j}) + r(2\mathbf{i} - \mathbf{j})$,
(b) $(2\mathbf{i} + 7\mathbf{j}) + r(4\mathbf{i} + 5\mathbf{j})$,
(c) from $(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) = 0$ obtain $(-\mathbf{i} + 4\mathbf{j}) + r(\mathbf{i} + \mathbf{j})$,
(d) $(4\mathbf{i} - 6\mathbf{j}) + r(\mathbf{i} + \mathbf{j})$.

5. (a) $(2\mathbf{i} + 7\mathbf{j}) + r[(4 - 2)\mathbf{i} + (-3 - 7)\mathbf{j}] = (2\mathbf{i} + 7\mathbf{j}) + r(2\mathbf{i} - 10\mathbf{j})$,
(b) $(5\mathbf{i} + \mathbf{j}) + r(\mathbf{i} - \mathbf{j})$.

6. (a) $\begin{aligned} 2a + 7b &= c \\ 4a - 3b &= c \end{aligned}$ has augmented matrix $\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \begin{bmatrix} 2 & 7 & -1 & 0 \\ 4 & -3 & -1 & 0 \end{bmatrix} \end{array} \rightarrow$

$\begin{bmatrix} 2 & 7 & -1 & 0 \\ 0 & -17 & 1 & 0 \end{bmatrix}$. A solution is $c = 17, b = 1, a = 5$; this gives $5x + y = 17$.

(b) $\begin{aligned} 5a + b &= c \\ 6a \quad &= c \end{aligned}$ has as a solution $a = 1,\ b = 1,\ c = 6$; this gives $x + y = 6$.

7. (a) $\mathbf{OR} = (2\mathbf{i} + 3\mathbf{j}) - \dfrac{(2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} - \mathbf{j})}{|\mathbf{i} - \mathbf{j}|^2}(\mathbf{i} - \mathbf{j}) = 5/2\,(\mathbf{i} + \mathbf{j})$ gives $R = (5/2, 5/2)$,

(b) $\mathbf{OR} = (\mathbf{i} - \mathbf{j}) - \dfrac{(\mathbf{i} - \mathbf{j}) \cdot (4\mathbf{i} + 3\mathbf{j})}{|4\mathbf{i} + 3\mathbf{j}|^2}(4\mathbf{i} + 3\mathbf{j}) = 7/25\,(3\mathbf{i} - 4\mathbf{j})$ gives
$R = (21/25,\ -28/25)$,

(c) $\mathbf{OR} = 7/(4 + 16)\,(2\mathbf{i} + 4\mathbf{j}) = 7/10\,(\mathbf{i} + 2\mathbf{j})$ gives $R = (7/10,\ 7/5)$.

8. (a) $= 5/2|\mathbf{i} + \mathbf{j}| = 5/\sqrt{2}$, (b) $= 7/25\,|3\mathbf{i} - 4\mathbf{j}| = 7/5$,
(c) $= 7/10\,|\mathbf{i} + 2\mathbf{j}| = 7\sqrt{5}/10$.

9. (a) = distance from origin to $[(\mathbf{i} - \mathbf{j}) - (\mathbf{i} + 3\mathbf{j})] + r(2\mathbf{i} + 3\mathbf{j}) = 8/\sqrt{13}$,
(b) = distance from origin to $(\mathbf{i} - 9\mathbf{j}) + r(5\mathbf{i} - \mathbf{j}) = 44/\sqrt{26}$.

IV B

1. (a) $1/\sqrt{6},\ -2/\sqrt{6},\ 1/\sqrt{6}$,
(b) $x = 1 + r,\ y = 3 - 2r,\ z = -1 + r$,
(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + r\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$,
(d) $(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - \dfrac{(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + \mathbf{k})}{|\mathbf{i} - 2\mathbf{j} + \mathbf{k}|^2}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j}$ gives $(2, 1, 0)$,
(e) $|2\mathbf{i} + \mathbf{j}| = \sqrt{5}$,
(f) $(\mathbf{i} + 4\mathbf{k}) + r(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

2. (a) $(2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + r(3\mathbf{i} - \mathbf{j} + \mathbf{k})$,
(b) $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + r[(3 - 1)\mathbf{i} + (1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k}] =$
$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + r(2\mathbf{i} - \mathbf{j} + 7\mathbf{k})$,
(c) $= (5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) + r(1/3\mathbf{i} + 2/3\,\mathbf{j} - 2/3\mathbf{k})$.

3. (a) $a = 2 - r_o$ $\quad a = -1 + 2r_o'$
$b = -1 + 3r_o$ $\quad b = 8 - 3r_o'$
$c = r_o$ $\quad c = 3 + r_o'$.

(b) $$\begin{aligned} 2 - r_o &= -1 + 2r_o' \\ -1 + 3r_o &= 8 - 3r_o' \\ r_o &= 3 + r_o' \end{aligned} \quad \text{gives} \quad \begin{array}{ccc} r_o & r_o' & \\ \end{array} \begin{bmatrix} -1 & -2 & -3 \\ 3 & 3 & 9 \\ 1 & -1 & 3 \end{bmatrix}.$$
The solution is $r_o = 3$, $r_o' = 0$,

(c) $a = 2 - 3 = -1$, $b = -1 + 9 = 8$, $c = 3$ gives $(-1, 8, 3)$.

4. $a = 4 + r_o \quad = 2r_o'$
$b = -5 + 4r_o = 3$ $\qquad$ has solution $r_o = 2$, $r_o' = 3$,
$c = -1 + 3r_o = 2 + r_o'$
which gives $(a, b, c) = (6, 3, 5)$.

6. (a) $(43/26, 23/26, -38/26)$, $7\sqrt{3/26}$;
(b) $(13/9, -5/9, 2/9)$, $\sqrt{22}/3$.

7. (a) $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + r(3\mathbf{i} + \mathbf{j} - \mathbf{k})$,
(b) $(4\mathbf{i} + 3\mathbf{j}) + r(2\mathbf{j} - \mathbf{k})$,
(c) $(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) + r(-1/3\ \mathbf{i} - 2/3\ \mathbf{j} + 2/3\ \mathbf{k})$.

8. $(5, 1, 1)$.

IV C

1. (a) $x = 1 + r$
$y = -1 + r + s$
$z = 1 \quad - s$,

(b) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(c) $x - y - z = 1$,
(d) $(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + r(\mathbf{i} + \mathbf{j}) + s(\mathbf{j} - \mathbf{k})$,
(e) $= (\mathbf{i} + 4\mathbf{j}) + 3\mathbf{k} + r[(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - \mathbf{k})] = (\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + r(-\mathbf{i} + \mathbf{j} + \mathbf{k})$.

2. (a) $= (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + r[(3 - 1)\mathbf{i} + (1 - (-1))\mathbf{j} + (0 - 2)\mathbf{k}] +$
$s[(2 - 1)\mathbf{i} + ((-1) - (-1))\mathbf{j} + (4 - 2)\mathbf{k}] = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) +$
$r(2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + s(\mathbf{i} + 2\mathbf{k})$,
(b) $(5\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) + r(-4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + s(\mathbf{i} - \mathbf{j} - 4\mathbf{k})$.

3. (a) $$\begin{aligned} a - b + 2c &= d \\ 3a + b \quad &= d \\ 2a - b + 4c &= d \end{aligned}$$ has the augmented matrix

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 1 & -1 & 2 & -1 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ 2 & -1 & 4 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -6 & -2 & 0 \end{bmatrix},$$
which gives $d = 3$, $c = -1$, $b = -3$, and $a = 2$. Thus an answer is $2x - 3y - z = 3$.

(b) From
$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 5 & 2 & 7 & -1 & 0 \\ 1 & 0 & 8 & -1 & 0 \\ 6 & 1 & 3 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 8 & -1 & 0 \\ 0 & 1 & -45 & 5 & 0 \\ 0 & 0 & 19 & -2 & 0 \end{bmatrix}$$
we have $d = 19$, $c = 2$, $b = -5$, $a = 3$. Thus $3x - 5y + 2z = 19$ is an answer.

4. (a) The plane has standard form $6x - 3y - z = 8$. Thus an answer is $(8(6)/46, 8(-3)/46, 8(-1)/46) = (24/23, -12/23, -4/23)$.
   (b) $= |4/23\langle 6, -3, -1\rangle| = 8/\sqrt{46}$.

5. $= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + r(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + s[(\mathbf{i} - 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})] = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + r(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + s(-2\mathbf{j} - \mathbf{k})$.

6. $(\mathbf{i} + \mathbf{k}) + r(3\mathbf{i} - \mathbf{j}) + s(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

7. (a) $4x - y = 6$, $2x - y = 8$ gives $x = -1$, $y = -10$,
   (b) $4x - y = 3$, $2x - y = 7$ gives $x = -2$, $y = -11$,
   (c) $(-\mathbf{i} - 10\mathbf{j}) + r(-\mathbf{i} - \mathbf{j} + \mathbf{k})$.

8. It contains $(5, 1, 0)$, $(5, 2, 1)$; thus an answer is $(5\mathbf{i} + \mathbf{j}) + r(\mathbf{j} + \mathbf{k})$.

IV D

1. (a) $= \langle 1, 2\rangle + [r(\langle 3, -4\rangle - \langle 1, 2\rangle)] = \langle 1, 2\rangle + [r\langle 2, -6\rangle]$,
   (b) $\langle 2, 5\rangle + [r\langle 5, -8\rangle]$,
   (c) $\langle 3, 1, 4\rangle + [r\langle 3, -1, -2\rangle]$,
   (d) $\langle 7, 0, -2\rangle + [r\langle -6, 1, 6\rangle]$.

2. (a) $\langle 1, 2\rangle + [r\langle 2, -2\rangle + s\langle 2, -3\rangle]$, $\langle 1, 2\rangle + [r\langle 2, -2\rangle + rs\langle 2, -3\rangle]$,
   (b) $\langle 4, -1\rangle + [r\langle -2, 6\rangle + s\langle 5, -2\rangle]$; $\langle 4, -1\rangle + [r\langle -2, 6\rangle + rs\langle 5, -2\rangle]$,
   (c) $\langle 1, 1, 2\rangle + [r\langle 2, -1, 4\rangle + s\langle 4, -1, -4\rangle]$; $\langle 1, 1, 2\rangle + [r\langle 2, -1, 4\rangle + rs\langle 4, -1, -4\rangle]$

3. (a) $\langle 2, 1, 6\rangle + [r\langle 1, 3, -8\rangle + s\langle 4, 1, 1\rangle + t\langle -3, -5, 7\rangle]$;
   $\langle 2, 1, 6\rangle + [r\langle 1, 3, -8\rangle + rs\langle 4, 1, 1\rangle + rst\langle -3, -5, 7\rangle]$.
   (b) $\langle 3, 0, 7\rangle + [r\langle -1, -5, -6\rangle + s\langle 4, 2, 3\rangle + t\langle -4, 10, -8\rangle]$;
   $\langle 3, 0, 7\rangle + [r\langle -1, -5, -6\rangle + rs\langle 4, 2, 3\rangle + rst\langle -4, 10, -8\rangle]$.

4. (a) $= |\langle 2, -4\rangle| = 2\sqrt{5}$,
   (b) $= |\langle 4, 2, 6\rangle| = 2\sqrt{14}$,
   (c) $= |\langle 2, 0, 1\rangle \times \langle 0, 2, 3\rangle| = |\langle -2, -6, 4\rangle| = 2\sqrt{14}$,
   (d) $= 1/2|\langle 1, 0, 2\rangle \times \langle -1, 1, 5\rangle| = 1/2|\langle -2, -7, 1\rangle| = \sqrt{54}/2$,
   (e) $|(\langle 1, -1, 2\rangle \times \langle 3, 0, 5\rangle) \cdot \langle 1, -1, 4\rangle| = 6$,
   (f) $= 1/6|(\langle 1, 0, 2\rangle \times \langle 1, -1, 5\rangle) \cdot \langle 2, 0, 7\rangle| = 1/2$.

5. (a) $= |\langle 1, 1, 0\rangle \times \langle 2, -1, 0\rangle| = 3$,
   (b) $= |\langle 2, 3, 0\rangle \times \langle 1, 0, 0\rangle| = 3$,
   (c) $= 1/2|\langle 1, 0, 0\rangle \times \langle 2, 7, 0\rangle| = 7/2$,
   (d) $= 1/2|\langle 2, -3, 0\rangle \times \langle 6, 5, 0\rangle| = 14$.

Chapter V

V A

1. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$,

   (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$.

2. (a) $\begin{aligned} x &= 2 + r + 2s \\ y &= 3 \phantom{{}+r} + 4s, \end{aligned}$
   (b) $\begin{aligned} x &= \phantom{-1+{}} 2r + 3s. \\ y &= -1 + r. \end{aligned}$

3. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$,

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} [r]$,

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$,

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} [r]$,

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$,

(f) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

4. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0-1 & 1-1 & 2-1 \\ 1-1 & 3-1 & 0-1 \\ 2-0 & 4-0 & 1-0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$,

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 5 \\ -3 & -6 & -3 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$.

5. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$,

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -11 & -6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$,

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$.

V B

1. (a) $\begin{bmatrix} 1+2 \\ 3+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, (b) $\begin{bmatrix} 4 & 7 \\ 3 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 3 \\ -1 & 3 \\ -2 & 9 \end{bmatrix}$,

(d) $\begin{bmatrix} 5 & 4 & 0 \\ -2 & 6 & 0 \\ 5 & 15 & -7 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 6(1) \\ 6(3) \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & 0 & 6 \\ 15 & 3 & 24 \end{bmatrix}$.

3. (a) $= \begin{bmatrix} 7 & 0 \\ 21 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 23 & 24 \end{bmatrix}$,

(b) $= \begin{bmatrix} 8 \\ 24 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$,

(c) $= \begin{bmatrix} 4 & 8 \\ 12 & 4 \\ 0 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 3 & 21 \\ 18 & 12 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 9 & -17 \\ -18 & 8 \end{bmatrix}$.

4. (a) $= \begin{bmatrix} 1+1 \\ 2+1 \end{bmatrix} + \begin{bmatrix} 0+1 & 1+2 \\ 5-3 & 3-5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}; \quad \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 15 & 9 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

(b) $\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 & 9 \\ 8 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}; \quad \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & 3 \\ 15 & 9 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

V C

1. (a) $= \begin{bmatrix} 1(4) + 2(2) \\ 3(4) + 1(2) \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$,

(b) $= \begin{bmatrix} 2(1) + 1(-1) & 2(2) + 1(4) \\ 3(1) + 1(-1) & 3(2) + 1(4) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 2 & 10 \end{bmatrix}$,

(c) $\begin{bmatrix} 6 & 8 \\ 8 & 19 \\ 4 & 13 \end{bmatrix}$.

2. (a) $= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 27 \\ 22 \end{bmatrix}$,

(b) $= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -14 \\ -17 \end{bmatrix} = \begin{bmatrix} -13 \\ -14 \end{bmatrix}$.

3. $= \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 6 \\ 0 & 5 & 7 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 14 \end{bmatrix}$.

4. $= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 5 \\ 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 15 \\ 10 \end{bmatrix}$.

5. (a) $= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right)$

$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix} + \begin{bmatrix} 4 & -5 \\ 27 & 30 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 13 \end{bmatrix} + \begin{bmatrix} 4 & -5 \\ 27 & 30 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$,

(b) $= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & 8 \end{bmatrix} \left(\begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right)$

$= \begin{bmatrix} 24 \\ 14 \end{bmatrix} + \begin{bmatrix} 19 & 7 \\ 77 & 23 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

6. (a) $$= \begin{bmatrix}1\\0\\6\end{bmatrix} + \begin{bmatrix}1 & 0\\2 & 4\\-5 & 3\end{bmatrix}\left(\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-1 & 2 & 0\\3 & -9 & 1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}\right)$$

$$= \begin{bmatrix}3\\8\\-1\end{bmatrix} + \begin{bmatrix}-1 & 2 & 0\\10 & -32 & 4\\14 & -37 & 3\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix},$$

(b) $$\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}1 & 1 & 9\\2 & 0 & 6\end{bmatrix}\left(\begin{bmatrix}3\\1\\1\end{bmatrix} + \begin{bmatrix}4\\1\\2\end{bmatrix}[r]\right)$$

$$= \begin{bmatrix}17\\12\end{bmatrix} + \begin{bmatrix}23\\20\end{bmatrix}[r].$$

7. (a) $$= \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}-2 & 0\\2 & 1\end{bmatrix}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1 & -2\\-1 & 0\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\right)$$

$$= \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}-2\\3\end{bmatrix} + \begin{bmatrix}10\\-11\end{bmatrix} = \begin{bmatrix}11\\-8\end{bmatrix},$$

(b) $$= \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\-3 & -4 & -1\end{bmatrix}\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\\-2 & -6\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-48\\1\end{bmatrix}.$$

V D

1. (a) $$= \frac{1}{10-(-3)}\begin{bmatrix}5 & -1\\3 & 2\end{bmatrix} = 1/13 \begin{bmatrix}5 & -1\\3 & 2\end{bmatrix},$$

(b) $$-1/21\begin{bmatrix}1 & -4\\-6 & 3\end{bmatrix}.$$

2. (a) $$\begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\1 & 0 & 2 & 0 & 0 & 1\end{bmatrix},$$

(b) $$\begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & -1 & 2 & -1 & 0 & 1\end{bmatrix},$$

(c) $$\begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 2 & -1 & 1 & 1\end{bmatrix},$$

(d) $$\begin{bmatrix}1 & 0 & 0 & 1 & -1 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 2 & -1 & 1 & 1\end{bmatrix},$$

(e) $$\begin{bmatrix}1 & 0 & 0 & 1 & -1 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & -1/2 & 1/2 & 1/2\end{bmatrix},$$

(f) $$\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\-1/2 & 1/2 & 1/2\end{bmatrix},$$

(g) $$AA^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

3. (a) $\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$

gives $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{bmatrix}$

gives $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 5 & -1 & -1 \\ 0 & 2 & 0 & 7 & -3 & -1 \\ 0 & 0 & 2 & -3 & 1 & 1 \end{bmatrix}$

gives 1/2 $\begin{bmatrix} 5 & -1 & -1 \\ 7 & -3 & -1 \\ -3 & 1 & 1 \end{bmatrix}$.

4. (a) $= -\begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$

$= -1/8 \begin{bmatrix} 4 \\ 7 \end{bmatrix} + 1/8 \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$,

(b) $= -\begin{bmatrix} 6 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$

$= 1/12 \begin{bmatrix} -5 \\ 6 \end{bmatrix} + 1/12 \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$,

(c) $= -\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Chapter VI

VI A

1. (a) $= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + r \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 13 \\ -10 \end{bmatrix} + r \begin{bmatrix} 7 \\ 8 \end{bmatrix}$,

(b) $= \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + r \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 17 \\ 6 \end{bmatrix} + r \begin{bmatrix} 11 \\ -2 \end{bmatrix}$.

2. (a) $= \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -5 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} + r\begin{bmatrix} 2 \\ -8 \\ -14 \end{bmatrix}$,

(b) $= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + r\begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$.

3. (a) $= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -6 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -10 \end{bmatrix} + r\begin{bmatrix} 14 \\ 10 \end{bmatrix}$,

(b) $= \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 6 \\ -1 & 0 & 1 \\ 3 & 2 & -7 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + r\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 17 \\ 5 \\ -13 \end{bmatrix} + r\begin{bmatrix} -4 \\ -2 \\ 10 \end{bmatrix}$.

4. (a) $= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 1 \\ -1 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + r\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 21 \\ 3 \\ 20 \end{bmatrix} + r\begin{bmatrix} 30 \\ 20 \\ 54 \end{bmatrix} + s\begin{bmatrix} 8 \\ 5 \\ 12 \end{bmatrix}$,

(b) $= \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 7 \\ -1 & 4 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + r\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 15 \\ 2 \end{bmatrix} + r\begin{bmatrix} 8 \\ 46 \\ 14 \end{bmatrix} + s\begin{bmatrix} -4 \\ 12 \\ 1 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + r\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right)$, $0 \leq r \leq 1$ goves $\langle 10, 14\rangle + [r\langle -6, -22\rangle]$,

(b) $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & -1 \\ 0 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + r\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right)$, $0 \leq r \leq 1$ gives $\langle 12, -2, 19\rangle$ $+ [r\langle -8, 10, -20\rangle]$.

6. (a) $\begin{bmatrix} 1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -5 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + r\begin{bmatrix} 3 \\ -2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 9 \end{bmatrix}\right)$,

$0 \leq r, s \leq 1$ gives $\langle 12, 5\rangle + [r\langle 1, -19\rangle + s\langle 42, 8\rangle]$,

(b) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 0 & -1 \\ 2 & -5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + r\begin{bmatrix} 3 \\ -2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 9 \end{bmatrix}\right)$, $0 \leq r, s \leq 1$

gives $\langle 12, -2, -6\rangle + [r\langle 1, 2, 16\rangle + s\langle 42, -9, -41\rangle]$.

7. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + r\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$, $0 \leq r, s, t \leq 1$

gives $\langle 16, 31, 18\rangle + [r\langle 6, -2, 14\rangle + s\langle 3, 16, 7\rangle + t\langle 5, 4, 7\rangle]$.

VI B

1. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (b) $= \begin{bmatrix} 1 & 0 + 4(1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. (a) E_{12}, (b) $E_{2+(-5)3}$, (c) $E_{(1/6)2}$.

4. (a) $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$, (c) $= \begin{bmatrix} 1 & 3 \\ 0+4(1) & 2+4(3) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 14 \end{bmatrix}$,

(d) $= \begin{bmatrix} 1+4(3) & 3 \\ 0+4(2) & 2 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ 8 & 2 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 3 \\ 0 & 12 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 18 \\ 0 & 12 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 3 & 4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3+3(1) & 4+3(2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 6 & 10 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 3 & 13 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 12 & 16 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 10 \\ 0 & 5 \\ 3 & 20 \end{bmatrix}$

VI C

1. (a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2. (a) $\to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(b) $\to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,

(c) $\to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

3. (a) $A \to A_1 = E_{1+(-3)2}A \to A_2 = A_1E_{2+(-2)1} \to A_2\,E_{(-1/2)2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_{1+(-3)2}\,AE_{2+(-2)1}E_{(-1/2)2}$,

(b) $A \to A_1 = E_{1+(-1)2}\,A \to A_2 = E_{2+(3)3}\,A_1 \to A_3 = A_2\,E_{2+(-2)1} \to A_3\,E_{(-1)2}$

gives $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{2+(3)3}\,E_{1+(-1)2}\,AE_{2+(-2)1}E_{(-1)2}$,

(c) $AE_{2+(-2)1}E_{3+(-3)1}E_{3+(-4)2}$.

4. (a) $C = E_{1+(-3)2} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $D = E_{2+(-2)1}E_{(-1/2)2} = \begin{bmatrix} 1 & 1 \\ 0 & -1/2 \end{bmatrix}$ gives

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1/2 \end{bmatrix}$,

(b) $C = E_{2+(3)3}\,E_{1+(-1)2} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix}$, $D = E_{2+(-2)1}E_{(-1)2} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

gives $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

5. $= E_{1+(-1)2}{}^{-1} E_{(1/2)1}{}^{-1} E_{2+(-1)1}{}^{-1} = E_{1+(1)2} E_{(2)1} E_{2+(1)1}$.

6. (a) $\to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_{1+(-2)2} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} E_{(1/3)2}$; hence,

$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = E_{1+(-2)2}{}^{-1} E_{(1/3)2}{}^{-1} = E_{1+(2)2} E_{(3)2}$.

(b) $\to \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives $E_{1+(1/2)2} E_{(4)2} E_{(2)1}$.

VI D

1. (a) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 6 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 5 \\ 1 & 5 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 5 & 7 \\ 0 & 5 & 3 & 6 \\ 2 & 7 & 6 & 4 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

$r = s = 3$, $r = 3, s = 2$, $r = s = 2$,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

$r = 3, s = 1$, $r = 2, s = 1$, $r = s = 1$.

4. (a) $\begin{bmatrix} 1 & 6 \\ 0 & 64 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. (a) $\to \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $r = 2, s = 1$.

(b) $\to \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $r = s = 1$.

(c) $\to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$;

$r = 3, s = 2$.

6. (a) $\to \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 1 & -1/2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & -1/2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

$r = 2, s = 1$.

(b) $\to \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \\ 5 & 3 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$;

$r = 3, s = 1$.

Chapter VII

VII A

1. (a) $= 2(-4) - 3(6) = -26$, (b) $= 1(-3) - 7(2) = -17$.
2. (a) $= 1(4)(6) = 24$, (b) $= 2(-3)(4) = -24$.
3. $\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 2 & 2 & 6 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 2 & 6 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} =$

$\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} = 4$.

4. (a) $= -\det \begin{bmatrix} 1 & 0 & 3 \\ 4 & 2 & 6 \\ 7 & 3 & 9 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & -6 \\ 7 & 3 & -12 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 7 & 3 & -3 \end{bmatrix} =$

$-(-6) = 6$.

(b) $= \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 8 & 7 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \to \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & -15 & -11 \\ 0 & 0 & -10 & -8 \end{bmatrix} =$

$1/6 \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & -30 & -22 \\ 0 & 0 & -30 & -24 \end{bmatrix} = 1/6 \det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & -30 & -22 \\ 0 & 0 & 0 & -2 \end{bmatrix} =$

$1/6\,(60) = 10$.

5. (a) $= (-1)^{3+2}(4)\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 12$,

(b) $= (-1)^{2+2}(-2)\det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = 10$,

(c) $= (-1)^{1+3}(3)\det \begin{bmatrix} 4 & 1 & 1 \\ 5 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} = 3(-1)^{3+1}(8)\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 24$.

6. (a) $= \det \begin{bmatrix} 1 & 3 & 4 \\ 0 & -5 & -3 \\ 0 & 4 & 2 \end{bmatrix} = (1)\det \begin{bmatrix} -5 & -3 \\ 4 & 2 \end{bmatrix} = 2$,

(b) $$=\det\begin{bmatrix}1&3&2&7\\0&-5&2&-14\\0&4&3&2\\0&5&1&5\end{bmatrix}=\det\begin{bmatrix}-5&2&-14\\4&3&2\\5&1&5\end{bmatrix}$$

$$=\det\begin{bmatrix}-15&0&-24\\-11&0&-13\\5&1&5\end{bmatrix}=(-1)\det\begin{bmatrix}-15&-24\\-11&-13\end{bmatrix}=-(195-264)=69,$$

(c) $$=(4)\det\begin{bmatrix}4&5&4\\2&-1&3\\6&7&11\end{bmatrix}=(4)\det\begin{bmatrix}14&0&19\\2&-1&3\\20&0&32\end{bmatrix}=(-4)\det\begin{bmatrix}14&19\\20&32\end{bmatrix}$$

$$=-4(448-380)=-272.$$

VII B

1. (a) $[1\quad 3]\begin{bmatrix}1\\3\end{bmatrix}=10$ gives $\sqrt{10}$ by B.1; $\sqrt{1^2+3^2}=\sqrt{10}$ by B.2,

(b) $[1\quad 2\quad 5]\begin{bmatrix}1\\2\\5\end{bmatrix}=30$ gives $\sqrt{30}$ by B.1; $\sqrt{1^2+2^2+5^2}=\sqrt{30}$ by B.2.

(c) $\det\left(\begin{bmatrix}1&-1\\2&4\end{bmatrix}\begin{bmatrix}1&2\\-1&4\end{bmatrix}\right)=\det\begin{bmatrix}2&-2\\-2&20\end{bmatrix}=36$ gives 6 by B.1;
$|4-(-2)|=6$ by B.2.

(d) $\det\left(\begin{bmatrix}1&2&4\\3&3&2\end{bmatrix}\begin{bmatrix}1&3\\2&3\\4&2\end{bmatrix}\right)=\det\begin{bmatrix}21&17\\17&22\end{bmatrix}=173$ gives $\sqrt{173}$ by B.1;

$\det\begin{bmatrix}1&3\\2&3\end{bmatrix}=-3$, $\det\begin{bmatrix}1&3\\4&2\end{bmatrix}=-10$, $\det\begin{bmatrix}2&3\\4&2\end{bmatrix}=-8$ gives
$\sqrt{9+100+64}=\sqrt{173}$ by B.2.

(e) $\det\left(\begin{bmatrix}1&6&2\\4&2&1\\5&3&4\end{bmatrix}\begin{bmatrix}1&4&5\\6&2&3\\2&1&4\end{bmatrix}\right)=\det\begin{bmatrix}41&18&31\\18&21&30\\31&30&50\end{bmatrix}=3249$ gives $\sqrt{3249}=57$

by B.1; $\det\begin{bmatrix}1&4&5\\6&2&3\\2&1&4\end{bmatrix}=-57$ gives $|-57|=57$ by B.2.

2. (a) $\det\left(\begin{bmatrix}1&1&0&1\\0&1&1&0\\1&0&2&1\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\0&1&2\\1&0&1\end{bmatrix}\right)=\det\begin{bmatrix}3&1&2\\1&2&2\\2&2&6\end{bmatrix}=18$ gives $3\sqrt{2}$;

(b) $\det\left(\begin{bmatrix}0&2&0&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}0&1\\2&2\\0&3\\1&4\end{bmatrix}\right)=\det\begin{bmatrix}5&8\\8&30\end{bmatrix}=86$ gives $\sqrt{86}$.

VII C

1. (a) $=\left\|\begin{bmatrix}2\\1\end{bmatrix}\right\|=\sqrt{5}$, (b) $\left\|\begin{bmatrix}1&1\\-1&2\end{bmatrix}\right\|=3$.

(c) $= \left\| \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 1 & 0 \end{bmatrix} \right\| = \sqrt{22}$, (d) $\left\| \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 3 \\ 0 & 1 & 1 \end{bmatrix} \right\| = 16.$

2. (a) $\mathbf{PQ} = \langle 2, -3 \rangle$ gives $\left\| \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\| = \sqrt{13}$,

   (b) $\mathbf{PQ} = \langle 2, -3 \rangle$, $\mathbf{QR} = \langle 1, -2 \rangle$ gives $\left\| \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \right\| = 1.$

3. (a) $\mathbf{PQ} = \langle 1, 1, -9 \rangle$ gives $\left\| \begin{bmatrix} 1 \\ 1 \\ -9 \end{bmatrix} \right\| = \sqrt{83}$,

   (b) $\mathbf{PQ} = \langle 1, 1, -9 \rangle$, $\mathbf{QR} = \langle 1, 0, 8 \rangle$ gives $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -9 & 8 \end{bmatrix} \right\| = \sqrt{354}$,

   (c) $= \left\| \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -9 & 8 & -2 \end{bmatrix} \right\| = 3.$

4. (a) $= 1/2 \left\| \begin{bmatrix} 2 & 3 \\ -2 & -5 \end{bmatrix} \right\| = 2$, (b) $= 1/2 \left\| \begin{bmatrix} 1 & -2 \\ 9 & 2 \end{bmatrix} \right\| = 10$,

   (c) $= 1/2 \left\| \begin{bmatrix} 5 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix} \right\| = \sqrt{158}/2$, (d) $= 1/2 \left\| \begin{bmatrix} 1 & -1 \\ -2 & 8 \\ -2 & -3 \end{bmatrix} \right\| = \sqrt{545}/2.$

5. (a) $= 1/6 \left\| \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -3 \\ 0 & 5 & -1 \end{bmatrix} \right\| = 7/2$,

   (b) $= 1/6 \left\| \begin{bmatrix} 2 & 4 & -2 \\ 3 & -6 & 8 \\ -2 & 4 & -4 \end{bmatrix} \right\| = 16/3.$

Chapter VIII

VIII A

1. (a) $\det\left(\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} 1-\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix} = 0$
   gives $\lambda^2 - 7 = 0$;

   (b) $\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = 0$ gives $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

2. (a) Characteristic equation $(2-\lambda)(1-\lambda) = 0$ gives 1, 2,
   (b) Characteristic equation $(2-\lambda)(2-\lambda) = 0$ gives 2,
   (c) $\lambda^2 - 3\lambda + 12 = 0$ gives no (real) eigenvalues.

3. In 2(a) the eigenvalue 1 gives $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$,

   and this yields equations $\begin{aligned} x \quad &= x \\ x + 2y &= y \end{aligned}$ with solution $x = c$, $y = -c$ for c arbitrary.

Thus $c\langle 1, -1\rangle$ describes eigenvectors belonging to 1. Similarly $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $= 2\begin{bmatrix} x \\ y \end{bmatrix}$ gives $c\langle 0, 1\rangle$ as eigenvectors.

In 2 (b) the eigenvalue 2 gives $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, and this yields $\begin{matrix} 2x + y = 2x \\ 2y = 2y \end{matrix}$ which has as its solution $x = c$, $y = 0$. Thus $c\langle 1, 0\rangle$ describes the eigenvectors belonging to 2.

4. (a) The characteristic equation $\lambda^2 - 6\lambda + 5 = 0$ gives eigenvalues 1 and 5. The equations $\begin{matrix} 2x + \ \ y = x \\ 3x + 4y = y \end{matrix}$ give $c\langle -1, 1\rangle$ as eigenvectors belonging to 1. The equations $\begin{matrix} 2x + \ \ y = 5x \\ 3x + 4y = 5y \end{matrix}$ give $c\langle 1, 3\rangle$ as eigenvectors belonging to 5.

(b) $\lambda^2 - 5\lambda = 0$ gives eigenvalues 0 and 5. The equations $\begin{matrix} 4x + y = 0x \\ 4x + y = 0y \end{matrix}$ give $c\langle 1, -4\rangle$ as eigenvectors for 0. The equations $\begin{matrix} 4x + y = 5x \\ 4x + y = 5y \end{matrix}$ give $c\langle 1, 1\rangle$ as eigenvectors for 5.

VIII B

1. (a) $2x^2 + 14xy - 4y^2$, (b) $x^2 + 6xy + 12xz + y^2 - 2yz + 4z^2$.

2. (a) $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & -1 \end{bmatrix}$.

3. (a) positive-semidefinite, (b) indefinite,
(c) positive–definite, (d) negative–definite,
(e) indefinite, (f) negative–semidefinite.

4. (a) Indefinite since there is a positive and a negative diagonal entry;

(b) $\to \begin{bmatrix} 9 & -6 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$ gives positive–semidefinite;

(c) $\to \begin{bmatrix} -1 & 3 \\ 0 & 18 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & 18 \end{bmatrix}$ gives indefinite;

(d) $\to \begin{bmatrix} 6 & 2 & 2 \\ 0 & 4/3 & -5/3 \\ 2 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 2 \\ 0 & 4/3 & -5/3 \\ 2 & -5/3 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

gives positive-definite.

VIII C

1. (a) $x^2 + 8xy - y^2$ gives $\begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 4 & -1/2 \\ -1/2 & 0 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 3\lambda + 1 = 0$ with roots $(3 \pm \sqrt{5})/2$. Thus, the eigenvalues are both positive, and the conic is an ellipse.

(b) $\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 3\lambda - 2 = 0$ with roots $(3 \pm \sqrt{17})/2$. The eigenvalues differ in sign and the conic is a hyperbola.

(c) $\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4\lambda = 0$. The eigenvalues 0, 4 imply that the conic is a parabola.

(d) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ has characteristic equation $(\lambda - 4)^2 = 0$; the conic is a circle.

3. (a) $\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$ has eigenvalues 7 and -3 and eigenvectors $c\langle 3, 1\rangle$, $c\langle -1, 3\rangle$. Thus $\langle 3/\sqrt{10}, 1/\sqrt{10}\rangle$ and $\langle -1/\sqrt{10}, 3/\sqrt{10}\rangle$ are the desired unit vectors; the equation relative to axes in the direction of these vectors is $7(x')^2 - 3(y')^2 - 21 = 0$.

(b) $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ has eigenvalues 6 and 1 and eigenvectors $c\langle 2, 1\rangle$, $c\langle -1, 2\rangle$. Thus $\langle 2/\sqrt{5}, 1/\sqrt{5}\rangle$ and $\langle -1/\sqrt{5}, 2/\sqrt{5}\rangle$ are the desired unit vectors, and the corresponding conic equation is $6(x')^2 + (y')^2 - 6 = 0$.

(c) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues 5 and 0 and eigenvectors $c\langle 2, 1\rangle$, $c\langle -1, 2\rangle$. The desired unit vectors are $\langle 2/\sqrt{5}, 1/\sqrt{5}\rangle$, $\langle -1/\sqrt{5}, 2/\sqrt{5}\rangle$. Substitution from $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & \sqrt{2}/\sqrt{5} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ into the given equation gives $5(x')^2 + x' - 3y' - 9 = 0$.

Chapter IX

IX A

1. (a) $=(xy + xz)\mathbf{i} + (xz - yz)\mathbf{j} = x(y + z)\mathbf{i} + z(x - y)\mathbf{j}$,
   (b) $3xy\mathbf{i} + 3xz\mathbf{j}$,
   (c) $=(xy + 2xz)\mathbf{i} + (xz - 2yz)\mathbf{j} = x(y + 2z)\mathbf{i} + z(x - 2y)\mathbf{j}$,
   (d) $x(4y - z)\mathbf{i} + z(4x + y)\mathbf{j}$.

2. (a) $x(y + 1)\mathbf{i} + (x^2 - y^2)\mathbf{k}$,
   (b) $x(3y + 2)\mathbf{i} - yz\mathbf{j} + (3x^2 - 2y^2)\mathbf{k}$,
   (c) $x(2y - 1)\mathbf{i} - 3yz\mathbf{j} + (2x^2 + y^2)\mathbf{k}$.

3. (a) $x^3y\mathbf{i} + x^2y^2\mathbf{j}$, (b) $3xy^3\mathbf{i} - 3x^2y^2\mathbf{j}$,
   (c) $=x^2y^2 - x^2y^2 = 0$, (d) $=x^3y^3 - x^3y^3 = 0$.

4. (a) $x^2y\mathbf{i} - xy^2\mathbf{j} + x^2yz\mathbf{k}$, (b) $=xy + xy + xyz^2 = xy(2 + z^2)$,
   (c) $=\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & -y & xz \\ y & -x & yz \end{bmatrix} = z(x^2 - y^2)\mathbf{i} + (y^2 - x^2)\mathbf{k}$,
   (d) $=x^2y^2 - x^2y^2 + x^2y^2z^2 = x^2y^2z^2$,
   (e) $=\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ xy^2 & -x^2y & xy^2z \\ x & -y & xz \end{bmatrix} = xyz(y^2 - x^2)\mathbf{i} + xy(x^2 - y^2)\mathbf{k}$,
   (f) 0.

5. $(r - s)^2(rs)^2\mathbf{i} + (r - s)\mathbf{j} - rs\mathbf{k}$.

6. (a) $(r^2s - r + s)\mathbf{i} + r^2s(r - s)\mathbf{j}$, (b) $r(1 - r)\mathbf{i} + r\ln r^2\mathbf{j}$
   (c) $rs(s - r)\cos r^2$, (d) $-r^4s^2t^2\mathbf{i} - e^{rst}\mathbf{j}$.

IX B

1. (a) $y - x^3 + 3x - 2 = 0$,
   (b) $x = r, y = r^3 - 3r + 2$,
   (c) $\mathbf{f}(r) = r\mathbf{i} + (r^3 - 3r + 2)\mathbf{j}$.
2. (a) $x - 4y^2 + 2y - 7 = 0$,
   (b) $x = 4r^2 - 2r + 7, y = r$,
   (c) $\mathbf{f}(r) = (4r^2 - 2r + 7)\mathbf{i} + r\mathbf{j}$.
3. (a) $x = 2\cos\theta, y = 2\sin\theta$,
   (b) $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j}$.
   (c) Since $y \geq 0$ for each point (x, y) above the x axis, the equation $y = \sqrt{4 - x^2}$ is the correct choice.
   (d) Since $x \leq 0$ for each point (x, y) to the left of the y axis, the equation $x = -\sqrt{4 - y^2}$ is the correct choice.
4. (a) $x = 2\cos\theta, y = 3\sin\theta$,
   (b) $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$,
   (c) $x = 2/3\sqrt{9 - y^2}$.
5. (a) $x \geq 0$ and $y \geq 0$ gives $x = 2\cosh r, y = 3\sinh r$; $\mathbf{f}(r) = 2\cosh r\,\mathbf{i} + 3\sinh r\,\mathbf{j}$; $y = 3/2\sqrt{x^2 - 4}$.
   (b) $x \leq 0, y \geq 0$ gives $x = -2\cosh r, y = 3\sinh r$; $\mathbf{f}(r) = -2\cosh r\,\mathbf{i} + 3\sinh r\,\mathbf{j}$; $y = 3/2\sqrt{x^2 - 4}$.
   (c) $x \leq 0, y \leq 0$ gives $x = -2\cosh r, y = 3\sinh r$; $\mathbf{f}(r) = -2\cosh r\,\mathbf{i} + 3\sinh r\,\mathbf{j}$; $y = -3/2\sqrt{x^2 - 4}$. The choice $y = -3\sinh r$ for the parametric equations is also correct.
6. (a) $x = 3\sinh r, y = 4\cosh r$; $\mathbf{f}(r) = 3\sinh r\,\mathbf{i} + 4\cosh r\,\mathbf{j}$; $y = 4/3\sqrt{x^2 + 9}$.
   (b) Same answer as (a).
   (c) $x = 3\sinh r, y = -4\cosh r$; $\mathbf{f}(r) = 3\sinh r\,\mathbf{i} - 4\cosh r\,\mathbf{j}$; $y = -4/3\sqrt{x^2 + 9}$
7. (a) $y - 3x^2 = 0$ (b) $x = r, y = 3r^2$ (c) $\mathbf{f}(r) = r\mathbf{i} + 3r^2\mathbf{j}$.
8. (a) $x + 5y^2 = 0$ (b) $x = -5r^2, y = r$, (c) $\mathbf{f}(r) = -5r^2\mathbf{i} + r\mathbf{j}$.
9. (a) $y = -2 + (x - 3)^2 + 3(x - 3) - 2$ gives $y = x^2 - 3x - 4$,
   (b) $x = -2 + 4(y - 1)^2$ gives $x = 4y^2 - 8y + 2$.
10. (a) $(x - 2)^2 + (y - 3)^2 = 4$,
    (b) $(x - (-3))^2/4 + y^2/9 = 1$ gives $(x + 3)^2/4 + y^2/9 = 1$,
    (c) $(x - 1)^2/4 - (y + 2)^2/9 = 1$.
11. (a) $\mathbf{f}(\theta) = (2\cos\theta + 2)\mathbf{i} + (2\sin\theta + 3)\mathbf{j}$,
    (b) $\mathbf{f}(\theta) = (2\cos\theta - 3)\mathbf{i} + 3\sin\theta\,\mathbf{j}$,
    (c) $\mathbf{f}(r) = (2\cosh r + 1)\mathbf{i} + (3\sinh r - 2)\mathbf{j}$.
12. (a) This is the curve $y = 4x^2$ shifted 2 units to the right and 3 units up. Since $y = 4x^2$ has vector equation $\mathbf{f}(r) = r\mathbf{i} + 4r^2\mathbf{j}$, an answer is $\mathbf{f}(r) = (r + 2)\mathbf{i} + (4r^2 + 3)\mathbf{j}$.
    (b) This is $x^2 + y^2 = 16$, or $\mathbf{f}(\theta) = 4\cos\theta\,\mathbf{i} + 4\sin\theta\,\mathbf{j}$, shifted 4 units to the right and 1 unit up. Thus an answer is $\mathbf{f}(\theta) = (4\cos\theta + 4)\mathbf{i} + (4\sin\theta + 1)\mathbf{j}$.
    (c) This is $x^2/4 + y^2/9 = 1$, or $\mathbf{f}(\theta) = 2\cos\theta\mathbf{i} + 3\sin\theta\,\mathbf{j}$, shifted 1 unit to the left and 3 units up. An answer is $\mathbf{f}(\theta) = (2\cos\theta - 1)\mathbf{i} + (3\sin\theta + 3)\mathbf{j}$.
    (d) This is $x^2/1 - y^2/4 = 1$ shifted 5 units up. An answer is $\mathbf{f}(r) = -\cosh r\,\mathbf{i} + (2\sinh r + 5)\mathbf{j}$.

IX C

1. (a) $r\mathbf{i} + s\mathbf{j} + r^2e^s\mathbf{k}$, (b) $r^2 \cos rs\, \mathbf{i} + r\mathbf{j} + s\mathbf{k}$
2. Solving for z (≥ 0) gives $z = \sqrt{-3x^2 - x + y^2}$.
3. (a) $y^2 \sin \phi\, \mathbf{i} + y\mathbf{j} + y^2 \cos \phi\, \mathbf{k}$, (b) $x^2 + z^2 - y^4 = 0$.
4. (a) $y \sin \phi\, \mathbf{i} + y \cos \phi\, \mathbf{j} + y^2\mathbf{k}$,
   (b) $z = (\sqrt{x^2 + y^2})^2$ gives $z = x^2 + y^2$.
5. (a) The y coordinate is not changed. The x and z coordinates are determined as for the polar coordinate point $(r, 0)$ rotated through an angle ϕ in the xz plane. The answer is $(x_o \cos \phi, x_o{}^2, x_o \sin \phi)$.
   (b) $x \cos \phi\, \mathbf{i} + x^2\mathbf{j} + x \sin \phi\, \mathbf{k}$.
   (c) $y = x^2 + z^2$.
6. (a) $(x_o, x_o{}^2 \cos \phi, x_o^2 \sin \phi)$, (b) $x\mathbf{i} + x^2 \cos \phi\, \mathbf{j} + x^2 \sin \phi\, \mathbf{k}$,
   (c) $x^4 - y^2 - z^2 = 0$.
7. Revolving the point $(0, y_o, 3y_o)$ through an angle ϕ about the z axis gives $(y_o \sin \phi, y_o \cos \phi, 3y_o)$. Thus an answer is $y \sin \phi\, \mathbf{i} + y \cos \phi\, \mathbf{j} + 3y\mathbf{k}$.
8. Revolving the point $(4, 0, z_o)$ through an angle ϕ about the z axis gives $(4 \cos \phi, 4 \sin \phi, z_o)$. Thus an answer is $4 \cos \phi\, \mathbf{i} + 4 \sin \phi\, \mathbf{j} + zk$.
9. (b) $z = c \cos \phi$,
   (c) $x = a \sin \phi \cos \theta$, $y = b \sin \phi \sin \theta$.
10. $z = c \sinh \phi$; $x^2/(a \cosh \phi)^2 + y^2/(b \cosh \phi)^2 = 1$ gives $x = a \cosh \phi \cos \theta$, $y = b \cosh \phi \sin \theta$.
11. (a) Set $z = r$; then $x^2/(2r)^2 + y^2/(3r)^2 = 1$ gives $x = 2r \cos \theta$, $y = 3r \sin \theta$.
    (b) The standard equations are $x = r$, $y = s$, $z = 2r^2 + 3s^2$.

Chapter X

X A

1. (a) $2xy^3$; $3x^2y^2$.
   (b) $-xy \sin xy + \cos xy$; $-x^2 \sin xy$.
   (c) $1/(x + \cos y)$; $-\sin y/(x + \cos y)$.
2. (a) $2xy^3z^4$; $3x^2y^2z^4$; $4x^2y^3z^3$. (b) $y^2e^{xy^2+z}$; $2xye^{xy^2+z}$; e^{xy^2+z}.
3. $g_r = g_x \bar{x}_r + g_y \bar{y}_r + g_z \bar{z}_r$.
4. (a) $2r^2s + (r + s)^2$, (b) $4rs + 2(r + s)$,
   (c) $= 2(2rs) + 2\bar{y}(1) = 4rs + 2(r + s)$.
5. (a) $g_r = 2\bar{x}e^s + 1(2r) = 2re^{2s} + 2r$; $g_r(1, 0) = 4$.
   (b) $g_s = \bar{y}(0) + \bar{x}(1) + (-1)(-\sin s) = r^2 + \sin s$; $g_s(1, \pi) = 1$.
6. (a) $g_x \cos \theta + g_y \sin \theta$, (b) $-rg_x \sin \theta + rg_y \cos \theta$.
7. (a) $g_x \cos \theta + g_y \sin \theta$, (b) $-rg_x \sin \theta + rg_y \cos \theta$,
   (c) g_z (the symbol z is used ambiguously here).
8. (a) $g_x \sin \phi \cos \theta + g_y \sin \phi \sin \theta + g_z \cos \phi$,

(b) $\rho\, g_x \cos\phi \cos\theta + \rho g_y \cos\phi \sin\theta - \rho\, g_z \sin\phi$,
(c) $-\rho\, g_x \sin\phi \sin\theta + \rho\, g_y \sin\phi \cos\theta$.

9. (a) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$, (b) $A^{-1} = 1/r\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix}$
gives $g_x = \cos\theta\, g_r - 1/r \sin\theta\, g_\theta$ and $g_y = \sin\theta\, g_r + 1/r\cos\theta\, g_\theta$.

10. $\begin{bmatrix} g_r \\ g_s \end{bmatrix} = \begin{bmatrix} 2r & 1 \\ -1 & 2s \end{bmatrix}\begin{bmatrix} g_x \\ g_y \end{bmatrix}$ and $\begin{bmatrix} 2r & 1 \\ -1 & 2s \end{bmatrix}^{-1} = 1/(4rs+1)\begin{bmatrix} 2s & -1 \\ 1 & 2r \end{bmatrix}$ gives
$g_x = (2sg_r - g_s)/(4rs+1)$ and $g_y = (g_r + 2rg_s)/(4rs+1)$.

X B

1. (a) $x^3 + (h(x))^3 - 2 = 0$, (b) $3x^2 + 3h^2\, dh/dx = 0$,
(c) $dh/dx = -x^2/h^2 = -1/1 = -1$ at $(1, 1)$.

2. Differentiating $xe^{h(x)} + (h(x))^2 - 2 = 0$ gives $xe^h\, dh/dx + e^h + 2h\, dh/dx = 0$.
Substituting $x = 2$, $h = 0$ gives $2\, dh/dx(2) + 1 = 0$, from which $dh/dx(2) = -1/2$.

3. (a) $2\bar{x}\bar{x}_r + 2\bar{y}\bar{y}_r + s = 0 \qquad 2\bar{x}\bar{x}_s + 2\bar{y}\bar{y}_s + r = 0$
$\bar{x}\bar{y}_r + \bar{x}_r\bar{y} - s^2 = 0; \qquad \bar{x}\bar{y}_s + \bar{x}_s\bar{y} - 2rs = 0.$
(b) $2\bar{x}_r + 4\bar{y}_r + 1 = 0 \qquad 2\bar{x}_s + 4\bar{y}_s + 1 = 0$
$2\bar{x}_r + \bar{y}_r - 1 = 0; \qquad 2\bar{x}_s + \bar{y}_s - 2 = 0$
give $\bar{x}_r = 5/6$, $\bar{y}_r = -2/3$, $\bar{x}_s = 3/2$, $\bar{y}_s = -1$.

4. From $2\bar{x}^2 + \bar{y}^2 - rs - 4 = 0$ and $\bar{x}\bar{y} - rs = 0$ obtain
$4\bar{x}\bar{x}_r + 2\bar{y}\bar{y}_r - s = 0 \qquad 4\bar{x}\bar{x}_s + 2\bar{y}\bar{y}_s - r = 0$
$\bar{x}\bar{y}_r + \bar{x}_r\bar{y} - s = 0; \qquad \bar{x}\bar{y}_s + \bar{x}_s\bar{y} - r = 0.$
At $(r, s, \bar{x}, \bar{y}) = (1, 2, 1, 2)$ this gives
$4\bar{x}_r + 4\bar{y}_r - 2 = 0 \qquad 4\bar{x}_s + 4\bar{y}_s - 1 = 0$
$2\bar{x}_r + \bar{y}_r - 2 = 0; \qquad 2\bar{x}_s + \bar{y}_s - 1 = 0.$
Hence $\bar{x}_r = 3/2$, $\bar{y}_r = -1$, $\bar{x}_s = 3/4$, $\bar{y}_s = -1/2$.

X C

1. (a) $= (x^2y^3)_x\mathbf{i} + (x^2y^3)_y\mathbf{j} = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$,
(b) $y^2e^{xy^2}\mathbf{i} + 2xye^{xy^2}\mathbf{j}$,
(c) $(y^2 + z)\mathbf{i} + 2xy\mathbf{j} + x\mathbf{k}$,
(d) $e^x \sin yz\, \mathbf{i} + ze^x \cos yz\, \mathbf{j} + ye^x \cos yz\, \mathbf{k}$.

2. (a) $= 2(1)(3)^3\mathbf{i} + 3(1)^2(3)^2\mathbf{j} = 54\mathbf{i} + 27\mathbf{j}$,
(b) $9e^9\mathbf{i} + 6e^9\mathbf{j}$.

3. (a) $2\mathbf{i}$, (b) $2\mathbf{j}$.

4. (a) $= \langle y^2, 2xy\rangle|_{\langle 1,2\rangle} \cdot \langle 3, -1\rangle = \langle 4, 4\rangle \cdot \langle 3, -1\rangle = 8$,
(b) $= ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}\,|_{\langle 0,1\rangle} \cdot (2\mathbf{i} + 3\mathbf{j}) = 2$,
(c) $= \langle \ln yz, x/y, x/z\rangle|_{\langle 2,4,1\rangle} \cdot \langle 0, 2, 3\rangle = 7$.

5. Divide each answer in 4 by $|\mathbf{v}_o|$: (a) $8/\sqrt{10}$, (b) $2/\sqrt{13}$, (c) $7/\sqrt{13}$.

6. (a) $= \nabla f(\mathbf{u}_o) = \langle 54, 27\rangle$; $|\langle 54, 27\rangle| = 27\sqrt{5}$,
(b) $= \langle xy\cos xy + \sin xy,\ x^2\cos xy\rangle|_{\langle 1,\pi\rangle} = \langle -\pi, -1\rangle$; $|\langle -\pi, -1\rangle|$
$= \sqrt{\pi^2 + 1}$,
(c) $= \langle 2xye^z, x^2e^z, x^2ye^z\rangle|_{\langle 2,1,0\rangle} = \langle 4, 4, 4\rangle$; $4\sqrt{3}$.

X D

1. (a) $= 64^{1/3} = 4, 0, = 16/3\ (64)^{-2/3} = 1/3;$
   (b) $f(x_1, y_1) \approx f(x_o, y_o) + \Delta f \approx 4 + 0(.001) + 1/3\ (.0003)$ gives 4.0001.
2. Let $f(x, y) = (7x^3 + y^2)^{1/4}$, $(x_o, y_o) = (1, 3)$, $(x_1, y_1) = (1.001, 2.999)$. Then $f(1, 3) = 2$, $f_x(1, 3) = 21/32$, $f_y(1, 3) = 3/16$ give $2 + 21/32\ (.001) + 3/16\ (-.001) \approx 2.0005$.
3. Let $f(x, y, z) = (x^2 + y^2 + 3z^4)^{2/3}$, $(x_o, y_o, z_o) = (5, 6, 1)$ and $(x_1, y_1, z_1) = (5.0001, 6.001, 1.001)$. Then $f(5, 6, 1) = 16$, $f_x(5, 6, 1) = 10/6$, $f_y(5, 6, 1) = 2$, $f_z(5,6,1) = 2$ give $16 + 10/6\ (.0001) + 2(.001) + 2(.001) \approx 16.004$.
4. Let A denote area and x and y the lengths of the sides. Then $A = xy$ and the maximum error in A is estimated by $\Delta A \approx A_x\,\Delta x + A_y\,\Delta y = y\Delta x + x\Delta y = 7(.1) + 4(.1) = 1.1$.
5. Let V denote the volume, r the base radius, and h the height. Then $V = \pi r^2 h$ gives $\Delta V \approx A_r\,\Delta r + A_h\,\Delta h = (2\pi rh)\Delta r + (\pi r^2)\Delta h = 80\pi(.1) + 100\pi(.1) = 18\pi$. Thus $\pi(10^2)4 + 18\pi = 418\pi$ is the answer.
6. Let d denote distance, t time, and v average speed. Then $v = d/t$ gives
$$\Delta v \approx v_d\,\Delta d + v_t\,\Delta t = 1/t\Delta d - d/t^2\ \Delta t = \frac{.1}{500/150} - \frac{500}{(500/150)^2}(-1/3600) = .0425$$
   miles per hour. The negative sign was chosen for Δt, since the maximum error occurs when $v_d\,\Delta d$ and $v_t\,\Delta t$ have the same sign.

Chapter XI

XI A

1. (a) $= \begin{bmatrix} (xy^3)_x & (xy^3)_y \\ -(x^2+y)_x & -(x^2+y)_y \end{bmatrix} = \begin{bmatrix} y^3 & 3xy^2 \\ -2x & -1 \end{bmatrix}$,
   (b) $\begin{bmatrix} ye^x & e^x \\ -y^2 & -2xy \end{bmatrix}$.
2. (a) $= \begin{bmatrix} 2^3 & 3(0)(2^2) \\ -2(0) & -2(2) \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}$.
3. (a) $\begin{bmatrix} 2xyz & x^2z & x^2y \\ -z & 0 & -x \end{bmatrix}$, $\begin{bmatrix} 4 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix}$;
   (b) $\begin{bmatrix} y & x \\ -1 & 0 \\ 0 & 2y \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 6 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2yz & y^2 \\ -y & -x & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.
4. (a) $= \begin{bmatrix} y & x \\ 0 & -2y \end{bmatrix}_{\langle 1,2\rangle} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ gives $\langle 4, 8\rangle$,
   (b) $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$ gives $\langle 7, 0\rangle$,

(c) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix}$ gives $\langle 6, 3, -9 \rangle$,

(d) $= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ gives $\langle 1, -1, 2 \rangle$.

XI B

1. (a) $(dr^2/dr\ \mathbf{i} + d(3r)/dr\ \mathbf{j})|_{r=5} = 10\mathbf{i} + 3\mathbf{j}$ gives $(25\mathbf{i} + 15\mathbf{j}) + r(10\mathbf{i} + 3\mathbf{j})$ for the tangent and $(25\mathbf{i} + 15\mathbf{j}) + r(3\mathbf{i} - 10\mathbf{j})$ for the normal.
   (b) The direction vector of the tangent is $-\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j}$ evaluated when $\cos\theta = 1/2$ and $\sin\theta = \sqrt{3}/2$. This gives $(1/2\ \mathbf{i} + \sqrt{3}/2\ \mathbf{j}) + r(-\sqrt{3}/2\ \mathbf{i} + 1/2\ \mathbf{j}) = (1/2\ \mathbf{i} + \sqrt{3}/2\ \mathbf{j}) + r(\sqrt{3}\mathbf{i} - \mathbf{j})$ for the tangent and $(1/2\ \mathbf{i} + \sqrt{3}/2\ \mathbf{j}) + r(\mathbf{i} + \sqrt{3}\mathbf{j})$ for the normal.
2. (a) From $\mathbf{f}(r) = (2 + r)\mathbf{i} + (2 + 4r^2)\mathbf{j}$ and $r_o = 1$ obtain $(3\mathbf{i} + 6\mathbf{j}) + r(\mathbf{i} + 8\mathbf{j})$ for the tangent and $(3\mathbf{i} + 6\mathbf{j}) + r(8\mathbf{i} - \mathbf{j})$ for the normal.
   (b) From $\mathbf{f}(r) = (4 + \sqrt{2}\cos\theta)\mathbf{i} + 2\sqrt{2}\sin\theta\ \mathbf{j}$ evaluated when $\cos\theta = -1/\sqrt{2}$, $\sin\theta = 1/\sqrt{2}$ obtain $(3\mathbf{i} + 2\mathbf{j}) + r(\mathbf{i} + 2\mathbf{j})$ for the tangent and $(3\mathbf{i} + 2\mathbf{j}) + r(2\mathbf{i} - \mathbf{j})$ for the normal.
3. (a) From $\mathbf{f}(x) = x\mathbf{i} + 3e^{2x}\mathbf{j}$ obtain $3\mathbf{j} + r(\mathbf{i} + 6\mathbf{j})$ for the tangent and $3\mathbf{j} + r(6\mathbf{i} - \mathbf{j})$ for the normal.
   (b) $\mathbf{f}(x) = x\mathbf{i} + \cos^2 x\ \mathbf{j}$ gives $\mathbf{j} + r\mathbf{i}$ and $\mathbf{j} + r\mathbf{j}$.
4. (a) $= \mathbf{i} + r(e^0\mathbf{i} + 1/(0 + 1)\ \mathbf{j} + 0^2\mathbf{k}) = \mathbf{i} + r(\mathbf{i} + \mathbf{j})$ for the tangent. The normal plane is $\mathbf{i} + r(\mathbf{i} - \mathbf{j}) + s\mathbf{k}$.
   (b) $= (\mathbf{j} + \pi/2\ \mathbf{k}) + r(-\sin\pi/2\ \mathbf{i} + \cos\pi/2\ \mathbf{j} + \mathbf{k}) = (\mathbf{j} + \pi/2\ \mathbf{k}) + r(-\mathbf{i} + \mathbf{k})$ for the tangent. The normal plane is $(\mathbf{j} + \pi/2\ \mathbf{k}) + r\mathbf{j} + s(\mathbf{i} + \mathbf{k})$.
5. (a) $\mathbf{f}_r(1, 2) = [(2r + 2s)\mathbf{i} + \mathbf{j} + (3r^2 + 4s)\mathbf{k}]|_{\langle 1, 2\rangle} = 6\mathbf{i} + \mathbf{j} + 11\mathbf{k}$, $\mathbf{f}_s(1, 2) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ give $(5\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}) + r(6\mathbf{i} + \mathbf{j} + 11\mathbf{k}) + s(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ as the tangent plane. From $(6\mathbf{i} + \mathbf{j} + 11\mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 37\mathbf{i} - 2\mathbf{j} - 20\mathbf{k}$ the normal line is $(5\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}) + r(37\mathbf{i} - 2\mathbf{j} - 20\mathbf{k})$.
   (b) $\mathbf{f}_r(0, 2) = 2\mathbf{i}$, $\mathbf{f}_s(0, 2) = -4\mathbf{k}$ give for the tangent plane $(\mathbf{i} + \mathbf{j} - e^4\mathbf{k}) + r\mathbf{i} + s\mathbf{k}$; the normal line is $(\mathbf{i} + \mathbf{j} - e^4\mathbf{k}) + r\mathbf{j}$.
6. (a) $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j} + xy^3\mathbf{k}$ at $(1, 3)$ gives $(\mathbf{i} + 3\mathbf{j} + 27\mathbf{k}) + r(\mathbf{i} + 27\mathbf{k}) + s(\mathbf{j} + 27\mathbf{k})$ and $(\mathbf{i} + 3\mathbf{j} + 27\mathbf{k}) + r(-27\mathbf{i} - 27\mathbf{j} + \mathbf{k})$.
   (b) $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j} + e^{x+y^2}\mathbf{k}$ gives $(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + r(\mathbf{i} + \mathbf{k}) + s(\mathbf{j} + 2\mathbf{k})$ and $(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + r(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.
7. (a) $\mathbf{f}(x, \phi) = x\mathbf{i} + x^2\cos\phi\ \mathbf{j} + x^2\sin\phi\ \mathbf{k}$ gives $(\mathbf{i} + \sqrt{3}/2\ \mathbf{j} - 1/2\ \mathbf{k}) + r(\mathbf{i} + \sqrt{3}\ \mathbf{j} - \mathbf{k}) + s(\mathbf{j} + \sqrt{3}\mathbf{k})$ and $(\mathbf{i} + \sqrt{3}/2\ \mathbf{j} - 1/2\ \mathbf{k}) + r(4\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k})$.
   (b) $\mathbf{f}(y, \phi) = y\sin\phi\ \mathbf{i} + y\cos\phi\ \mathbf{j} + e^y\mathbf{k}$ gives $(\sqrt{2}/2\ \mathbf{i} + \sqrt{2}/2\ \mathbf{j} + e\mathbf{k}) + r(\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 2e\mathbf{k}) + s(\mathbf{i} - \mathbf{j})$ and $(\sqrt{2}/2\ \mathbf{i} + \sqrt{2}/2\ \mathbf{j} + e\mathbf{k}) + r(e\mathbf{i} + e\mathbf{j} - \sqrt{2}\mathbf{k})$.

XI C

1. (b) $re^{r\bar{y}(r)} - (\bar{y}(r))^2 + 1 = 0$;
   (c) $r(r\ d\bar{y}/dr + \bar{y})e^{r\bar{y}} + e^{r\bar{y}} - 2\bar{y}\ d\bar{y}/dr = 0$ with $r = 0$, $\bar{y} = 1$ gives $d\bar{y}/dr = 1/2$;
   (d) $= \mathbf{j} + r(\mathbf{i} + 1/2\ \mathbf{j}) = \mathbf{j} + r(2\mathbf{i} + \mathbf{j})$ and $\mathbf{j} + r(\mathbf{i} - 2\mathbf{j})$.

2. Differentiating implicitly $r^3\bar{y}(r) + r(\bar{y}(r))^3 - 2 = 0$ gives $r^3\, d\bar{y}/dr + 3r^2\bar{y} + 3r\bar{y}^2\, d\bar{y}/dr + \bar{y}^3 = 0$. Evaluation at $r = 1$, $\bar{y} = 1$ gives $d\bar{y}/dr = -1$. The tangent is $(\mathbf{i} + \mathbf{j}) + r(\mathbf{i} - \mathbf{j})$; the normal is $(\mathbf{i} + \mathbf{j}) + r(\mathbf{i} + \mathbf{j})$.
3. Differentiating implicitly $r^2 + (\bar{y}(r))^2 + (\bar{z}(r))^2 - 6 = 0$ and $r + 3\bar{y}(r) + \bar{z}(r) - 6 = 0$ gives $2r + 2\bar{y}\, d\bar{y}/dr + 2\bar{z}\, d\bar{z}/dr = 0$ and $1 + 3\, d\bar{y}/dr + d\bar{z}/dr = 0$. Substitution of $r = 1$, $\bar{y} = 1$, $\bar{z} = 2$ gives $2 + 2d\bar{y}/dr + 4\, d\bar{z}/dr = 0$ and $1 + 3\, d\bar{y}/dr + d\bar{z}/dr = 0$, which have the solution $d\bar{y}/dr = -1/5$, $d\bar{z}/dr = -2/5$. Thus the tangent and normal are respectively $(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + r(5\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + r(\mathbf{i} + 5\mathbf{j}) + s(2\mathbf{i} + 5\mathbf{k})$.
4. Differentiating implicitly $r^3 + (\bar{y}(r))^2 + 4\bar{z}(r) - 9 = 0$ and $r + 2\bar{y}(r) + \bar{z}(r) - 6 = 0$ gives $3r^2 + 2\bar{y}\, d\bar{y}/dr + 4\, d\bar{z}/dr = 0$ and $1 + 2\, d\bar{y}/dr + d\bar{z}/dr = 0$. Substitution of $r = 1$, $\bar{y} = 2$, $\bar{z} = 1$ gives $d\bar{y}/dr = -1/4$ and $d\bar{z}/dr = -1/2$. Thus the tangent and normal are respectively $(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + r(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + r(\mathbf{i} + 4\mathbf{j}) + s(\mathbf{i} + 2\mathbf{k})$.
5. From $r^2 + s^2 + (\bar{z}(r, s))^2 - 3 = 0$ obtain by implicit differentiation $2r + 2\bar{z}\bar{z}_r = 0$ and $2s + 2\bar{z}\bar{z}_s = 0$.
   Substitution of $r = 1$, $s = 1$, and $\bar{z} = 1$ gives $\bar{z}_r = -1$ and $\bar{z}_s = -1$. Thus the tangent and normal are respectively $(\mathbf{i} + \mathbf{j} + \mathbf{k}) + r(\mathbf{i} - \mathbf{k}) + s(\mathbf{j} - \mathbf{k})$ and $(\mathbf{i} + \mathbf{j} + \mathbf{k}) + r(\mathbf{i} + \mathbf{j} + \mathbf{k})$.
6. Let $\mathbf{f}(r, s) = r\mathbf{i} + s\mathbf{j} + \bar{z}(r, s)\mathbf{k}$ describe the surface.
   Then $rs^2 + r^3\bar{z}(r, s) - s\bar{z}(r, s) - 2 = 0$ gives $s^2 + r^3\bar{z}_r + 3r^2\bar{z} - s\bar{z}_r = 0$ and $2rs + r^3\bar{z}_s - s\bar{z}_s - \bar{z} = 0$. Evaluation at $r = 1$, $s = 0$, $\bar{z} = 2$ gives $\bar{z}_r = -6$ and $\bar{z}_s = 2$. Hence, the tangent and normal are $(\mathbf{i} + 2\mathbf{k}) + r(\mathbf{i} - 6\mathbf{k}) + s(\mathbf{j} + 2\mathbf{k})$ and $(\mathbf{i} + 2\mathbf{k}) + r(6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

XI D

1. (a) $\mathbf{v} = dt^2/dt\, \mathbf{i} - dt^3/dt\, \mathbf{j} = 2t\mathbf{i} - 3t^2\mathbf{j}$,
   $\mathbf{a} = d(2t)/dt\, \mathbf{i} - d(3t^2)/dt\, \mathbf{j} = 2\mathbf{i} - 6t\mathbf{j}$,
   speed $= |2t\mathbf{i} - 3t^2\mathbf{j}| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$.
   (b) $\mathbf{v} = -2\sin 2t\, \mathbf{i} + 2\cos 2t\, \mathbf{j}$; $\mathbf{a} = -4\cos 2t\, \mathbf{i} - 4\sin 2t\, \mathbf{j}$;
   speed $= \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$.
   (c) $\mathbf{v} = e^t\mathbf{i} + 2e^{2t}\mathbf{j} + 3e^{3t}\mathbf{k}$; $\mathbf{a} = e^t\mathbf{i} + 4e^{2t}\mathbf{j} + 9e^{3t}\mathbf{k}$;
   speed $= \sqrt{e^{2t} + 4e^{4t} + 9e^{6t}} = e^t\sqrt{1 + 4e^{2t} + 9e^{4t}}$.
2. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$; $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j}$; speed $= \sqrt{13}$.
3. (a) From $\mathbf{v}(t) = \int \mathbf{a}\, dt = \int \mathbf{F}/m\, dt = \int (10t^3\mathbf{i} - \mathbf{j} + 2t\mathbf{k})\, dt$ obtain $\mathbf{v} = 5t^4/2\, \mathbf{i} - t\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector. From $10\mathbf{j} = \mathbf{v}(0) = \mathbf{C}$ obtain $\mathbf{v} = 5t^4/2\, \mathbf{i} + (10 - t)\mathbf{j} + t^2\mathbf{k}$.
   (b) $\mathbf{f}(t) = \int \mathbf{v}\, dt$ gives $\mathbf{f} = t^5/2\, \mathbf{i} + (10t - t^2/2)\mathbf{j} + t^3/3\, \mathbf{k} + \mathbf{C}'$. From $\mathbf{i} - \mathbf{j} + \mathbf{k} = \mathbf{f}(0) = \mathbf{C}'$ obtain $\mathbf{f}(t) = (1 + t^5/2)\mathbf{i} + (-1 + 10t - t^2/2)\mathbf{j} + (1 + t^3/3)\mathbf{k}$.
4. $\mathbf{a} = \mathbf{F}/m = \mathbf{i} + t\mathbf{j} - 16\mathbf{k}$; $\mathbf{v}(t) = \int (\mathbf{i} + t\mathbf{j} - 16\mathbf{k})dt$ and $\mathbf{v}(0) = -10\mathbf{k}$ give $\mathbf{v} = t\mathbf{i} + t^2/2\, \mathbf{j} - (10 + 16t)\mathbf{k}$. Thus $\mathbf{f}(t) = \int \mathbf{v}\, dt$ and $\mathbf{f}(0) = 200\mathbf{k}$ give $\mathbf{f}(t) = t^2/2\, \mathbf{i} + t^3/6\, \mathbf{j} + (200 - 10t - 8t^2)\mathbf{k}$. Substituting $t = 1$ gives
   (a) $\mathbf{a} = \mathbf{i} + \mathbf{j} - 16\mathbf{k}$,
   (b) $\mathbf{v} = \mathbf{i} + 1/2\, \mathbf{j} - 26\mathbf{k}$,
   (c) $\mathbf{f} = 1/2\, \mathbf{i} + 1/6\, \mathbf{j} + 182\mathbf{k}$.

5. (a) The velocity vector is $\mathbf{v} = 500\sqrt{2}/2\ \mathbf{i} + 500\sqrt{2}/2\ \mathbf{j} + 10\mathbf{k} = 250\sqrt{2}\mathbf{i} + 250\sqrt{2}\mathbf{j} + 10\mathbf{k}$.
   (b) The position is obtained from $\mathbf{f}(t) = \int \mathbf{v}\, dt$ and $\mathbf{f}(0) = 200\mathbf{i} + 100\mathbf{j} + 2\mathbf{k}$; it is $\mathbf{f}(t) = (200 + 250\sqrt{2}t)\mathbf{i} + (100 + 250\sqrt{2}t)\mathbf{j} + (2 + 10t)\mathbf{k}$.
   (c) $= |\mathbf{v}| = 10\sqrt{2501}$.

6. (a) $\mathbf{v} = 400\mathbf{j} + 5\mathbf{k}$,
   (b) $\mathbf{f}(t) = \int (400\mathbf{j} + 5\mathbf{k})\, dt$ and $\mathbf{f}(0) = 200\mathbf{i} + 3\mathbf{k}$ give $\mathbf{f}(t) = 200\mathbf{i} + 400t\mathbf{j} + (3 + 5t)\mathbf{k}$,
   (c) $= |400\mathbf{j} + 5\mathbf{k}| = \sqrt{160025}$.

7. (a) $\rho(t) = 3t$, $\theta(t) = t$,
   (b) $\mathbf{f}(t) = 3t \cos t\ \mathbf{i} + 3t \sin t\ \mathbf{j}$,
   (c) $\mathbf{v} = (-3t \sin t + 3 \cos t)\mathbf{i} + (3t \cos t + 3 \sin t)\mathbf{j}$; speed $= |\mathbf{v}| = 3\sqrt{t^2 + 1}$.

8. (a) The projection of the bug's position on the xy plane has polar coordinates $\rho(t) = 4 - 4t/5$, $\theta = 2t$; the height of the bug above the xy plane is $z(t) = 3t/5$. Thus $\mathbf{f}(t) = (4 - 4t/5) \cos 2t\ \mathbf{i} + (4 - 4t/5) \sin 2t\ \mathbf{j} + 3t/5\ \mathbf{k}$.
   (b) $\mathbf{v} = [8(t/5 - 1) \sin 2t - 4/5 \cos 2t]\mathbf{i} + [8(1 - t/5) \cos 2t - 4/5 \sin 2t]\mathbf{j} + 3/5\mathbf{k}$; speed $= |\mathbf{v}| = 1/5\ \sqrt{25 + 64(t - 5)^2}$.

Chapter XII

XII A

1. (a) $g_x = 2xy^3$, $g_y = 3x^2y^2$ give $g_{xx} = (2xy^3)_x = 2y^3$, $g_{xy} = (2xy^3)_y = 6xy^2$, $g_{yx} = (3x^2y^2)_x = 6xy^2$, $g_{yy} = (3x^2y^2)_y = 6x^2y$.
   (b) $g_{xx} = (y^2e^{xy^2})_x = y^4e^{xy^2}$, $g_{xy} = 2y(xy^2 + 1)e^{xy^2} = g_{yx}$, $g_{yy} = (2xye^{xy^2})_y = 2x(2xy^2 + 1)e^{xy^2}$.

2. $g_{xx} = 0$, $g_{yy} = 2xz^3$, $g_{zz} = 6xy^2z$, $g_{xy} = g_{yx} = 2yz^3$, $g_{xz} = g_{zx} = 3y^2z^2$, $g_{yz} = g_{zy} = 6xyz^2$.

3. (a) $\cos\theta\,(g_x \circ \mathbf{f}) + \sin\theta\,(g_y \circ \mathbf{f})$ by the chain rule;
   (b) $(-r \sin\theta)(g_{xx} \circ \mathbf{f}) + (r \cos\theta)(g_{xy} \circ \mathbf{f})$ by the chain rule;
   (c) $(-r \sin\theta)(g_{xy} \circ \mathbf{f}) + (r \cos\theta)(g_{yy} \circ \mathbf{f})$ by the chain rule;
   (d) $\cos\theta[(-r \sin\theta)g_{xx} + (r \cos\theta)g_{xy}] + (-\sin\theta)g_x + \sin\theta[(-r \sin\theta)g_{xy} + (r \cos\theta)g_{yy}] + (\cos\theta)g_y$ by applying the product rule to (a) and using (b), (c), and abbreviated notation;
   (e) $\cos\theta(g_{xx} \circ \mathbf{f}) + \sin\theta\,(g_{xy} \circ \mathbf{f})$ by the chain rule;
   (f) $\cos\theta\,(g_{yx} \circ \mathbf{f}) + \sin\theta\,(g_{yy} \circ \mathbf{f})$ by the chain rule;
   (g) $\cos\theta\,[(\cos\theta)g_{xx} + (\sin\theta)g_{xy}] + 0g_x + \sin\theta[(\cos\theta)g_{yx} + (\sin\theta)g_{yy}] + 0g_y$ by applying the product rule to (a) and using (e), (f) and abbreviated notation;
   (h) $(g \circ \mathbf{f})_\theta = (-r \sin\theta)(g_x \circ \mathbf{f}) + (r \cos\theta)(g_y \circ \mathbf{f})$ gives $(g \circ \mathbf{f})_{\theta\theta} = -r \sin\theta\,[(-r \sin\theta)g_{xx} + (r \cos\theta)g_{xy}] + (-r \cos\theta)g_x + r \cos\theta[(-r \sin\theta)g_{yx} + (r \cos\theta)g_{yy}] + (-r \sin\theta)g_y$.

4. Using abbreviated notation, $g_r = 2rg_x + 2sg_y$ and $g_s = -2sg_x + 2rg_y$ give:
   (a) $g_{rr} = 2r(2rg_{xx} + 2sg_{xy}) + 2g_x + 2s(2rg_{yx} + 2sg_{yy}) = 4r^2g_{xx} + 8rsg_{xy} + 4s^2g_{yy} + 2g_x$;
   (b) $g_{rs} = 2r(-2sg_{xx} + 2rg_{xy}) + 2s(-2sg_{yx} + 2rg_{yy}) + 2g_y = -4rsg_{xx} + 4(r^2 - s^2)g_{xy} + 4rsg_{yy} + 2g_y$,
   (c) $g_{ss} = -2s(-2sg_{xx} + 2rg_{xy}) - 2g_x + 2r(-2sg_{yx} + 2rg_{yy}) = 4s^2g_{xx} - 8rsg_{xy} + 4r^2g_{yy} - 2g_x$

XII B

1. (a) $=2^3(-1) + 3(2)^2(-1)(x-2) + (2)^3(y+1) + 1/2(6)\,(2)(-1)(x-2)^2 + 3(2)^2(x-2)(y+1) + 1/2(0)(y+1)^2 = -8 - 12(x-2) + 8(y+1) - 6(x-2)^2 + 12(x-2)(y+1)$.
   (b) $f_x = y^2e^{xy^2}, f_y = 2xye^{xy^2}, f_{xx} = y^4e^{xy^2}, f_{xy} = (2xy^3 + 2y)e^{xy^2}, f_{yy} = (4x^2y^2 + 2x)e^{xy^2}$ give $1 + y^2$.

2. (a) $\begin{bmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{bmatrix}$, (b) $\begin{bmatrix} 2yz^3 & 2xz^3 & 6xyz^2 \\ 2xz^3 & 0 & 3x^2z^2 \\ 6xyz^2 & 3x^2z^2 & 6x^2yz \end{bmatrix}$.

3. (a) constant = 8; linear = $\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle|_{\langle 1,1,2\rangle} \cdot \langle x-1, y-1, z-2\rangle$ $= 8(x-1) + 16(y-1) + 12(z-2)$; quadratic =

$$1/2[x-1 \quad y-1 \quad z-2]\begin{bmatrix} 0 & 2yz^3 & 3y^2z^2 \\ 2yz^3 & 2xz^3 & 6xyz^2 \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{bmatrix}_{\langle 1,1,2\rangle}\begin{bmatrix} x-1 \\ y-1 \\ z-2 \end{bmatrix}$$

$$= 8(y-1)^2 + 6(z-2)^2 + 16(x-1)(y-1) + 12(x-1)(z-2) + 24(y-1)(z-2).$$

   (b) constant = 1; linear = $\langle 2xe^{yz}, x^2ze^{yz}, x^2ye^{yz}\rangle|_{\langle 1,0,2\rangle} \cdot$ $\langle x-1, y, z-2\rangle = 2(x-1) + 2y$; quadratic =

$$1/2[x-1 \quad y \quad z-2]\begin{bmatrix} 2e^{yz} & 2xze^{yz} & 2xye^{yz} \\ 2xze^{yz} & x^2z^2e^{yz} & (x^2yz + x^2)e^{yz} \\ 2xye^{yz} & (x^2yz + x^2)e^{yz} & x^2y^2e^{yz} \end{bmatrix}_{\langle 1,0,2\rangle}\begin{bmatrix} x-1 \\ y \\ z-2 \end{bmatrix}$$

$$= (x-1)^2 + 2y^2 + 4(x-1)y + y(z-2).$$

XII C

1. (a) $g_x = 2x - 3y = 0$ and $g_y = 3y^2 - 3x = 0$ solved simultaneously give $x = 0$, $y = 0$ and $x = 9/4$, $y = 3/2$; hence, $\langle 0, 0\rangle$ and $\langle 9/4, 3/2\rangle$ are the critical vectors.
   (b) $2x - 4y = 0$, $2y - 4x = 0$ give $\langle 0, 0\rangle$.
   (c) $2x - 2y = 0$, $2y - 2x = 0$ give $\langle c, c\rangle$ for arbitrary c.
   (d) $3x^2 + 1 = 0$, $3y^2 = 0$ give no critical vectors.

2. (a) $y - z^2 = 0$, $x - 1 = 0$, $-2xz + 2 = 0$ give $\langle 1, 1, 1\rangle$;
   (b) $2xz - 2x = 0$, $-2yz + 6 = 0$, $x^2 - y^2 = 0$ give $\langle 3, 3, 1\rangle$ and $\langle -3, 3, 1\rangle$;
   (c) $2y - z + 1 = 0$, $2x + z - 4 = 0$, $y - x = 0$ give $\langle 3/4, 3/4, 5/2\rangle$.

3. (a) $H_g = \begin{bmatrix} 2 & -3 \\ -3 & 6y \end{bmatrix}$ gives $H_g(0, 0) = \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix}$ and

   $H_g(9/4, 3/2) = \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix}$ is indefinite; $\begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix}$ is positive-definite.

(c) g has a saddle point at $\langle 0, 0\rangle$ and a local minimum at $\langle 9/4, 3/2\rangle$.

4. In Problem 1(b), $H_g(0, 0) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$ is indefinite, and hence, g has a saddle point at $\langle 0, 0\rangle$.

In Problem 1(c), $H_g(c, c) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ is positive-semidefinite; the nature of (c, c) cannot be determined by our method. In 2(a), $H_g(1, 1, 1) =$ $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix}$ is indefinite and hence g has a saddle point at $\langle 1, 1, 1\rangle$

In 2(b), $H_g(3, 3, 1) = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -2 & -6 \\ 6 & -6 & 0 \end{bmatrix}$ and $H_g(-3, 3, 1) = \begin{bmatrix} 0 & 0 & -6 \\ 0 & -2 & -6 \\ -6 & -6 & 0 \end{bmatrix}$ are both indefinite, and hence, g has a saddle point at $\langle 3, 3, 1\rangle$ and at $\langle -3, 3, 1\rangle$.

In 2(c), $H_g = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ is indefinite, and hence, g has a saddle point at $\langle 3/4, 3/4, 5/2\rangle$.

5. (a) $3x^2 - 6y = 0$, $3y^2 - 6x = 0$ give $\langle 0, 0\rangle$ and $\langle 2, 2\rangle$ as critical vectors. Therefore $H_g(0, 0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$ is indefinite and $H_g(2, 2) = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}$ is positive-definite; it follows that $\langle 0, 0\rangle$ gives a saddle point and $\langle 2, 2\rangle$ a local minimum.

(b) $2x - y - 6 = 0$, $2y - x = 0$; $3z^2 - 3 = 0$ give $\langle 4, 2, 1\rangle$ and $\langle 4, 2, -1\rangle$ as critical vectors, Since $H_g(4, 2, 1) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is positive-definite and

$H_g(4, 2, -1) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ is indefinite, it follows that $\langle 4, 2, 1\rangle$ gives a local minimum and $\langle 4, 2, -1\rangle$ gives a saddle point.

XII D

1. (a) $g \circ \mathbf{f} = \cos^2 \theta + \sin \theta$ and $d(g \circ \mathbf{f})/d\theta = -2 \sin \theta \cos \theta + \cos \theta = 0$ gives $\cos \theta = 0$, $\sin \theta = \pm 1$ or $\sin \theta = 1/2$, $\cos \theta = \pm\sqrt{3}/2$. Then $d^2(g \circ \mathbf{f})/d\theta^2 = -2 \cos^2 \theta + 2 \sin^2 \theta - \sin \theta$ is positive if $\cos \theta = 0$ and $\sin \theta = \pm 1$ and negative if $\sin \theta = 1/2$ and $\cos \theta = \pm\sqrt{3}/2$. Therefore, $\langle 0, 1\rangle$, $\langle 0, -1\rangle$ give local minima and $\langle \sqrt{3}/2, 1/2\rangle$, $\langle -\sqrt{3}/2, 1/2\rangle$ give local maxima for g.

(b) Set $\mathbf{f}(x) = x\mathbf{i} + 3x^2\mathbf{j}$; then $g \circ \mathbf{f} = 24x - 6x^2$ and $d(g \circ \mathbf{f})/dx = 24 - 12x = 0$ give $\langle 2, 12\rangle$ as the critical vector of $g \circ \mathbf{f}$. From $d^2(g \circ \mathbf{f})/dx^2 < 0$, g has a local maximum at $\langle 2, 12\rangle$.

(c) Set $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$; then $g \circ \mathbf{f} = 2\cos\theta + 9\sin^2\theta$ and $d(g \circ \mathbf{f})/d\theta = -2\sin\theta + 18\sin\theta\cos\theta = 0$ give $\sin\theta = 0$ $\cos\theta = \pm 1$ or $\cos\theta = 1/9$, $\sin\theta = \pm 4\sqrt{5}/9$. The second derivative $d^2(g \circ \mathbf{f})/d\theta^2 = -2\cos\theta + 18(\cos^2\theta - \sin^2\theta)$ is positive when $\sin\theta = 0$ and negative if $\cos\theta = 1/9$. Therefore, g has a local minima at $\langle 2, 0\rangle$ and $\langle -2, 0\rangle$ and local maxima at $\langle 2/9, 4\sqrt{5}/3\rangle$ and $\langle 2/9, -4\sqrt{5}/3\rangle$.

2. $g \circ \mathbf{f} = -r^2 - 3r$ and $d(g \circ \mathbf{f})/dr = -2r - 3 = 0$ give $r = -3/2$. By the second derivative test, g has a local maximum at $\langle -3/2, 9/4, 3/2\rangle$.

3. (a) Let $\mathbf{f} = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$; then $g \circ \mathbf{f} = x^3 + y^2 - 6xy$ and $(g \circ \mathbf{f})_x = 3x^2 - 6y = 0$, $(g \circ \mathbf{f})_y = 2y - 6x = 0$ give $x = 0$, $y = 0$ and $x = 6$, $y = 18$. Since $H_{g\circ\mathbf{f}}(0, 0) = \begin{bmatrix} 0 & -6 \\ -6 & 2 \end{bmatrix}$ is indefinite and $H_{g\circ\mathbf{f}}(6, 18) = \begin{bmatrix} 36 & -6 \\ -6 & 2 \end{bmatrix}$ is positive-definite, g has a local minimum at $\langle 6, 18, 108\rangle$.

(b) $g \circ \mathbf{f} = r^3 - 12r + s^3 - 6s^2$ and $(g \circ \mathbf{f})_r = 3r^2 - 12 = 0$, $(g \circ \mathbf{f})_s = 3s^2 - 12s = 0$ give $\langle r, s\rangle = \langle 2, 0\rangle, \langle 2, 4\rangle, \langle -2, 0\rangle$, and $\langle -2, 4\rangle$. From $H_{g\circ\mathbf{f}} = \begin{bmatrix} 6r & 0 \\ 0 & 6s - 12 \end{bmatrix}$ it is seen that $H_{g\circ\mathbf{f}}$ is indefinite at $\langle 2, 0\rangle, \langle -2, 4\rangle$, positive-definite at $\langle 2, 4\rangle$, and negative-definite at $\langle -2, 0\rangle$. Hence, g has a local minimum at $\mathbf{f}(2, 4) = \langle 2, -8, -32\rangle$ and a local maximum at $\mathbf{f}(-2, 0) = \langle -2, -8, 0\rangle$.

(c) Let $\mathbf{f}(x, \phi) = x\mathbf{i} + x\cos\phi\,\mathbf{j} + x\sin\phi\,\mathbf{k}$; then $g \circ \mathbf{f} = x^3 - 3x\sin\phi$ and $(g \circ \mathbf{f})_x = 3x^2 - 3\sin\phi = 0$, $(g \circ \mathbf{f})_\phi = -3x\cos\phi = 0$ give $\langle x, \phi\rangle = \langle 0, 0\rangle$, $\langle 0, \pi\rangle$, $\langle 1, \pi/2\rangle$, and $\langle -1, \pi/2\rangle$. From $H_{g\circ\mathbf{f}} = \begin{bmatrix} 6x & -3\cos\phi \\ -3\cos\phi & 3x\sin\phi \end{bmatrix}$ it is seen that g has a local minimum at $\mathbf{f}(1, \pi/2) = \langle 1, 0, 1\rangle$, and a local maximum at $\mathbf{f}(-1, \pi/2) = \langle -1, 0, -1\rangle$.

XII E

1. (a) $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ $0 \le \theta \le \pi$,

(b) $g \circ \mathbf{f} = \cos^2\theta + \sin\theta$ and $d(g \circ \mathbf{f})/d\theta = -2\sin\theta\cos\theta + \cos\theta = 0$ give $\theta = \pi/6, \pi/2$, and $5\pi/6$. Comparing $g \circ \mathbf{f}$ at $0, \pi/6, \pi/2, 5\pi/6$, and π gives $5/4$ for the maximum and 1 for the minimum value of g.

2. (a) $f(x) = x\mathbf{i} + x^2\mathbf{j}$, $0 \le x \le 1$ gives $g \circ \mathbf{f} = x - x^2$ and $d(g \circ \mathbf{f})/dx = 1 - 2x = 0$ when $x = 1/2$. From $g \circ \mathbf{f}(0) = 0$ $g \circ \mathbf{f}(1) = 0$ and $g \circ \mathbf{f}(1/2) = 1/4$ the maximum of g is $1/4$ and the minimum is 0.

(b) $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$ $0 \le \theta \le 2\pi$ gives $g \circ \mathbf{f} = 2\cos\theta - 6\sin\theta$; then $d(g \circ \mathbf{f})/d\theta = -2\sin\theta - 6\cos\theta = 0$ when $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$ or $\cos\theta = -1/\sqrt{10}$, $\sin\theta = 3/\sqrt{10}$. Comparison of g at $\langle 2/\sqrt{10}, -9/\sqrt{10}\rangle$ and $\langle -2/\sqrt{10}, 9/\sqrt{10}\rangle$, $\mathbf{f}(0) = \langle 2, 0\rangle$ shows that the maximum is $2\sqrt{10}$ and the minimum is $-2\sqrt{10}$.

3. (a) Let $g = x^2 + y^2$; then $g \circ \mathbf{f} = (r^2 - 8)^2 + (4r + 3)^2$ and $d(g \circ \mathbf{f})/dr = 4r^3 + 24 = 0$ when $r = -6^{1/3}$. The desired distance is $\sqrt{g \circ \mathbf{f}(-6^{1/3})} = \sqrt{73 - 18(6^{1/3})}$.

(b) $\mathbf{f}(\theta) = (3 + \cos\theta)\mathbf{i} + (4 + \sin\theta)\mathbf{j}$ and $g = x^2 + y^2$ gives $g \circ \mathbf{f} = 26 + 6\cos\theta + 8\sin\theta$. Then $d(g \circ \mathbf{f})/d\theta = 0$ when $\tan\theta = 4/3$, which implies that $\cos\theta = 3/5$, $\sin\theta = 4/5$ or $\cos\theta = -3/5$, $\sin\theta = -4/5$. The latter gives the distance, which is $\sqrt{26 - 18/5 - 32/5} = 4$.

(c) Let $\mathbf{f}(x) = x\mathbf{i} + (x^2 + x)\mathbf{j}$ and $g = (x + 1)^2 + (y - 1)^2$. Then $g \circ \mathbf{f} = x^4 + 2x^3 + 2$ and $d(g \circ \mathbf{f})/dx = 4x^3 + 6x^2 = 0$ give $x = 0$, $x = -3/2$. The distance is $\sqrt{g \circ \mathbf{f}(-3/2)} = \sqrt{5}/4$.

4. $g(x, y, z) = x^2 + y^2 + z^2$ gives $g \circ \mathbf{f} = 2r^4 + 4r + 5$. Solving $d(g \circ \mathbf{f})/dr = 0$ gives $r = -2^{-1/3}$, and the desired distance is $\sqrt{g \circ \mathbf{f}(-2^{-1/3})} = \sqrt{5 - 3(2^{-1/3})}$.

5. The boundary $x^2 + y^2 = 1$ is described by $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$. Then $g \circ \mathbf{f} = \cos^2\theta - 2\sin^2\theta$, and $d(g \circ \mathbf{f})/d\theta = 0$ is satisfied by $\theta = 0, \pi/2, \pi$, or $3\pi/2$. The maximum and minimum on the boundary are respectively $g \circ \mathbf{f}(0) = 1$ and $g \circ \mathbf{f}(\pi/2) = -2$. In the interior, $g_x = 2x = 0$, $g_y = -4y = 0$ give $\langle 0, 0\rangle$ as the critical vector. Since $g(0, 0) = 0$, the maximum of g is 1 and the minimum is -2.

6. Let $\mathbf{f}(x) = x\mathbf{i}$, $0 \le x \le 1$ represent the segment from (0, 0) to (1, 0). Then $g \circ \mathbf{f} = x^2$ gives a maximum of 1 and minimum of 0 on this segment. The segment from (1, 0) to (1, 1) is represented by $\mathbf{f}(y) = \mathbf{i} + y\mathbf{j}$, $0 \le y \le 1$. Then $g \circ \mathbf{f} = 1 - 2y$ has a maximum of 1 and minimum of -1 on this segment. Continuing counterclockwise, g has maximums of 2, 2 and minimums of -1, 0 on the other two segments. In the interior $g_x = 2x - 4y = 0$, $g_y = -4x + 2 = 0$ give the critical vector $\langle 1/2, 1/4\rangle$. Since $g(1/2, 1/4) = 1/4$ it follows that 2 is the maximum and -1 the minimum.

7. The segment from (0, 0) to (1, 1) is represented by $\mathbf{f}(x) = x\mathbf{i} + x\mathbf{j}$, $0 \le x \le 1$. On this segment $g \circ \mathbf{f} = 10x^2 - 8x$ has maximum 1 and minimum $-8/5$. Proceeding counterclockwise g has maxima 2, 1 and minima 1/5, $-4/5$ on the other segments. In the interior $g_x = 10x - 4 = 0$, $g_y = 10y - 4 = 0$ has critical vector $\langle 2/5, 2/5\rangle$, and $g(2/5, 2/5) = -8/5$. Hence, g has maximum 2 and minimum $-8/5$.

XII F

1. (a) $\det\begin{bmatrix} 2x & 1 \\ 2x & 2y \end{bmatrix} = 4xy - 2x = 0$ and $x^2 + y^2 - 1 = 0$ give $\langle x, y\rangle = \langle 0, 1\rangle$, $\langle 0, -1\rangle$, $\langle \sqrt{3}/2, 1/2\rangle$, and $\langle -\sqrt{3}/2, 1/2\rangle$. Comparison shows that $g(\sqrt{3}/2, 1/2) = 5/4$ is the maximum and $g(0, -1) = -1$ is the minimum.

(b) $\det\begin{bmatrix} 2x & 2y \\ 10x + 6y & 6x + 10y \end{bmatrix} = 12x^2 - 12y^2 = 0$ and $5x^2 + 6xy + 5y^2 - 8 = 0$ give $\langle x, y\rangle = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$, $\langle -1/\sqrt{2}, -1/\sqrt{2}\rangle$, $\langle \sqrt{2}, -\sqrt{2}\rangle$, and $\langle -\sqrt{2}, \sqrt{2}\rangle$. Comparison shows 4 is the maximum and 1 is the minimum.

2. $\det\begin{bmatrix} 2x & 2y \\ 2x + y & x + 2y \end{bmatrix} = 2x^2 - 2y^2 = 0$ and $x^2 + xy + y^2 = 1$ give 2/3 for the minimum value of g. The distance is $\sqrt{2/3}$.

3. (a) $\det\begin{bmatrix} 0 & 1 & -1 \\ 2x & 2y & 0 \\ 1 & 2 & 1 \end{bmatrix} = 2y - 6x = 0$, $x^2 + y^2 = 40$, $x + 2y + z = 8$ has solution $\langle 2, 6, -6\rangle$ and $\langle -2, -6, 22\rangle$. Comparison shows the maximum is 12 and the minimum is -28.

   (b) $\det\begin{bmatrix} 2x & 4y & 6z \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = 6z - 2x = 0$, $x + y + z = 3$, $x + 2y + z = 2$ give $x = 3$, $y = -1$, and $z = 1$. The minimum is $g(3, -1, 1) = 14$.

4. Let $g = x^2 + y^2 + z^2$; then $\det\begin{bmatrix} 2x & 2y & 2z \\ 2x & 4y & 2z \\ 1 & 3 & 2 \end{bmatrix} = 4y(2x - z) = 0$, $x^2 + 2y^2 + z^2 = 5$, $x + 3y + 2z = 0$ give $\langle x, y, z\rangle = \langle 2, 0, -1\rangle, \langle -2, 0, 1\rangle$, $1/\sqrt{19}\langle 3, -5, 6\rangle$, $1/\sqrt{19}\langle -3, 5, -6\rangle$. The minimum of g is 70/19; the distance is $\sqrt{70/19}$.

5. $\langle 2x, 4y, 6z\rangle \times \langle 1, 4, 3\rangle = \langle 12y - 24z, 6z - 6x, 8x - 4y\rangle = \langle 0, 0, 0\rangle$ and $x + 4y + 3z = 24$ give $x = 2$, $y = 4$, $z = 2$. The minimum is $g(2, 4, 2) = 48$.

6. Let $g = x^2 + y^2 + z^2$. Then $\langle 2x, 2y, 2z\rangle \times \langle yz^2, xz^2, 2xyz\rangle = \langle 0, 0, 0\rangle$ and $xyz^2 = 32$ give $\langle x, y, z\rangle = \langle 2, 2, 2\sqrt{2}\rangle, \langle 2, 2, -2\sqrt{2}\rangle, \langle -2, -2, 2\sqrt{2}\rangle$, and $\langle -2, -2, -2\sqrt{2}\rangle$. The minimum of g is 16; the distance is 4.

7. Let the first-quadrant vertex of the rectangle be (x, y). Then the area is $A = 4xy$ where $x^2/9 + y^2/16 = 1$. From $\det\begin{bmatrix} 4y & 4x \\ 2x/9 & y/8 \end{bmatrix} = y^2/2 - 8x^2/9 = 0$ and $x^2/9 + y^2/16 = 1$, the values $x = 3/\sqrt{2}$, $y = 4/\sqrt{2}$ are obtained. Thus $A = 4(3/\sqrt{2})(4/\sqrt{2}) = 24$ is the maximum area.

8. Let the right angle vertex be $(0, y)$. It may be assumed that the other vertices are $(0, -4)$ and (x, y). Then the area is $A = 1/2\, x(y + 4)$ where $x^2/9 + y^2/16 = 1$. From $\det\begin{bmatrix} 2 + y/2 & x/2 \\ 2x/9 & y/8 \end{bmatrix} = 0$ and $x^2/9 + y^2/16 = 1$ the values $x = 3\sqrt{3}/2$ and $y = 2$ are obtained. Thus $A = 1/2\,(3\sqrt{3}/2)(6) = 9\sqrt{3}/2$ is the maximum area.

9. Let (x, y, z) be the vertex in the first octant; then $V = 8xyz$ where $x^2/1 + y^2/4 + z^2/9 = 1$. Thus $\langle 8yz, 8xz, 8xy\rangle \times \langle 2x, y/2, 2z/9\rangle = \langle 0, 0, 0\rangle$ and $x^2/1 + y^2/4 + z^2/9 = 1$ give the values $x = 1/\sqrt{3}$, $y = 2/\sqrt{3}$, $z = 3/\sqrt{3}$. The maximum volume is $16/\sqrt{3}$.

10. Let x and y be the base dimensions and z the height. Then the surface area is $A = xy + 2xz + 2yz$ where $xyz = 500$. Thus $\langle y + 2z, x + 2z, 2x + 2y\rangle \times \langle yz, xz, xy\rangle = \langle 0, 0, 0\rangle$ and $xyz = 500$ give the values $x = 10$, $y = 10$, and $z = 5$. The maximum area is 300.

11. Let x and y be the base dimensions and z the height. The cost is $C = 6xy + 4xz + 4yz$ where $xyz = 12$. Then $\langle 6y + 4z, 6x + 4z, 4z + 4y\rangle \times \langle yz, xz, xy\rangle = \langle 0, 0, 0\rangle$ and $xyz = 12$ give $x = 2$, $y = 2$, and $z = 3$. The minimum cost is 72 cents.

Chapter XIII

XIII A

1. (a) $x^2y^4/4$, (b) $=x^2(2^4)/4 - x^2(1^4)/4 = 15x^2/4$,
(c) $=15x^3/12|_0^1 = 5/4$, (d) $x^3y^3/3$
(e) $y^3/3$ (f) $y^4/12|_1^2 = 5/4$.

2. (a) $= \int_1^2 dx \int_0^3 xy^2\, dy = \int_1^2 xy^3/3 \Big|_{y=0}^{y=3} dx = \int_1^2 9x\, dx = 27/2;$
$= \int_0^3 dy \int_1^2 xy^2\, dx = \int_0^3 x^2y^2/2 \Big|_{x=1}^{x=2} dy = \int_0^3 3y^2/2\, dy = 27/2.$
(b) $= \int_0^1 dx \int_0^2 e^{x+2y}\, dy = \int_0^1 (e^4 - 1)e^x/2\, dx = (e^4 - 1)(e - 1)/2;$
$= \int_0^2 dy \int_0^1 e^{x+2y}\, dx = \int_0^2 (e^{2y+1} - e^{2y})dy = (e^4 - 1)(e - 1)/2.$

3. (a) $= \int_0^1 dx \int_0^1 xe^{xy}\, dy = \int_0^1 (e^x - 1)dx = e - 2$. The other order will require integration by parts, or tables.
(b) $= \int_0^{\pi/4} dy \int_0^1 y \cos xy\, dx = \int_0^{\pi/4} \sin y\, dy = 1 - \sqrt{2}/2.$

4. (a) $= \left(\int_0^2 x\, dx\right)\left(\int_2^4 y\, dy\right)\left(\int_0^3 z^2\, dz\right) = (2)(6)(9) = 108,$
(b) $= \int_0^1 dy \int_0^1 dz \int_0^1 ze^{xz+y}\, dx = \int_0^1 dy \int_0^1 (e^{z+y} - e^y)dz =$
$\left(\int_0^1 e^y\, dy\right)\left(\int_0^1 (e^z - 1)dz\right) = (e - 1)(e - 2).$

Chapter XIV

XIV A

1. (a) $\mathbf{f}(x) = x\mathbf{i} + x^2\mathbf{j},\ 1 \le x \le 2.$
(b) $g \circ \mathbf{f} = x^3/x^2 = x;\ |J_{\mathbf{f}}| = \sqrt{1^2 + (2x)^2} = \sqrt{1 + 4x^2}.$
(c) $= \int_1^2 x\sqrt{1 + 4x^2}\, dx = (17^{3/2} - 5^{3/2})/12.$

2. (a) Let $\mathbf{f} = 2 \cos \theta\, \mathbf{i} + 2 \sin \theta\, \mathbf{j},\ -\pi/2 \le \theta \le \pi/2$; then $g \circ \mathbf{f} = 8 \cos \theta \sin^2 \theta$ and $|J_{\mathbf{f}}| = 2$. Thus $\int_{\mathscr{C}} g\, dL = \int_{-\pi/2}^{\pi/2} 16 \cos \theta \sin^2 \theta\, d\theta = 32/3.$
(b) From $\langle 1, 2\rangle + [r(\langle 7, 5\rangle - \langle 1, 2\rangle]$ obtain $\mathbf{f}(r) = (1 + 6r)\mathbf{i} + (2 + 3r)\mathbf{j}$, $0 \le r \le 1$. Then $g \circ \mathbf{f} = e^{3+9r}$ and $|J_{\mathbf{f}}| = 3\sqrt{5}$. Thus
$\int_{\mathscr{C}} g\, dL = 3\sqrt{5}e^3 \int_0^1 e^{9r}\, dr = \sqrt{5}e^3(e^9 - 1)/3.$

(c) Let $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$, $0 \le \theta \le \pi/2$; then $g \circ \mathbf{f} = 6\cos\theta\sin\theta$ and $|J_{\mathbf{f}}| = \sqrt{4\sin^2\theta + 9\cos^2\theta} = \sqrt{9 - 5\sin^2\theta}$. Thus
$$\int_{\mathscr{C}} g\,dL = \int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{9 - 5\sin^2\theta}\,d\theta = 38/5.$$

3. (a) $= \int_0^{\pi/2} (x/\sqrt{2 - \cos^2 x})\sqrt{1 + \sin^2 x}\,dx = \int_0^{\pi/2} x\,dx = \pi^2/8$,

(b) $= \int_0^1 \sqrt{1 + e^{2x}}\sqrt{1 + (e^x)^2}\,dx = (1 + e^2)/2$.

4. (a) $= \int_\pi^{2\pi} (\sqrt{\theta^2 + 4}/\theta)\sqrt{\theta^4 + 4\theta^2}\,d\theta = 7\pi^3/3 + 4\pi$,

(b) $= \int_0^\pi 3\theta\sqrt{\theta^2 + 1}\,d\theta = (\pi^2 + 1)^{3/2} - 1$.

5. (a) $= \int_0^\pi r(3r^2)(6r^3)\sqrt{1 + 36r^2 + 324r^4}\,dr = 270/7$,

(b) $= \int_0^1 \sqrt{1 + e^{2r}}\sqrt{1 + e^{2r}}\,dr = (1 + e^2)/2$.

6. (a) $= \int_0^1 \sqrt{4r^2 + 9r^4}\,dr = \int_0^1 r\sqrt{4 + 9r^2}\,dr = (13^{3/2} - 8)/27$,

(b) $\int_0^1 \sqrt{1 + 36r^2 + 324r^4}\,dr = \int_0^1 (1 + 18r^2)\,dr = 7$.

7. (a) $\mathbf{f}(\theta) = 4\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$ gives $\int_0^{2\pi} \sqrt{16\sin^2\theta + 9\cos^2\theta}\,d\theta$,

(b) $\mathbf{f}(x) = x\mathbf{i} + xe^x\mathbf{j}$, $0 \le x \le 4$ gives $\int_0^4 \sqrt{1 + e^{2x}(x + 1)^2}\,dx$.

XIV B

1. $g \circ \mathbf{f} = s^2$, $|J_{\mathbf{f}}| = |s(0) - r(1)| = r$ gives $\int_{\mathscr{R}} g\,dA = \int_0^1 dr \int_1^2 s^2 r\,ds = 7/6$.

2. (a) $\mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $\mathbf{I} = [0, 2] \times [0, 2\pi]$ gives $g \circ \mathbf{f} = 2r\cos\theta + 1$, $|J_{\mathbf{f}}| = r$. Thus $\int_{\mathscr{R}} g\,dA = \int_0^2 dr \int_0^{2\pi} (2r\cos\theta + 1)r\,d\theta = 4\pi$.

(b) $\mathbf{f}(r, \theta) = 2r\cos\theta\,\mathbf{i} + 3r\sin\theta\,\mathbf{j}$, $[0, 1] \times [0, 2\pi]$ give $g \circ \mathbf{f} = 36r^2$ and $|J_{\mathbf{f}}| = 6r$. Thus $\int_{\mathscr{R}} g\,dA = \int_0^1 dr \int_0^{2\pi} 216r^3\,d\theta = 108\pi$.

3. (a) $\langle 1, 4\rangle + [r(\langle 2, 5\rangle - \langle 1, 4\rangle + s(\langle 4, 3\rangle - \langle 2, 5\rangle)]$ gives $\mathbf{f}(r, s) = (1 + r + 2s)\mathbf{i} + 4(4 + r - 2s)\mathbf{j}$, $[0, 1] \times [0, 1]$. Thus
$$\int_{\mathscr{R}} g\,dA = \int_0^1 dr \int_0^1 (5 + 2r)(4)ds = 24.$$

(b) $\langle 1, 5\rangle + [r(\langle 2, 0\rangle - \langle 1, 5\rangle) + s(\langle 7, 3\rangle - \langle 2, 0\rangle)]$ gives $\mathbf{f}(r, s) = (1 + r + 5s)\mathbf{i} + (5 - 5r + 3s)\mathbf{j}$, $[0, 1] \times [0, 1]$. Thus
$$\int_{\mathscr{R}} g\,dA = \int_0^1 dr \int_0^1 (-22 + 28r)28r\,ds = -140/3.$$

4. (a) $= \int_0^1 dx \int_x^{x^2+2} xy\,dy = \int_0^1 xy^2/2\Big|_{y=x}^{y=x^2+2} dx = \int_0^1 (x^5/2 + 3x^3/2 + 2x)dx =$ $35/24$.

(b) $= \int_{-1}^1 dx \int_{x^2}^1 x^2 y\,dy = 4/21$.

5. $= \int_{-1}^1 dy \int_{-3}^{y^{1/3}} xy^2\,dx = -30/11$.

6. (a) $= \int_0^{\pi/2} d\theta \int_0^{\theta} (\rho\theta)\rho\,d\rho = \pi^5/480$,

(b) $= \int_0^{\pi/2} d\theta \int_0^3 (\cos\theta)\rho\,d\rho = 9/2$.

7. (a) $= \int_0^1 (x - x^4)\,dx = 3/10$, (b) $= \int_{-2}^2 (4 - y^2)dy = 32/3$,

(c) $= \int_0^2 (e^x - x)dx = e^2 - 3$, (d) $=1/2 \int_0^{\pi/2} \cos\theta\,d\theta = 1/2$.

XIV C

1. $|J_{\mathbf{f}}| = \left\| \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 5 \end{bmatrix} \right\| = \sqrt{62}$ and $g \circ \mathbf{f} = 6rs$ give $\int_0^1 dr \int_0^2 6\sqrt{62}rs\,ds = 6\sqrt{62}$.

2. (a) $\langle 2, 1, 5\rangle + [r(\langle 3, 2, 0\rangle - \langle 2, 1, 5\rangle) + s(\langle 7, 1, -2\rangle - \langle 3, 2, 0\rangle)]$ gives $\mathbf{f}(r, s) = (2 + r + 4s)\mathbf{i} + (1 + r - s)\mathbf{j} + (5 - 5r - 2s)\mathbf{k}$, $[0, 1] \times [0, 1]$. Then $\int_{\mathscr{S}} g\,dA = \int_0^1 dr \int_0^1 (1 + 5s)\sqrt{398}\,ds = 7\sqrt{398}/2$.

(b) $\mathbf{f}(r, s) = (2 + r + 4rs)\mathbf{i} + (1 + r - rs)\mathbf{j} + (5 - 5r - 2rs)\mathbf{k}$, $[0, 1] \times [0, 1]$ gives $\int_0^1 dr \int_0^1 (9 - 2r)\sqrt{398}r\,ds = 23\sqrt{398}/6$.

3. (a) $= x + y^2 + y - (x + y^2) = y$,

(b) $f_x = 1, f_y = 2y$ gives $\sqrt{1 + 1^2 + (2y)^2} = \sqrt{2 + 4y^2}$,

(c) $= \int_0^1 dx \int_0^2 y\sqrt{2 + 4y^2}\,dy = 13\sqrt{2}/3$.

4. $= \int_0^1 dx \int_{x^2}^{x} 3x\sqrt{11}\,dy = \sqrt{11}/4$.

5. (a) $= x$; note that y and z do not occur in the rule for g and hence there is no explicit substitution.

(b) $= \int_0^{2\pi} yx\,d\phi = 2\pi xy$

(c) $= \int_{\mathscr{C}} 2\pi xy\,dL$ where $\mathscr{C}$ is represented by $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le \pi/2$.

Thus $\int_{\mathscr{S}} g\,dA = \int_0^{\pi/2} 2\pi\cos\theta\sin\theta\,d\theta = \pi$.

6. $\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dL \int_0^{2\pi} y^3 \sin^2 \phi/x\,d\phi = \int_{\mathscr{C}} \pi y^3/x\,dL$ where $\mathscr{C}$ is represented by $\mathbf{f}(x) = y^2\mathbf{i} + y\mathbf{j}$, $1 \le y \le 2$. Thus $\int_{\mathscr{S}} g\,dA = \int_1^2 \pi y\sqrt{4y^2 + 1}\,dy = \pi(17^{3/2} - 5^{3/2})/12$.

7. (a) $\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dL \int_3^4 y/x\,dz = \int_{\mathscr{C}} y/x\,dL$ where $\mathscr{C}$ is represented by $\mathbf{f}(x) = x\mathbf{i} + x^2\mathbf{j}$, $0 \le x \le 1$. Thus $\int_{\mathscr{S}} g\,dA = \int_1^2 x\sqrt{1 + 4x^2}\,dx = (17^{3/2} - 5^{3/2})/12$.

(b) $\int_{\mathscr{S}} g\,dA = \int_{\mathscr{C}} dL \int_{x-y}^{x+y+6} y\,dz = \int_{\mathscr{C}} (2y^2 + 6y)dL = \int_0^{2\pi}$ $(8\sin^2\theta + 12\sin\theta)(2)\,d\theta = 16\pi$.

8. (a) $= \int_0^2 dx \int_0^1 \sqrt{6}\,dy = 2\sqrt{6}$,

(b) $= \int_0^1 dr \int_0^{2\pi} \sqrt{3}r\,d\theta = \pi\sqrt{3}$,

(c) $\mathbf{f}(\theta) = (2 + \cos\theta)\mathbf{i} + (3 + \sin\theta)\mathbf{j}$, $0 \le \theta \le 2\pi$ for the circle gives $\int_0^{2\pi} 2\pi(3 + \sin\theta)\,d\theta = 12\pi^2$,

(d) $= \int_0^{2\pi} (2\sin\theta - \cos\theta + 7)\,d\theta = 14\pi$.

XIV D

1. (a) $g \circ \mathbf{f} = 3(r + s)$ and $|J_{\mathbf{f}}| = \left\| \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 3 & 0 \end{bmatrix} \right\| = 9$ give $\int_0^1 dr \int_0^1 ds \int_0^2 27(r + s)dt = 54$.

(b) $\langle 1, 0, 2\rangle + [r\mathbf{PQ} + s\mathbf{QR} + t\mathbf{RS}]$ gives $\mathbf{f}(r, s) = (1 + r + s - t)\mathbf{i} + (-r + 2s - 2t)\mathbf{j} + (2 + r + 4s - 2t)\mathbf{k}$, $[0, 1] \times [0, 1] \times [0, 1]$. Then $\int_{\mathscr{T}} g\,dV = \int_0^1 dr \int_0^1 ds \int_0^1 (1 + 3s - 3t)6\,dt = 6$.

2. (a) $= \int_{\mathscr{R}} dA \int_0^{2\pi} yx^2\,d\phi = \int_{\mathscr{R}} 2\pi\,yx^2\,dA = 2\pi \int_0^1 dx \int_0^{x^2} yx^2\,dy = \pi/7$.

(b) $= \int_{\mathscr{R}} dA \int_0^{2\pi} xy^3\,d\phi = \int_{\mathscr{R}} 2\pi xy^3\,dA = 2\pi \int_1^2 dy \int_0^{y^2} xy^3\,dx = 255\pi/8$.

3. (a) $= \int_{\mathscr{R}} dA \int_{x-y}^{3+x} (x^2 + y^2)dz = \int_{\mathscr{R}} (3 + y)(x^2 + y^2)dA = \int_0^1 dr \int_0^{2\pi}$ $(3 + r\sin\theta)r^2 r\,d\theta = 3\pi/2$.

(b) $= \int_{\mathscr{R}} dA \int_{xy}^{y(x+1)} (x + y)dz = \int_{\mathscr{R}} y(x + y)dA = \int_0^1 dr \int_0^1 r(1 + rs)r\,ds = 11/24$ (using the representation $\mathbf{f}(r, s) = (1 - r + rs)\mathbf{i} + r\mathbf{j}$ for the triangle).

4. (a) $=2\pi\int_{\mathscr{R}} y\,dA = 2\pi\int_0^1 dr\int_0^{2\pi}(2 + r\sin\theta)r\,d\theta = 4\pi^2$,

(b) $=2\pi\int_{\mathscr{R}} y\,dA = 2\pi\int_0^1 dx\int_0^{x^3} y\,dy = \pi/7$,

(c) $=\int_0^1 dx\int_0^1 (2x + y + 3)\,dy = 9/2$.

XIV E

1. (a) $M_x = \int_0^1 dx\int_0^2 y\,dy = 2$; $M_y = \int_0^1 dx\int_0^2 x\,dy = 1$.

(b) $M_x = \int_0^1 dr\int_0^1 (3r + 2rs)(11r)\,ds = 44/3$; $M_y = \int_0^1 dr\int_0^1 (1 + r - 3rs)(11r)\,ds$ $= 11/3$.

(c) $M_x = \int_0^1 dr\int_0^\pi (r\sin\theta)r\,d\theta = 2/3$; $M_y = \int_0^1 dr\int_0^\pi (r\cos\theta)r\,d\theta = 0$. (Note: The result $M_y = 0$ may also be observed by the symmetry of $\mathscr{R}$ about the y axis.)

2. (a) $M_x = 3$, $M_y = 2$, $A = 3$ give $\langle 2/3, 1\rangle$,

(b) $M_x = \int_0^1 dr\int_0^{\pi/6} r^2\sin\theta\,d\theta = (2-\sqrt{3})/6$, $M_y = 1/6$. $A = \pi/12$ give $\langle 2/\pi, (4-2\sqrt{3})/\pi\rangle$,

(c) $M_x = \int_0^1 dx\int_{x^2}^{x} y\,dy = 1/15$, $M_y = 1/12$, $A = 1/6$ give $\langle 1/2, 2/5\rangle$.

3. (a) $M_{xy} = \int_0^1 dx\int_1^2 dy\int_2^3 z\,dz = 5/2$, $M_{xz} = 3/2$, $M_{yz} = 1/2$,

(b) $M_{xy} = M_{xz} = 0$ by symmetry. $M_{yz} = \int_{\mathscr{R}} dA\int_0^{2\pi} xy\,d\phi = 2\pi\int_{\mathscr{R}} xy\,dA =$ $2\pi\int_0^1 dx\int_0^x xy\,dy = \pi/4$.

(c) $M_{xy} = M_{xz} = 0$ by symmetry. Let $\mathscr{R}$ be the first quadrant of $x^2 + y^2 \le 1$. Then $M_{yz} = 2\pi\int_{\mathscr{R}} xy\,dA = 2\pi\int_0^1 dr\int_0^{\pi/2} r^3\cos\theta\sin\theta\,d\theta = \pi/4$.

4. (a) $M_{yz} = 2\pi\int_0^1 dx\int_0^{2x} xy\,dy = \pi$ and $V = 4\pi/3$ give $\langle 3/4, 0, 0\rangle$.

(b) $M_{xy} = \int_{\mathscr{R}} dA\int_0^{x+3} z\,dz = \int_{\mathscr{R}} (x+3)^2/2\,dA = \int_0^1 dr\int_0^{2\pi} (r\cos\theta + 3)^2 r/2\,d\theta$ $= 37\pi/8$, $M_{xz} = 0$, $M_{yz} = \pi/4$ and $V = 3\pi$ give $\langle 1/12, 0, 37/24\rangle$.

5. (a) $I_x = \int_0^1 dx\int_0^2 y^2\,dy = 8/3$; $I_y = \int_0^1 dx\int_0^2 x^2\,dy = 2/3$.

(b) $I_x = \int_0^1 dx\int_x^1 y^2\,dy = 1/4$; $I_y = 1/12$.

(c) $I_x = \int_0^1 dr\int_{-\pi/2}^{\pi/2} r^3\sin^2\theta\,d\theta = \pi/8$, $I_y = \pi/8$.

6. (a) $I_{xy} = \int_{\mathscr{R}} dA \int_0^1 z^2\,dz = \int_0^1 dr \int_0^{2\pi} r/3\,d\theta = \pi/3$; $I_{xz} = \int_0^1 dr \int_0^{2\pi} r^3 \sin^2\theta\,dr = \pi/4$; $I_{yz} = \pi/4$.
   (b) $I_{xy} = \int_{\mathscr{R}} dA \int_0^{2\pi} y^3 \sin^2\phi\,d\theta = \int_0^1 dx \int_0^{x^3} \pi y^3\,dy = \pi/52$; $I_{xz} = \pi/52$; $I_{yz} = \pi/9$.
7. (a) $I_x = 7/3$, $I_y = 1/3$, $A = 1$ give $\sqrt{7/3}$, $\sqrt{1/3}$;
   (b) $I_x = \pi/8$, $I_y = \pi/8$, $A = \pi/2$ give $1/2$, $1/2$.

XIV F

1. Representations $\mathbf{f}_1(r) = r\mathbf{i} + r\mathbf{j}$, $0 \le r \le 1$ for $\mathscr{C}_1$ and $\mathbf{f}_2(r) = (1+r)\mathbf{i} + (1-r)\mathbf{j}$, $0 \le r \le 1$ for $\mathscr{C}_2$ give $\int_{\mathscr{C}} g\,dL = \int_0^1 \sqrt{2}r^3\,dr + \int_0^1 \sqrt{2}(1+r)(1-r)^2\,dr = \sqrt{2}/4 + 5\sqrt{2}/12 = 2\sqrt{2}/3$.
2. Proceeding counterclockwise from the origin around the perimeter gives for the four segments $\int_0^1 x^2(0)\,dx + \int_0^1 (1)^2\,dy + \int_0^1 x^2(1)\,dx + \int_0^1 (0)^2 y\,dy = 0 + 1/2 + 1/3 + 0 = 5/6$.
3. $= \int_0^1 dr \int_0^1 (1+r)\,ds + \int_0^1 dr \int_0^1 2\,ds + \int_0^1 dr \int_0^1 (2-r)\sqrt{2}\,ds = 3\sqrt{2} + 2 + 3\sqrt{2}/2 = (7 + 3\sqrt{2})/2$.
4. Writing $\mathscr{R}$ as the union of $[0, 1] \times [0, 2]$ and $[1, 2] \times [1, 2]$ gives $\int_0^1 dx \int_0^2 xy\,dy + \int_1^2 dx \int_1^2 xy\,dy = 1 + 9/4 = 13/4$. Alternatively $\mathscr{R}$ can be written as the union of $[0, 1] \times [0, 1]$ and $[0, 2] \times [1, 2]$.
5. $= \int_0^1 dr \int_0^{\pi} (r \sin\theta) r\,d\theta + \int_{-1}^0 dx \int_{-x-1}^0 y\,dy + \int_0^1 dx \int_{x-1}^0 y\,dy = 2/3 - 1/6 - 1/6 = 1/3$.
6. (a) $= \int_0^1 dx \int_1^Y x/y^2\,dy = \int_0^1 x(1 - 1/Y)\,dx = 1/2 - 1/2Y$
   (b) $= 1/2$.
7. (a) $= \lim_{X\to\infty} \int_0^X dx \int_1^3 ye^{-x}\,dy = \lim_{X\to\infty} 4(1 - e^{-X}) = 4$,
   (b) $= \lim_{X\to\infty} \int_1^2 dy \int_{-X}^0 xye^{-x^2y}\,dx = -1/2$.
8. $= \lim_{X\to\infty} \int_1^X dx \int_0^{1/x} xy^2\,dy = \lim_{X\to\infty} (X-1)/3X = 1/3$.
9. (a) For $X > 1$ let $\mathscr{C}_X$ be the curve $y = x^2$ from $(1, 1)$ to (X, X^2). Then $\mathscr{C}_X$ is represented by $\mathbf{f}(x) = x\mathbf{i} + x^2\mathbf{j}$, $1 \le x \le X$, and $\int_{\mathscr{C}} g\,dL = \lim_{X\to\infty} \int_{\mathscr{C}_X} g\,dL = \lim_{X\to\infty} \int_1^X 1/x^3\,dx = 1/2$.
   (b) $= \lim_{X\to\infty} \int_2^X (1/x^5)\sqrt{2}\,dx = \sqrt{2}/64$.

10. (a) $= \int_X^1 dx \int_1^2 y/\sqrt{x}\, dx = 3(1 - \sqrt{X})$,
(b) $= 3$.

11. (a) $= \lim_{X\to 0} \int_1^8 dx \int_X^1 (xy)^{-1/3}\, dy = 27/4$.
(b) $= \lim_{X\to 0} \int_X^1 dx \int_X^1 (xy).^{-1/2}\, dy = 4$,
(c) $= \lim_{R\to 0} \int_R^1 dr \int_0^{2\pi} r^{1/3}\, d\theta = 3\pi/2$.

12. $= \lim_{X\to 0} \int_X^1 (\sqrt{2})2^{-1/3}x^{-2/3}\, dx = 3(2^{1/6})$.

Chapter XV

XV A

1. (a) $= r(r^2)\mathbf{i} - r^2\mathbf{j} = r^3\mathbf{i} - r^2\mathbf{j}$,
(b) $= dr/dr\ \mathbf{i} = dr^2/dr\ \mathbf{j} = \mathbf{i} + 2r\mathbf{j}$,
(c) $= \int_0^2 (r^3\mathbf{i} - r^2\mathbf{j}) \cdot (\mathbf{i} + 2r\mathbf{j})\, dr = -4$.

2. (a) $= \int_0^1 (r^7\mathbf{i} + r^6\mathbf{j}) \cdot (2r\mathbf{i} + 3r^2\mathbf{j})\, dr = 5/9$,
(b) $= \int_0^{2\pi} [(\cos\theta - 2\sin\theta)\mathbf{i} + (3\cos\theta + 4\sin\theta)\mathbf{j}]\cdot$
$(-\sin\theta\,\mathbf{i} + 2\cos\theta\,\mathbf{j})\, d\theta = 8\pi$.

3. (a) $= \int_0^1 (r^4\mathbf{i} + r^4\mathbf{j} + r^5\mathbf{k}) \cdot (\mathbf{i} + 2r\mathbf{j} + 3r^2\mathbf{k})\, dr = 109/120$,
(b) $= \int_0^1 [-r^3\mathbf{i} - (3r - 1)\mathbf{k}] \cdot (\mathbf{i} - 2r\mathbf{j} + 3\mathbf{k})\, dr = -7/4$.

4. (a) $= rs\mathbf{i} + (3r - s + r + 2s)\mathbf{j} + [2(r + 2s) - (3r - s)]\mathbf{k} =$
$rs\mathbf{i} + (10r - s)\mathbf{j} + (-r + 5s)\mathbf{k}$,
(b) $= (s\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (r\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 7\mathbf{i} + (r - 2s)\mathbf{j} - (3r + s)\mathbf{k}$,
(c) $= \int_0^1 dr \int_0^2 [7rs + (10r - s)(r - 2s) + (r - 5s)(3r + s)]\, ds = -82/3$.

5. (a) $= \int_0^1 dr \int_0^1 [s^2\mathbf{i} + (r^2s^2 + 2r)\mathbf{j}] \cdot [2s\mathbf{i} - 2s^2\mathbf{j} - r\mathbf{k}]\, ds = -3/10$,
(b) $= \int_0^1 dr \int_0^2 [(2r - s)\mathbf{i} + rs\mathbf{k}] \cdot [(r + s)\mathbf{i} - r\mathbf{j} - \mathbf{k}] = -4/3$.

XV B

1. (a) $= (xy + xy^2)dx + (x^3y - xy^2)dy = xy(1 + y)dx + xy(x^2 - y)dy$,
(b) $xy(y^2 + x)\, dx\, dy$,
(c) $4xy\, dx + 4y^2\, dy$,
(d) $-2e^{xy}\, dx\, dy$,
(e) $x(3 - 2x)\, dx + y(3 + 2y)\, dy$,
(f) $x^2(3 - x)\, dx\, dy$.

2. (a) $(3x + z)\,dx + z(y - 3x)\,dy + z(3y - x)\,dz$,
   (b) $(x + 2y)\,dy\,dz + dz\,dx + x(z - 2x)\,dx\,dy$,
   (c) $2x(3z + 2y)\,dx\,dy\,dz$.

3. (a) $xy^3\,dx\,dy$, (b) $-x^3y\,dx\,dy$,
   (c) $x^2y\,dx\,dy$, (d) $x^2y\,dx + xy^2\,dy$,
   (e) 0 (f) $xy^4\,dx\,dy$
   (g) $-xz^3\,dx\,dy\,dz$, (h) $-xy^5z\,dx\,dy\,dz$,
   (i) $x^3y^2\,dy\,dz - y^3\,dz\,dx + x^4y\,dy\,dx$,
   (j) 0, (k) $x^2yz\,dx\,dy\,dz$.

4. (a) $=(x^3y)_x dx + (x^3y)_y\,dy = 3x^2y\,dx + x^3\,dy$,
   (b) $=(3x^2y\,dx + x^3\,dy)\cdot dx = -x^3\,dx\,dy$,
   (c) $=(y^3\,dx + 3xy^2\,dy)\cdot dx + dx\cdot dy = (1 - 3xy^2)\,dx\,dy$
   (d) 0

5. (a) $yz^2\,dx + xz^2\,dy + 2xyz\,dz$,
   (b) $=(y\,dx + x\,dy)\cdot dx + (z\,dy + y\,dz)\cdot dz = z\,dy\,dz - x\,dx\,dy$,
   (c) $-z\,dx\,dy\,dz$,
   (d) 0.

XV C

1. (a) The representation $\mathbf{f}(x) = x\mathbf{i} + x^2\mathbf{j}$, $1 \le x \le 2$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ give
   $\int_{\mathscr{C}^0} \nabla g \cdot d\mathbf{f} = \int_1^2 (2x^3\mathbf{i} + x^2\mathbf{j})\cdot(\mathbf{i} + 2x\mathbf{j})\,dx = 15$. Also
   $g(\mathbf{f}(2)) = g(\mathbf{f}(1)) = g(2, 4) - g(1, 1) = 15$.
   (b) The representation $\mathbf{f}(\theta) = 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j}$, $0 \le \theta \le \pi$ and $\nabla g = \mathbf{i}$ give
   $\int_0^\pi \mathbf{i}\cdot(-2\sin\theta\,\mathbf{i} + 2\cos\theta\,\mathbf{j})\,d\theta = -4$. Also $g(-2, 0) - g(2, 0) = -4$.

2. From $g_x = 2xye^{x^2y}$ obtain $g = e^{x^2y} + H(y)$. From $g_y = x^2e^{x^2y} + dH/dy = x^2e^{x^2y}$ obtain $dH/dy = 0$ which has a solution $H = 0$. Thus $g = e^{x^2y}$ is an answer.

3. (a) $(x^2 + 4)_x = (2xy + 3)_y = 2x$. From $g_x = 2xy + 3$ obtain $g = x^2y + 3x + H(y)$; then $x^2 + dH/dy = g_y = x^2 + 4$ give $H = 4y$, and hence, $g = x^2y + 3x + 4y$.
   (b) $(2y + x\cos y)_x = (\sin y)_y = \cos y$. From $g_x = \sin y$ obtain $g = x\sin y + H(y)$; then $x\cos y + dH/dy = g_y = 2y + x\cos y$ gives $H = y^2$, and hence, $g = x\sin y + y^2$.

4. (a) $x^3\mathbf{i} + y^3\mathbf{j} = \nabla[(x^4 + y^4)/4]$ gives $1/4\,[(1^4 + (-2)^4] - (2^4 + 3^4)] = -20$.
   (b) $\mathbf{h} = \nabla(-\cos xy)$ gives $-\cos 3\pi/2 - (-\cos 0) = 1$.

5. (a) The representations $\mathbf{f}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, $0 \le \theta \le 2\pi$ and
   $\mathbf{f}'(\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $[0, 1] \times [0, 2\pi]$ give
   $\int_0^{2\pi} (\cos\theta\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = -\pi$ and
   $\int_0^1 dr \int_0^{2\pi} (-1 - r\cos\theta)r\,d\theta = -\pi$.
   (b) The area integral is $\int_0^1 dx \int_0^1 0\,dy = 0$. The line integral using representing functions $r\mathbf{i}$, $\mathbf{i} + r\mathbf{j}$, $(1 - r)\mathbf{i} + \mathbf{j}$, and $(1 - r)\mathbf{j}$; $0 \le r \le 1$, for the four sides

of the boundary is $\int_0^1 (2r\mathbf{i}) \cdot \mathbf{i} + \int_0^1 (2\mathbf{i} + r\mathbf{j}) \cdot \mathbf{j}\, dr + \int_0^1 [(2 - 2r)\mathbf{i} + \mathbf{j}] \cdot (-\mathbf{i})\, dr$ $+ \int_0^1 (1 - r)\mathbf{j} \cdot (-\mathbf{j})\, dr = 1 + 1/2 - 1 - 1/2 = 0.$

(c) The area integral is $\int_{\mathscr{R}} -x\, dA = \int_0^1 dr \int_0^1 (-rs)r\, ds = -1/6$. The boundary line integral using representations $r\mathbf{i} + r\mathbf{j}$, $(1 - r)\mathbf{i} + \mathbf{j}$, and $(1 - r)\mathbf{j}$; $0 \le r \le 1$, is $\int_0^1 (r^2\mathbf{i} - r^3\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})\, dr + \int_0^1 [(1 - r)\mathbf{i} - \mathbf{j}] \cdot (-\mathbf{i})\, dr +$ $\int_0^1 -(1 - r)^3\mathbf{j} \cdot (-\mathbf{j})\, dr = 1/12 - 1/2 + 1/4 = -1/6.$

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